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Algebraic geometry

Ulrich bundles on blowing up (and an erratum)

Fibrés de Ulrich sur les éclatements (et un erratum)

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A B S T R A C T

We deal with the behaviour of Ulrich bundles with respect to push-forward and pull-back via blowing-up points. We also correct a wrong statement in [11].

R É S U M É

Nous décrivons le comportement des faisceaux d'Ulrich en ce qui concerne leur image directe et réciproque par rapport aux éclatements des points. Nous corrigeons aussi un énoncé incorrect dans [11].

1. Introduction and notation

Throughout this note, we will work on an algebraically closed field k of characteristic 0, and $\mathbb{P}^N$ will denote the projective space over k of dimension N. A surface is a smooth connected projective scheme of dimension 2.

Let $X \subseteq \mathbb{P}^N$ be a smooth n-dimensional *variety*, i.e. a closed integral subscheme, and set $\mathcal{O}_X(h) := \mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{O}_X$. We are interested in studying vector bundles on $X \subseteq \mathbb{P}^N$. Recently, many authors focused their attention on *Ulrich bundles (with respect to* $\mathcal{O}_X(h)$*)*, i.e. vector bundles on X such that

$$H^i\big(X, \mathcal{F}(-ih)\big) = H^j\big(X, \mathcal{F}(-(j+1)h)\big) = 0$$

for each $i > 0$ and $j < n$.

Serre duality immediately yields that $\mathcal{F}$ is Ulrich if and only if $\mathcal{F}^\vee(K_X + (n+1)h)$ is Ulrich as well (here K_X denotes the canonical divisor on X). Moreover, each Ulrich bundle $\mathcal{F}$ is globally generated and *aCM*, i.e. $h^i\big(X, \mathcal{F}(th)\big) = 0$ for each $i = 1, \dots, n-1$ and $t \in \mathbb{Z}$: we refer the interested reader to the paper by D. Eisenbud, F.-O. Schreyer, and J. Weyman [8].

It is clear that each direct summand of an Ulrich bundle is also Ulrich. Thus, one can restrict the attention to *indecomposable* bundles, i.e. bundles that do not split as direct sum of bundles of smaller ranks. The study of indecomposable Ulrich bundles is a particularly intriguing problem that could give some suggestions on the complexity of the embedding

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$X \subseteq \mathbb{P}^N$. For instance, it is natural to ask whether a bound on the dimensions of the families of indecomposable Ulrich bundles supported on X actually exists. Indeed, the known examples suggest that a lot of varieties are *Ulrich-wild*, i.e. support p-dimensional families of pairwise non-isomorphic, indecomposable Ulrich bundles for arbitrary large p.

In the present paper, we slightly improve some results from [11] (see Theorem 0.1 and Corollary 0.2), also correcting a wrong statement about Ulrich bundles on smooth surfaces (see Theorem 0.3), whose proof contains a gap pointed out by the first author.

More precisely, in Section 2 we prove some general facts about the push-forward and the pull-back of a vector bundle via a blow-up map $\sigma : \widetilde{X} \to X$ at a point $P \in X$. The proofs are quite standard, and generalize some results due to P. Coronica and R.L.E. Schwarzenberger (see [7], [13]).

In Section 3, we use Theorem 0.1 of [11] to prove that all the surfaces of degree at least 4 in $\mathbb{P}^4$, which are obtained by *inner projection* (i.e. by projecting a surface in $\mathbb{P}^5$ from one of its points: see [3] for details), are Ulrich-wild, with the possible exception of some special $K3$ surfaces. This result extends a similar one, proved by the first author in [5] for rational surfaces.

In Section 4, we show that if $\mathcal{E}$ is an Ulrich bundle such that $\mathcal{E}(E)$ is trivial on E, then $\sigma_*\mathcal{E}(E)$ is also Ulrich, where $E = \sigma^{-1}(P)$ is the exceptional divisor. This result has been proved in [11] when $n = r = 2$ and $\mathcal{E}$ satisfies another technical restriction without the triviality hypothesis. Nevertheless, the proof therein contains a gap, and we show with two examples that we cannot remove the hypothesis from our corrected statement.

2. General results

Let X be a smooth variety of dimension n, and let $\sigma : \widetilde{X} \to X$ be the blow up at $P \in X$. When $n = 1$, σ is an isomorphism, hence we will assume $n \geq 2$ in the following lines.

