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On a constant in the energy estimate

Sur une constante dans l'estimation d'énergie

Rafał Czyż, Van Thien Nguyen

Institute of Mathematics, Faculty of Mathematics and Computer Science, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland

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ABSTRACT

In this note, we prove that the constant $D(p,m)$ in the energy estimate, for m-subharmonic function with bounded p-energy, is strictly bigger than 1, for $p>0$, $p\neq 1$.

RÉSUMÉ

Dans cette note, nous prouvons que la constante $D(p,m)$ dans l'estimation d'énergie, pour les fonctions m-sous-harmoniques avec p-énergie finie, est strictement supérieure à 1, pour $p>0$, $p\neq 1$.

Version française abrégée

Soit Ω un domaine m-hyperconvexe de $\mathbb{C}^n$. On note $\mathcal{E}_{0,m}(\Omega)$ la classe des fonctions m-sous-harmoniques bornées dans Ω telles que

$$\lim_{z\to\partial\Omega} u(z)=0 \text{ and } \int_\Omega H_m(u)<+\infty,$$

où $\beta=dd^c|z|^2$ est la forme kählérinne standard sur $\mathbb{C}^n$ et $H_m(\cdot)=(dd^c(\cdot))^m\wedge\beta^{n-m}$ est l'opérateur hessien m-complexe. Pour chaque $p>0$, on note $\mathcal{E}_{p,m}(\Omega)$ la classe des fonctions m-sous-harmoniques négatives u telles qu'il existe une suite décroissante $\{u_j\}\subset\mathcal{E}_{0,m}(\Omega)$ vérifiant

(i) $\lim_{j\to\infty} u_j=u$,
(ii) $\sup_j \int_\Omega(-u_j)^p H_m(u_j)=\sup_j e_{p,m}(u_j)<+\infty$.

E-mail addresses: Rafal.Czyz@im.uj.edu.pl (R. Czyż), Thien.Van.Nguyen@im.uj.edu.pl (V.T. Nguyen).

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Nous remarquons que $e_{p,m}(u) = \int_\Omega (-u)^p H_m(u)$, la p-énergie m-pluricomplexe de la fonction u, est bornée pour tout $u \in \mathcal{E}_{p,m}(\Omega)$ (voir [12]). Il résulte de [12] que l'opérateur hessien m-complexe est clairement défini pour la classe $\mathcal{E}_{p,m}(\Omega)$.

Nous rappellerons l'estimation d'énergie cruciale pour la classe $\mathcal{E}_{p,m}(\Omega)$.

Théorème 0.1. *Soient $u_0,\ u_1, \dots, u_m \in \mathcal{E}_{p,m}(\Omega)$. Il existe une constante $D(p,m) \geq 1$ telle que*

$$\int_\Omega (-u_0)^p dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge \beta^{n-m}$$
$$\leq D(p,m) e_{p,m}(u_0)^{\frac{p}{p+m}} e_{p,m}(u_1)^{\frac{1}{p+m}} \cdots e_{p,m}(u_m)^{\frac{1}{p+m}},$$

où

$$D(p,m) = \begin{cases} p^{-\frac{\alpha(p,m)}{1-p}}, & \text{if } 0 < p < 1, \\ 1, & \text{if } p = 1, \\ p^{\frac{p\alpha(p,m)}{p-1}}, & \text{if } p > 1, \end{cases}$$

et $\alpha(p,m) = (p+2)\left(\frac{p+1}{p}\right)^{m-1} - (p+1)$.

La preuve de ce théorème se trouve dans [15] (voir aussi [3,12,14]). Il est important, pour la théorie des fonctions δ-plurisousharmoniques, de savoir si la constante $D(p,m)$ est égale à ou plus grande que 1. Si $D(p,m) = 1$ pour toutes les fonctions dans $\mathcal{E}_{p,m}(\Omega)$, alors l'espace vectoriel $\delta\mathcal{E}_{p,m}(\Omega) = \mathcal{E}_{p,m}(\Omega) - \mathcal{E}_{p,m}(\Omega)$ muni d'une certaine norme serait un espace de Banach (voir [2,15]). De plus, les preuves dans [2,15] pourraient être simplifiées, tandis que certaines seraient superflues. Pour le cas $m = n$, Åhag and Czyż ont montré qu'il existe des fonctions telles que $D(p,n)$ soit strictement supérieur à 1. En conséquence, nous nous posons la même question lorque $m < n$. Dans cette note, nous montrons que cela est encore vrai.