We have $\operatorname{Pic}(\widetilde{X}) \cong \sigma^* \operatorname{Pic}(X) \oplus \mathbb{Z}\mathcal{O}_{\widetilde{X}}(E)$. Moreover, the exceptional divisor $E := \sigma^{-1}(P)$ is isomorphic to $\mathbb{P}^{n-1}$, hence $\operatorname{Pic}(E)$ is principal and generated by an ample line bundle isomorphic to $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$. If we denote by $\mathcal{I}$ the ideal sheaf of E inside $\widetilde{X}$, then $\mathcal{I}/\mathcal{I}^2 \cong \mathcal{O}_{\widetilde{X}}(-E) \otimes \mathcal{O}_E \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)$, hence $E^n = (-1)^{n-1}$ and $\mathcal{I}^m/\mathcal{I}^{m+1} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(m)$ for each $m \geq 1$. Finally

$$\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) = \sigma^*\mathcal{O}_X(K_X) \otimes \mathcal{O}_{\widetilde{X}}((n-1)E) \tag{1}$$

(see [9], Exercise II.8.5). We deduce that $\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) \otimes \mathcal{O}_E \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1-n)$.

Let $\mathcal{E}$ be a vector bundle on $\widetilde{X}$. In general $\mathcal{E} \otimes \mathcal{O}_E$ could be indecomposable (e.g., see Example 2 below) unless $n = 2$. Indeed, in this case, a theorem of Grothendieck yields $\mathcal{E} \otimes \mathcal{O}_E \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(\alpha_i)$.

The result below extends [7], Section 1.3 (see also Theorem 5 of [13] and its proof).

Theorem 2.1. *Let X be a smooth variety and let $\mathcal{E}$ be a vector bundle of rank r on $\widetilde{X}$.*

Assume $\mathcal{E} \otimes \mathcal{O}_E \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^{n-1}}(\alpha_i)$, where $-n \leq \alpha_i \leq \alpha_{i+1}$ for $i = 1, \dots, r-1$, and denote by s the maximum integer such that $\alpha_s \leq 0$. Then $R^j\sigma_*\mathcal{E} = 0$ for $j \geq 1$ and

$$\dim_{k(x)}(\sigma_*\mathcal{E} \otimes k(x)) = \begin{cases} r & \text{if } x \neq P, \\ s + \sum_{i=s+1}^r \binom{\alpha_i + n - 1}{n-1} & \text{if } x = P. \end{cases} \tag{2}$$

Proof. If $n = 1$, the statement is trivial. Thus we assume $n \geq 2$ from now on.

Since σ induces an isomorphism $\widetilde{X} \setminus E \cong X \setminus \{P\}$, it suffices to check that $(R^j\sigma_*\mathcal{E})_P = 0$ for $j \geq 1$ and

$$\dim_{k(P)}(\sigma_*\mathcal{E} \otimes k(P)) = s + \sum_{i=s+1}^r \binom{\alpha_i + n - 1}{n-1}.$$

The proof of the above facts runs along the same lines of the proof of Proposition 1.3.8 in [7].

First we have to show that $(R^j\sigma_*\mathcal{E})_P = 0$ for each $j \geq 1$. To this purpose, we use the isomorphism

$$\widehat{(R^j\sigma_*\mathcal{E})}_P \cong \varprojlim H^j\big(E_m, \mathcal{E} \otimes \mathcal{O}_{E_m}\big)$$

where $\mathcal{O} := \mathcal{O}_{X,P}$, $\mathfrak{m} \subseteq \mathcal{O}$ is the maximal ideal and $E_m := \widetilde{X} \times_X \operatorname{Spec}(\mathcal{O}/\mathfrak{m}^m)$: hence $E_1 = E \cong \mathbb{P}^{n-1}$. We have an exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(m) \longrightarrow \mathcal{O}_{E_{m+1}} \longrightarrow \mathcal{O}_{E_m} \longrightarrow 0. \tag{3}$$

For each $t \in \mathbb{Z}$, we have

$$H^0\big(E_1, \mathcal{E} \otimes \mathcal{O}_{E_1}(t)\big) = \bigoplus_{i=1}^r k[x_0, \dots, x_{n-1}]_{t+\alpha_i},$$

$$H^j\big(E_1, \mathcal{E}\otimes\mathcal{O}_{E_1}(t)\big) = 0, \qquad j \ge 1,\ j \ne n-1,$$
$$H^{n-1}\big(E_1, \mathcal{E}\otimes\mathcal{O}_{E_1}(t)\big) = \bigoplus_{i=1}^{r} k[x_0,\dots,x_{n-1}]_{-n-t-\alpha_i},$$

where the images of $x_0,\dots,x_{n-1}$ inside $\mathcal{O}$ is a system of regular local parameters generating $\mathfrak{m}$.

Tensoring Sequence (3) by $\mathcal{E}$, we obtain that

$$h^j\big(E_m, \mathcal{E}\otimes\mathcal{O}_{E_m}\big) = h^j\big(E_1, \mathcal{E}\otimes\mathcal{O}_{E_1}\big) = 0, \qquad j = 1,\dots,n-2.$$

By the induction on m, we deduce that

$$H^0\big(E_m, \mathcal{E}\otimes\mathcal{O}_{E_m}\big) = \bigoplus_{i=1}^{r}\bigoplus_{t=0}^{m-1} k[x_0,\dots,x_{n-1}]_{t+\alpha_i},$$
$$H^j\big(E_m, \mathcal{E}\otimes\mathcal{O}_{E_m}\big) = 0, \qquad j = 1,\dots,n-2.$$

Moreover, the hypothesis on the α_i's implies that $H^{n-1}\big(E_1, \mathcal{E}\otimes\mathcal{O}_{E_1}(t)\big) = 0$ for every positive t, hence

$$H^{n-1}\big(E_m, \mathcal{E}\otimes\mathcal{O}_{E_m}\big) = \bigoplus_{i=1}^{r}\bigoplus_{t=0}^{m-1} k[x_0,\dots,x_{n-1}]_{-n-t-\alpha_i}.$$

Trivially $(\widehat{R^j\sigma_*\mathcal{E}})_P = 0$ for $j = 1,\dots,n-2$. A case by case analysis shows that

$$\widehat{(\sigma_*\mathcal{E})}_P \cong \left(\bigoplus_{\alpha_i\le 0} k \oplus \bigoplus_{\alpha_i\ge 1}(x_0,\dots,x_{n-1})^{\alpha_i}\right)\otimes k[[x_0,\dots,x_{n-1}]],$$
$$(\widehat{R^{n-1}\sigma_*\mathcal{E}})_P \cong \bigoplus_{\alpha_i\le -n-1} k[x_0,\dots,x_{n-1}]/(x_0,\dots,x_{n-1})^{-\alpha_i-n}.$$

Thanks to the hypothesis on the α_i's, we have $(R^j\sigma_*\mathcal{E})_P = 0$ for each $j \ge 1$. Finally, the statement on $\sigma_*\mathcal{E}\otimes k(P)$ is an easy computation. □

Corollary 2.2. *Let X be a smooth variety and let $\mathcal{E}$ be a vector bundle of rank r on $\widetilde{X}$.*

Assume $\mathcal{E}\otimes\mathcal{O}_E \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^{n-1}}(\alpha_i)$, where $-n \le \alpha_i$ for $i = 1,\dots,r$. Then $\sigma_*\mathcal{E}$ is a vector bundle on X if and only if $\alpha_i \le 0$ for $i = 1,\dots,r$.

Proof. Thanks to Nakayama's lemma, $\sigma_*\mathcal{E}$ is a vector bundle if and only if $\dim_{k(x)}(\sigma_*\mathcal{E}\otimes k(x)) = r$ for each $x \in X$. Thus the statement is an immediate consequence of Equalities (2). □

Corollary 2.3. *Let X be a smooth variety and let $\mathcal{E}$ be a vector bundle of rank r on $\widetilde{X}$.*

Assume $\mathcal{E}\otimes\mathcal{O}_E \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^{n-1}}(\alpha_i)$, where $-n \le \alpha_i \le 0$ for $i = 1,\dots,r$. Then the following assertions are equivalent.

(a) $\mathcal{E} \cong \sigma^*\mathcal{F}$ where $\mathcal{F}$ is a vector bundle on X.
(b) $\alpha_i = 0$ for $i = 1,\dots,r$.
(c) The natural morphism $\sigma^*\sigma_*\mathcal{E} \to \mathcal{E}$ is an isomorphism.

Proof. Notice that $\sigma_*\mathcal{E}$ is a vector bundle, thanks to Corollary 2.2. If $\mathcal{E} \cong \sigma^*\mathcal{F}$ for some bundle $\mathcal{F}$ on X, then $\mathcal{E}\otimes\mathcal{O}_E \cong \sigma^*\mathcal{F}\otimes\sigma^*k(P) \cong \sigma^*k(P)^{\oplus r} \cong \mathcal{O}_{\mathbb{P}^{n-1}}^{\oplus r}$.

Let $\mathcal{E}\otimes\mathcal{O}_E \cong \mathcal{O}_{\mathbb{P}^{n-1}}^{\oplus r}$. The morphism $\sigma^*\sigma_*\mathcal{E} \to \mathcal{E}$ is trivially an isomorphism outside E. By the hypothesis, its restriction to E is an isomorphism. Thus it is surjective at each point of E, hence the assertion follows because a surjective map of vector bundles of the same rank is an isomorphism.

Finally, if $\sigma^*\sigma_*\mathcal{E} \to \mathcal{E}$ is an isomorphism, then $\mathcal{E} \cong \sigma^*\mathcal{F}$ where $\mathcal{F} := \sigma_*\mathcal{E}$. □

3. Pulling back Ulrich bundles

If $X \cong \mathbb{P}^n$ and $\mathcal{O}_X(h) \cong \mathcal{O}_{\mathbb{P}^n}(1)$, the unique Ulrich bundle on X is $\mathcal{O}_X$, thanks to the Horrocks theorem. Thus, from now on we will assume that $\deg(X) \ge 2$.