Théorème 0.2. *Pour $1 \leq m \leq n$, $p > 0$ $(p \neq 1)$, il existe des fonctions dans $\mathcal{E}_{p,m}(\mathbb{B})$, où $\mathbb{B}$ est la boule unité de $\mathbb{C}^n$, telles que la constante $D(p,m) > 1$.*

1. Introduction

A bounded domain $\Omega \subset \mathbb{C}^n$ is said to be a m-hyperconvex domain if there exists a continuous m-subharmonic function $\rho \colon \Omega \to \mathbb{R}^-$ such that $\{\rho < -c\} \Subset \Omega$, for all $c > 0$. Let $\mathcal{E}_{0,m}(\Omega)$ denote the set of all bounded m-subharmonic functions u defined on Ω such that

$$\lim_{z \to \partial\Omega} u(z) = 0 \text{ and } \int_\Omega H_m(u) < +\infty,$$

where $\beta = dd^c|z|^2$ is the canonical Kähler form in $\mathbb{C}^n$ and $H_m(\cdot) = (dd^c(\cdot))^m \wedge \beta^{n-m}$ is the m-complex Hessian operator. For each $p > 0$, we define $\mathcal{E}_{p,m}(\Omega)$ to be the class of all negative m-subharmonic functions u such that there exists a decreasing sequence $\{u_j\} \subset \mathcal{E}_{0,m}(\Omega)$ such that

(i) $\lim_{j\to\infty} u_j = u$,
(ii) $\sup_j \int_\Omega (-u_j)^p H_m(u_j) = \sup_j e_{p,m}(u_j) < +\infty$.

We note that $e_{p,m}(u) = \int_\Omega (-u)^p H_m(u)$, the m-pluricomplex p-energy of the function u, is bounded for any u in $\mathcal{E}_{p,m}(\Omega)$ (see [12]). It follows from [12] that the complex Hessian operator is well defined on $\mathcal{E}_{p,m}(\Omega)$. For further information about the complex Hessian operator, we refer the reader to [6,9,13] (see also [4,5,7,8,10,11]).

We recall a crucial energy estimate for the class $\mathcal{E}_{p,m}(\Omega)$.

Theorem 1.1. *Let $u_0,\ u_1, \dots, u_m \in \mathcal{E}_{p,m}(\Omega)$. Then there exists a constant $D(p,m) \geq 1$ depending only on p and m such that*

$$\int_\Omega (-u_0)^p dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge \beta^{n-m}$$
$$\leq D(p,m) e_{p,m}(u_0)^{\frac{p}{p+m}} e_{p,m}(u_1)^{\frac{1}{p+m}} \cdots e_{p,m}(u_m)^{\frac{1}{p+m}},$$

where

$$D(p,m)=\begin{cases} p^{-\frac{\alpha(p,m)}{1-p}}, & \text{if } 0<p<1,\\ 1, & \text{if } p=1,\\ p^{\frac{p\alpha(p,m)}{p-1}}, & \text{if } p>1,\end{cases}$$

and $\alpha(p,m)=(p+2)\left(\frac{p+1}{p}\right)^{m-1}-(p+1)$.

The proof of this theorem can be found in [15] (see also [3,12,14]). It is important, for the theory of δ-plurisubharmonic functions, to know if the constant $D(p,m)$ is equal to 1 or bigger than 1. If $D(p,m)=1$ for all functions in $\mathcal{E}_{p,m}(\Omega)$, then the vector space $\delta\mathcal{E}_{p,m}(\Omega)=\mathcal{E}_{p,m}(\Omega)-\mathcal{E}_{p,m}(\Omega)$, with a certain norm, would be a Banach space (see [2,15]). Furthermore, proofs in [2,15] could be simplified, and some of them would be superfluous. For the case $m=n$, Åhag and Czyż showed that there are functions such that, for all $n\in\mathbb{N}$ and all $p>0$ $(p\neq 1)$, the constant $D(p,n)$ is strictly greater than 1. For more details, we refer the reader to [1]. Thus, one may ask the same question when $m<n$. In this note, we show that it is still true by using the inequality for Beta function in [1].

Theorem 1.2. *For* $1\le m\le n$, $p>0$ $(p\neq 1)$, *there are functions in* $\mathcal{E}_{p,m}(\mathbb{B})$, *where* $\mathbb{B}$ *is the unit ball in* $\mathbb{C}^n$, *such that the constant* $D(p,m)>1$.