Let $h^0\big(X, \mathcal{O}_X(h)\big) = N+1$: we will always assume that $\mathcal{O}_{\widetilde{X}}(\widetilde{h}) := \sigma^*\mathcal{O}_X(h)\otimes\mathcal{O}_{\widetilde{X}}(-E)$ is very ample: then $\widetilde{X}$ is embedded in $\mathbb{P}^{N-1}$.

Example 1. If $\mathcal{F}$ is Ulrich with respect to $\mathcal{O}_X(h)$, it is not true in general that $\sigma^*\mathcal{F}$ is Ulrich with respect to $\mathcal{O}_{\widetilde{X}}(\widetilde{h})$.

Indeed, let X be a surface, $\mathcal{F}$ an Ulrich bundle of rank r on X and set $\mathcal{E} := \sigma^*\mathcal{F}$. Consider the standard sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}}(-E) \longrightarrow \mathcal{O}_{\widetilde{X}} \longrightarrow \mathcal{O}_E \longrightarrow 0. \tag{4}$$

Tensoring Sequence (4) by $\mathcal{E}(E) \otimes \sigma^*\mathcal{O}_X(-2h)$ and $\mathcal{E}(-2\widetilde{h})$, we obtain two exact sequences

$$0 \longrightarrow \mathcal{E} \otimes \sigma^*\mathcal{O}_X(-2h) \longrightarrow \mathcal{E}(E) \otimes \sigma^*\mathcal{O}_X(-2h) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus r} \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{E}(E) \otimes \sigma^*\mathcal{O}_X(-2h) \longrightarrow \mathcal{E}(-2\widetilde{h}) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus r} \longrightarrow 0.$$

We obtain $h^i\big(\widetilde{X}, \mathcal{E}(E) \otimes \sigma^*\mathcal{O}_X(-2h)\big) = 0$ for $i \geq 0$, from the cohomology of the first sequence and the projection formula. Thus $h^1\big(\widetilde{X}, \mathcal{E}(-2\widetilde{h})\big) = h^1\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus r}\big) = r \neq 0$ from the cohomology of the second exact sequence.

If $\mathcal{F}$ is any bundle on X, then $\widetilde{\mathcal{F}} := \sigma^*\mathcal{F}(-E)$ is trivially a bundle on $\widetilde{X}$. Notice that $\sigma_*\widetilde{\mathcal{F}}(E) \cong \mathcal{F}$. For the following result, see Theorem 0.1 of [11].

Theorem 3.1. *Let X be a smooth variety endowed with a very ample line bundle $\mathcal{O}_X(h)$. Assume that $\mathcal{O}_{\widetilde{X}}(\widetilde{h})$ is also very ample. If $\mathcal{F}$ is an Ulrich bundle with respect to $\mathcal{O}_X(h)$, then $\widetilde{\mathcal{F}}$ is an Ulrich bundle with respect to $\mathcal{O}_{\widetilde{X}}(\widetilde{h})$.*

Below we deal with a by-product of the above theorem. Recall that a surface $\widetilde{X} \subseteq \mathbb{P}^4$ is said to be obtained by inner projection if there is a surface $X \subseteq \mathbb{P}^5$ and $P \in X$ such that $\widetilde{X}$ is the closure of the image of X via the projection from $\mathbb{P}^5$ to $\mathbb{P}^4$ with centre P. A surface $\widetilde{X} \subseteq \mathbb{P}^4$ is said *non-degenerate* if it is not contained in any hyperplane.

Proposition 3.1. *Every non-degenerate surface of degree at least 4 in $\mathbb{P}^4$ obtained by an inner projection is Ulrich-wild, possibly except for some special $K3$ surfaces.*

Proof. A surface $\widetilde{X}$ as in the statement is abstractly isomorphic to the blow up of a surface X at $P \in X$. The rational projection map $\pi\colon X \dashrightarrow \widetilde{X}$ inverts the blow-up map σ outside the exceptional divisor E. These surfaces are described in Table A of [3], where they are classified in the eight classes (I), (II), (III), (IV), (V), (VI), (VII), and (VIII).

The surfaces in classes (II), (III), (IV), (V), and (VII) are rational, linearly normal and non-special (see [1], Théorème 1), hence they are Ulrich-wild thanks to [5], Example 5.1. Thus it remains to consider surfaces of types (VIII) and (VI).

Each surface in the class (VIII) is obtained by blowing up a point on an Enriques surface $X \subseteq \mathbb{P}^5$ of degree 10, which is again Ulrich-wild, thanks to Corollary 5.1 of [5].

Finally, let us examine the surfaces in the class (VI). They are obtained by blowing up a point on the complete intersection $X \subseteq \mathbb{P}^5$ of three quadrics, hence such X are $K3$ surfaces and $p_g(X) = 1$, $q(X) = 0$. In particular, surfaces in class (VI) have degree 7 and sectional genus 5 (see Table A of [3]). Thus, they correspond to the points of a subset $\mathcal{H}_0$ of the Hilbert scheme $\mathcal{H}$ of subschemes of $\mathbb{P}^4$ with the Hilbert polynomial $(7t^2 - t + 4)/2$. We will show below that $\mathcal{H}_0$ is irreducible, and that its generic point (which corresponds to a surface in class (VI)) is Ulrich-wild.

Let $\mathcal{G}$ be the Grassmannian of subspaces Σ of dimension 2 of $|\mathcal{O}_{\mathbb{P}^5}(2)| \cong \mathbb{P}^{20}$. We define

$$\mathcal{V} := \{\, (P, \Sigma) \mid P \in D,\ \forall D \in \Sigma \,\} \subseteq \mathbb{P}^5 \times \mathcal{G} \xrightarrow{\gamma} \mathcal{G}.$$

The fibre of the projection on $\mathbb{P}^5$ over $P \in \mathbb{P}^5$ is the Grassmannian $\mathcal{G}_P$ of subspaces of dimension 2 of the hyperplane inside $|\mathcal{O}_{\mathbb{P}^5}(2)|$ of quadrics through P. Since $\mathcal{G}_P$ is irreducible, it follows that $\mathcal{V}$ is irreducible as well.

Let $\mathcal{U}_0 \subseteq \mathcal{G}$ be the subset of points Σ such that $X_\Sigma := \bigcap_{Q \in \Sigma} Q$ is a smooth surface: notice that in this case X_Σ, being a complete intersection, is also connected. The scheme $\mathcal{V}_0 := \gamma^{-1}(\mathcal{U}_0)$ is naturally endowed with a flat family $\mathcal{X} \subseteq \mathbb{P}^5 \times \mathcal{V}_0$ of pointed surfaces, whose fibre over (P, Σ) is (X_Σ, P, Σ). The projection from P induces a natural map $\mathcal{X} \dashrightarrow \mathbb{P}^4 \times \mathcal{V}_0$ with image $\widetilde{\mathcal{X}}$, which can be still viewed as a family over $\mathcal{V}_0$.

Let $\mathcal{V}_0' \subseteq \mathcal{V}_0$ be the open subset over which the fibres of such a family are smooth: thus the fibres of $\widetilde{\mathcal{X}}$ are in $\mathcal{H}$, hence the family $\widetilde{\mathcal{X}}$ is flat over $\mathcal{V}_0'$ (see [9], Theorem III.9.9). Thanks to the universal property of the Hilbert scheme, we know the existence of a morphism $v\colon \mathcal{V}_0' \to \mathcal{H}_0$, which is surjective due to the definition of $\mathcal{H}_0$. We deduce that $\mathcal{H}_0$ is irreducible, thus its closure $\overline{\mathcal{H}}_0$ inside $\mathcal{H}$ is a projective variety.

Let $\mathcal{U}_1 \subseteq \mathcal{U}_0 \subseteq \mathcal{G}$ be the subset of points Σ such that the corresponding surface X_Σ has Picard group generated by the hyperplane class. The set $\mathcal{U}_1$ is the intersection of a countable family of open sets (see [12], Lemma III.2.2), hence it is dense and non-empty. Moreover, every surface represented by a point in $\mathcal{U}_1$ is Ulrich-wild, thanks to Theorem 2.7 of [2]. Thus $\mathcal{V}_1 := \gamma^{-1}(\mathcal{U}_1)$ is the intersection of a countable family of open sets, hence it is dense and non-empty. On the other hand, by Corollary III.10.7 of [9], there is a non-empty open subset $\mathcal{W} \subseteq \mathcal{H}_0$ such that $v_{|v^{-1}(\mathcal{W})}$ is smooth, hence flat and open. We infer that $\mathcal{H}_1 := v(\mathcal{V}_1 \cap v^{-1}(\mathcal{W}))$ is dense and non-empty inside $\overline{\mathcal{H}}_0$. It follows that each surface of class (VI) corresponding to a point of the dense subset $\mathcal{H}_1 \subseteq \mathcal{H}_0$ is the blow up of an Ulrich-wild surface.