2. Proof of Theorem 1.2

Let us recall some basic properties of the Beta function. For $x, y>0$, the Beta function $B(x,y)$ is defined by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t.$$

A key property of the Beta function is its relationship to the Gamma function as follows:

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where the Gamma function is given by

$$\Gamma(x)=\int_0^{+\infty} \mathrm{e}^{-t}t^{x-1}\mathrm{d}t.$$

This implies that the Beta function is symmetric. We can compute its partial derivative

$$\frac{\partial}{\partial y}B(x,y)=B(x,y)\left(\frac{\Gamma'(y)}{\Gamma(y)}-\frac{\Gamma'(x+y)}{\Gamma(x+y)}\right)=B(x,y)(\psi(y)-\psi(x+y)), \tag{1}$$

where $\psi(y)=\frac{\Gamma'(y)}{\Gamma(y)}$ is the digamma function. A crucial tool for the proof of Theorem 1.2 is the following lemma.

Lemma 2.1. *Let* $f:\mathbb{N}\times(0,+\infty)\to\mathbb{R}$ *be the function defined by*

$$f(p,n)=\frac{1}{n}+\frac{p}{p+n}+\psi(n)-\psi(n+p+1). \tag{2}$$

Then we have $f(p,n)\neq 0$ *for all* $n\in\mathbb{N}$ *and all* $p>0$ $(p\neq 1)$.

Proof. This follows from [1, Lemma 2.2]. □

Proof of Theorem 1.2. For $a>0$, we define the family of m-subharmonic functions in $\mathbb{B}$ by

$$u_a(z)=|z|^{2a}-1.$$

We can compute

$$\begin{aligned}\sigma_m(\lambda(\mathrm{Hess}(u_a)))&=\left[\binom{n}{m}+\binom{n-1}{m-1}(a-1)\right]a^m|z|^{2m(a-1)}\\ &=\frac{(n-1)!}{m!(n-m)!}[n+m(a-1)]a^m|z|^{2m(a-1)},\end{aligned}$$

where $\lambda(\mathrm{Hess}(u)) = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ are the eigenvalues of the complex Hessian matrix of u and $\sigma_m(\lambda)$ is the mth elementary symmetric polynomial with respect to $\lambda \in \mathbb{R}^n$. Thus

$$H_m(u_a) = (\mathrm{dd}^c u_a)^m \wedge \beta^{n-m} = C[n + m(a-1)]a^m |z|^{2m(a-1)} \mathrm{d}\lambda_n,$$

where C is the constant depending only on n, m, and $\mathrm{d}\lambda_n$ is the Lebesgue measure on $\mathbb{C}^n$. For $b > 0$, then we have

$$\begin{aligned}\int_{\mathbb{B}} (-u_a)^p H_m(u_b) &= C[n + m(b-1)]b^m \int_{\mathbb{B}} (1 - |z|^{2a})^p |z|^{2m(b-1)} \mathrm{d}\lambda_n \\ &= C[n + m(b-1)]b^m \int_{\partial\mathbb{B}} \mathrm{d}\sigma_n \int_0^1 (1 - t^{2a})^p t^{2m(b-1)} t^{2n-1} \mathrm{d}t \\ &= C[n + m(b-1)]b^m \sigma_n(\partial\mathbb{B}) \int_0^1 (1 - t^{2a})^p t^{2m(b-1)} t^{2n-1} \mathrm{d}t \\ &= C[n + m(b-1)] \frac{2\pi^n}{(n-1)!} \frac{b^m}{2a} \int_0^1 (1-s)^p s^{\frac{n+m(b-1)}{a} - 1} \mathrm{d}s \\ &= C_0[n + m(b-1)] \frac{b^m}{a} B\left(p+1, \frac{n + m(b-1)}{a}\right),\end{aligned} \tag{3}$$

where C_0 depends only on m and n. Letting $a = b$ in (3), we get

$$\begin{aligned}e_{p,m}(u_a) &= \int_{\mathbb{B}} (-u_a)^p H_m(u_a) \\ &= C_0[n + m(a-1)]a^{m-1} B\left(p+1, \frac{n + m(a-1)}{a}\right).\end{aligned}$$

Assume that $D(p, m) = 1$ in Theorem 1.1, then

$$\begin{aligned}&C_0[n + m(b-1)] \frac{b^m}{a} B\left(p+1, \frac{n + m(b-1)}{a}\right) \\ &\le C_0^{\frac{p}{p+m}} [n + m(a-1)]^{\frac{p}{p+m}} a^{\frac{(m-1)p}{p+m}} B\left(p+1, \frac{n + m(a-1)}{a}\right)^{\frac{p}{p+m}} \\ &\quad \times C_0^{\frac{m}{p+m}} [n + m(b-1)]^{\frac{m}{p+m}} b^{\frac{(m-1)m}{p+m}} B\left(p+1, \frac{n + m(b-1)}{b}\right)^{\frac{m}{p+m}}.\end{aligned}$$