Now let us fix a $\widetilde{X} \subseteq \mathbb{P}^4$ either in class (VIII) or in class (VI) obtained by inner projection from an Ulrich-wild surface $X \subseteq \mathbb{P}^5$. If $\mathfrak{F} \to B$ is a family of indecomposable and pairwise non-isomorphic Ulrich bundles on X, then $\widetilde{X}$ supports a family

$\widetilde{\mathfrak{F}} \to B$ of Ulrich bundles on $\widetilde{X}$, thanks to Theorem 3.1: if $\mathcal{F}$ is the fibre of $\mathfrak{F}$ over $b \in B$, then the fibre of $\widetilde{\mathfrak{F}}$ over b is $\widetilde{\mathcal{F}}$. If $\widetilde{\mathcal{F}}$ is decomposable, then the isomorphism $\mathcal{F} \cong \sigma_* \widetilde{\mathcal{F}}(E)$ would imply that $\mathcal{F}$ should also split. Similarly, one can easily check that if the bundles in $\mathfrak{F} \to B$ are pairwise non-isomorphic, then the same is true for the bundles in $\widetilde{\mathfrak{F}} \to B$.

Since the dimension of $\mathfrak{F} \to B$ can be arbitrarily large, the same is true for the family $\widetilde{\mathfrak{F}} \to B$. We deduce that $\widetilde{X}$ is Ulrich-wild also in this case. $\square$

Remark 1. Notice that each complete intersection $X \subseteq \mathbb{P}^5$ of three quadrics supports an Ulrich bundle thanks to Theorem 2.5 of [10]. It follows that every surface in the class (VI) always supports Ulrich bundles by Theorem 3.1.

4. Pushing forward Ulrich bundles

In this section, we will study the behaviour of the functor $\mathcal{E} \mapsto \sigma_* \mathcal{E}(E)$, where $\mathcal{E}$ is an Ulrich bundle on $\widetilde{X}$, which is the natural left inverse of $\mathcal{F} \mapsto \widetilde{\mathcal{F}}$.

We start with some comments that partially motivate the assumption $\mathcal{E} \otimes \mathcal{O}_E \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus r}$ that we often make in the statements.

Lemma 4.1. *Let X be a smooth variety of dimension $n \geq 2$ endowed with a very ample line bundle $\mathcal{O}_X(h)$. Assume that $\mathcal{O}_{\widetilde{X}}(\widetilde{h})$ is very ample too.*

Let $\mathcal{E}$ be an Ulrich bundle on $\widetilde{X}$ such that $\mathcal{E} \otimes \mathcal{O}_E \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^{n-1}}(\alpha_i)$. Then $0 \leq \alpha_i \leq 2$ for $i = 1, \dots, n$ and $\sum_{i=1}^r \alpha_i = r$.

Proof. Since $\mathcal{E}$ is globally generated, it follows that $\alpha_i \geq 0$ for $i = 1, \dots, r$. Moreover, $\mathcal{E}^\vee((n+1)\widetilde{h} + K_{\widetilde{X}})$ is Ulrich too, hence globally generated. Thanks to the definition of $\widetilde{h}$ and to Equality (1), we have

$$\mathcal{E}^\vee((n+1)\widetilde{h} + K_{\widetilde{X}}) \otimes \mathcal{O}_E \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^{n-1}}(2 - \alpha_i), \tag{5}$$

hence $\alpha_i \leq 2$ for $i = 1, \dots, r$.

Consider Sequence (4) and let C be the intersection of $X \subseteq \mathbb{P}^N$ with a general linear subspace of dimension $N - n + 1$. Since $CE = \widetilde{h}^{n-1} E = (-1)^{n-1} E^n = 1$, it follows that the restriction to C of the above sequence tensored by $\mathcal{E}$ and the equalities $c_1(\mathcal{E})\widetilde{h}^{n-1} = c_1(\mathcal{E} \otimes \mathcal{O}_C)$, $c_1(\mathcal{E}(-E)) = c_1(\mathcal{E}) - rE$ yield $\sum_{i=1}^r \alpha_i = r$. $\square$

When $n = 2$, $\mathcal{E} \otimes \mathcal{O}_E$ certainly splits and the above lemma implies that the numbers of 0's and of 2's in the sequence $\alpha_1, \dots, \alpha_r$ must coincide.

Example 2. It is not true in general that $\mathcal{E} \otimes \mathcal{O}_E$ is a sum of line bundles on E when $n \geq 3$.

Indeed, let $X = \mathbb{P}^3$ be the Veronese threefold in $\mathbb{P}^9$ embedded via $\mathcal{O}_X(h) := \mathcal{O}_{\mathbb{P}^3}(2)$. Consider the blow up $\sigma : \widetilde{X} \to X$ of a point $P \in X$ with the exceptional divisor E.

It is well known that the threefold $\widetilde{X}$ is isomorphic to a del Pezzo threefold of degree 7 in $\mathbb{P}^8$ embedded via the linear system $\sigma^* \mathcal{O}_X(h) \otimes \mathcal{O}_{\widetilde{X}}(-E)$.

As pointed out in [6], the threefold $\widetilde{X}$ is endowed with a natural isomorphisms $\widetilde{X} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$, thus there is a projection map $\pi \colon \widetilde{X} \to \mathbb{P}^2$. The group $\mathrm{Pic}(\widetilde{X})$ is freely generated by the classes ξ and f of $\mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))}(1)$ and $\pi^* \mathcal{O}_{\mathbb{P}^2}(1)$, respectively. The intersection theory on $\widetilde{X}$ is given by $\xi^3 = \xi^2 f = \xi f^2 = 1$ and $f^3 = 0$. Thus the class of E is $\xi - f$, $h = \xi + f$ and the class of a line ℓ on E is $Eh = \xi^2 - f^2$.

According to [6], there is an Ulrich bundle $\mathcal{E}$ of rank 2 with $c_1(\mathcal{E}) = 2\xi + 2f$ and $c_2(\mathcal{E}) = 3\xi^2 + 3f^2$. The Chern classes of $\mathcal{E} \otimes \mathcal{O}_E$ are $c_1(\mathcal{E} \otimes \mathcal{O}_E) = 2\xi^2 - 2f^2 = 2\ell$ and $c_2(\mathcal{E} \otimes \mathcal{O}_E) = 3$. If $\mathcal{E} \otimes \mathcal{O}_E \cong \mathcal{O}_{\mathbb{P}^2}(\alpha_1) \oplus \mathcal{O}_{\mathbb{P}^2}(\alpha_2)$ splits, then $\alpha_1 + \alpha_2 = 2$ and $\alpha_1 \alpha_2 = 3$, which is obviously impossible.

The following result inverts partially Theorem 3.1.

Theorem 4.2. *Let X be a smooth variety of dimension $n \geq 2$ endowed with a very ample line bundle $\mathcal{O}_X(h)$. Assume that $\mathcal{O}_{\widetilde{X}}(\widetilde{h})$ is also very ample. If $\mathcal{E}$ is an Ulrich bundle with respect to $\mathcal{O}_{\widetilde{X}}(\widetilde{h})$ such that $\mathcal{E} \otimes \mathcal{O}_E \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus r}$, then $\sigma_* \mathcal{E}(E)$ is Ulrich with respect to $\mathcal{O}_X(h)$.*

Proof. Since $\mathcal{E}(E) \otimes \mathcal{O}_E \cong \mathcal{O}_{\mathbb{P}^{n-1}}^{\oplus r}$ by hypothesis, it follows that $\mathcal{F} := \sigma_* \mathcal{E}(E)$ is a vector bundle (see Corollary 2.2) and $\mathcal{E} \cong \widetilde{\mathcal{F}}$ (see Corollary 2.3).

We will show that

$$h^i\big(X, \sigma_* \mathcal{E}(E) \otimes \mathcal{O}_X(-th)\big) = 0, \qquad i = 0, \dots, n,\ t = 1, \dots, n. \tag{6}$$

Recall that $R^i\sigma_*\mathcal{E}(E) = 0$ for $i \geq 1$, thanks to Theorem 2.1. It follows that for $t = 1, \dots, n$ and $i \geq 0$,

$$h^i\big(X, \sigma_*\mathcal{E}(E) \otimes \mathcal{O}_X(-th)\big) = h^i\big(\widetilde{X}, \mathcal{E}(-t\widetilde{h} - (t-1)E)\big) \tag{7}$$

whence we immediately deduce Equalities (6) when $t = 1$.

Now let us restrict to the case $t = 2, \dots, n$. Recall that $\mathcal{E}' := \mathcal{E}^\vee(K_{\widetilde{X}} + (n+1)\widetilde{h})$ is also Ulrich, hence

$$h^i\big(\widetilde{X}, \mathcal{E}'(-t\widetilde{h})\big) = 0, \qquad i = 0, \dots, n,\ t = 1, \dots, n. \tag{8}$$

Equalities (7) and the Serre duality on $\widetilde{X}$ give

$$h^i\big(X, \sigma_*\mathcal{E}(E) \otimes \mathcal{O}_X(-th)\big) = h^{n-i}\big(\widetilde{X}, \mathcal{E}'(-(n+1-t)\widetilde{h} + (t-1)E)\big), \tag{9}$$

for $t = 2, \dots, n$ and $i \geq 0$.

It follows from Equality (5) that $\mathcal{E}'(E) \otimes \mathcal{O}_E \cong \mathcal{O}_{\mathbb{P}^{n-1}}^{\oplus r}$. Thus for each t and λ

$$\mathcal{E}'(-(n+1-t)\widetilde{h} + \lambda E) \otimes \mathcal{O}_E \cong \mathcal{O}_{\mathbb{P}^{n-1}}(t - \lambda - n)^{\oplus r},$$

hence $h^{n-i}\big(\widetilde{X}, \mathcal{E}'(-(n+1-t)\widetilde{h} + \lambda E) \otimes \mathcal{O}_E\big) = 0$ in the range $t = 2, \dots, n$ and $\lambda = 1, \dots, t-1$ because $E \cong \mathbb{P}^{n-1}$.