By simplifying this inequality, we get

$$\begin{aligned}&\left[\frac{n + m(b-1)}{n + m(a-1)}\right]^{\frac{p}{p+m}} \left(\frac{b}{a}\right)^{\frac{mp+m}{p+m}} B\left(p+1, \frac{n + m(b-1)}{a}\right) \\ &\le B\left(p+1, \frac{n + m(a-1)}{a}\right)^{\frac{p}{p+m}} B\left(p+1, \frac{n + m(b-1)}{b}\right)^{\frac{m}{p+m}}.\end{aligned}$$

Consider the function $F \colon (0, +\infty) \times (0, +\infty) \to \mathbb{R}$ defined by

$$\begin{aligned}F(a, b) = &\left[\frac{n + m(b-1)}{n + m(a-1)}\right]^{\frac{p}{p+m}} \left(\frac{b}{a}\right)^{\frac{mp+m}{p+m}} B\left(p+1, \frac{n + m(b-1)}{a}\right) \\ &- B\left(p+1, \frac{n + m(a-1)}{a}\right)^{\frac{p}{p+m}} B\left(p+1, \frac{n + m(b-1)}{b}\right)^{\frac{m}{p+m}}.\end{aligned}$$

We can see that F is continuously differentiable and $F(1, 1) = 0$. Thus, to prove Theorem 1.2, it is enough to show that $\frac{\partial}{\partial b} F(1, 1) \neq 0$. Using formula (1), we have

$$\begin{aligned}\frac{\partial}{\partial b} B\left(p+1, \frac{n + m(b-1)}{a}\right) = &\frac{m}{a} B\left(p+1, \frac{n + m(b-1)}{a}\right) \\ &\times \left[\psi\left(\frac{n + m(b-1)}{a}\right) - \psi\left(p+1+\frac{n + m(b-1)}{a}\right)\right]\end{aligned}$$

and

$$\frac{\partial}{\partial b}B\left(p+1,\frac{n+m(b-1)}{b}\right)=-\frac{n-m}{b^2}B\left(p+1,\frac{n+m(b-1)}{b}\right)$$
$$\times\left[\psi\left(\frac{n+m(b-1)}{b}\right)-\psi\left(p+1+\frac{n+m(b-1)}{b}\right)\right].$$

Thus $\frac{\partial}{\partial b}F(a,b)$ is equal to

$$\frac{m}{n+m(a-1)}\frac{p}{p+m}\left[\frac{n+m(b-1)}{n+m(a-1)}\right]^{-\frac{m}{p+m}}\left(\frac{b}{a}\right)^{\frac{mp+m}{p+m}}B\left(p+1,\frac{n+m(b-1)}{a}\right)$$
$$+\frac{1}{a}\left[\frac{n+m(b-1)}{n+m(a-1)}\right]^{\frac{p}{p+m}}\frac{mp+m}{p+m}\left(\frac{b}{a}\right)^{\frac{mp-p}{p+m}}B\left(p+1,\frac{n+m(b-1)}{a}\right)$$
$$+\left[\frac{n+m(b-1)}{n+m(a-1)}\right]^{\frac{p}{p+m}}\left(\frac{b}{a}\right)^{\frac{mp+m}{p+m}}\frac{m}{a}B\left(p+1,\frac{n+m(b-1)}{a}\right)$$
$$\times\left[\psi\left(\frac{n+m(b-1)}{a}\right)-\psi\left(p+1+\frac{n+m(b-1)}{a}\right)\right]$$
$$+B\left(p+1,\frac{n+m(a-1)}{a}\right)^{\frac{p}{p+m}}B\left(p+1,\frac{n+m(b-1)}{b}\right)^{\frac{m}{p+m}}\frac{m}{p+m}$$
$$\times\frac{n-m}{b^2}\left[\psi\left(\frac{n+m(b-1)}{b}\right)-\psi\left(p+1+\frac{n+m(b-1)}{b}\right)\right].$$

Now we have

$$\frac{\partial}{\partial b}F(1,1)=\frac{mp+mn}{p+m}B(p+1,n)\left(\frac{1}{n}+\frac{p}{p+n}+\psi(n)-\psi(p+1+n)\right).$$

Lemma 2.1 implies that $\frac{\partial}{\partial b}F(1,1)\neq 0$. So Theorem 1.2 is proved. □

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