The cohomology of Sequence (4) tensored by $\mathcal{E}'(-(n+1-t)\widetilde{h} + \lambda E)$ yields that

$$h^{n-i}\big(\widetilde{X}, \mathcal{E}'(-(n+1-t)\widetilde{h} + (\lambda - 1)E)\big) = h^{n-i}\big(\widetilde{X}, \mathcal{E}'(-(n+1-t)\widetilde{h} + \lambda E)\big)$$

for each $i \geq 0$, $t = 2, \dots, n$ and $\lambda = 1, \dots, t-1$. It follows that

$$h^{n-i}\big(\widetilde{X}, \mathcal{E}'(-(n+1-t)\widetilde{h} + (t-1)E)\big) = h^{n-i}\big(\widetilde{X}, \mathcal{E}'(-(n+1-t)\widetilde{h})\big).$$

By combining the above identity with Equalities (8) and (9), we finally deduce that Equalities (6) hold also for $t = 2, \dots, n$. □

Example 3. The restriction $\mathcal{E} \otimes \mathcal{O}_E \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus r}$ cannot be removed from the hypothesis of the above theorem.

Indeed, let $3 \leq d \leq 9$. Recall that a del Pezzo surface X_d of degree d is the blow up of $\mathbb{P}^2$ at a set of $9 - d$ points $P_1, \dots, P_{9-d}$ in general position. In particular, X_{d-1} is the blow up $\widetilde{X}_d$ of X_d at a single point $P := P_{10-d}$, and we will denote by $\sigma_{d-1} \colon X_{d-1} = \widetilde{X}_d \to X_d$ such a blow-up map.

We recall that the linear system of cubics through $P_1, \dots, P_{9-d}$ is very ample and gives an embedding $X_d \subseteq \mathbb{P}^d$ with hyperplane class h. Moreover, each X_d contains a finite number of lines with respect to such an embedding. The group $\mathrm{Pic}(X_d)$ is freely generated by the class ℓ of the pull-back of a general line in $\mathbb{P}^2$ and by the classes $e_1, \dots, e_{9-d}$ of the exceptional divisors on X_d. In particular e_{9-d} is the class of the exceptional divisor E of σ_d. From now on, we will omit d in the subscripts, because we assume it fixed.

There exists a stable Ulrich bundle $\mathcal{E}$ of rank 2 fitting into an exact sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}}(3\ell - \sum_{i=1}^{8-d} e_i - 2e_{9-d}) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{Q|\widetilde{X}}(3\ell - \sum_{i=1}^{8-d} e_i) \longrightarrow 0,$$

where Q is a point in $\widetilde{X}$ not lying on any line (see Example 6.4 of [4]). Thus, restricting the above sequence to $E \cong \mathbb{P}^1$, whose class is e_{9-d}, we obtain an exact sequence of the form

$$\mathcal{O}_E(2) \longrightarrow \mathcal{E} \otimes \mathcal{O}_E \longrightarrow \mathcal{O}_E \longrightarrow 0,$$

which is trivially exact also on the left, because the kernel of $\mathcal{E} \otimes \mathcal{O}_E \to \mathcal{O}_E$ is an invertible sheaf. We conclude that $\mathcal{E} \otimes \mathcal{O}_E \cong \mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(2)$ since $\mathrm{Ext}^1_{\mathbb{P}^{n-1}}\big(\mathcal{O}_{\mathbb{P}^{n-1}}, \mathcal{O}_{\mathbb{P}^{n-1}}(2)\big) \cong H^1\big(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(2)\big) = 0$. It follows that $\sigma_*\mathcal{E}(E)$ is not locally free, due to Corollary 2.2.

Remark 2. Theorem 0.3 of [11] states that if $\sigma \colon \widetilde{X} \to X$ is a blow up at $P \in X$ and $\mathcal{E}$ is an Ulrich bundle of rank 2 which is *special* in the sense of [8] (i.e. $\mathcal{E} \cong \mathcal{E}^\vee(3\widetilde{h} + K_{\widetilde{X}})$), then $\sigma_*\mathcal{E}(E)$ is a special Ulrich bundle on S.

The proof therein contains a gap which cannot be overcome. For example, the bundle $\mathcal{E}$ described in the example satisfies $c_1(\mathcal{E}) = \mathcal{O}_{\widetilde{X}}(2h) \cong \mathcal{O}_{\widetilde{X}}(3\widetilde{h} + K_{\widetilde{X}})$, hence $\mathcal{E}$ is a special Ulrich bundle, but $\sigma_*\mathcal{E}(E)$ is not locally free.

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