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### Algebra/Group theory

# Metabelian $\mathbb{Q}_1$ -groups

## *Les* $Q_1$ *-groupes métabéliens*

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#### ABSTRACT

A finite group *G* is called a  $\mathbb{Q}_1$ -group if all of its non-linear irreducible characters are rational valued. In this paper, we will find the general structure of a metabelian  $\mathbb{Q}_1$ -group.  $\mathbb{C}$  2017 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

#### RÉSUMÉ

Un groupe fini *G* est appelé un  $\mathbb{Q}_1$ -groupe si les valeurs des caractères non linéaires sont rationnelles. Dans cet article, nous déterminons la structure des  $\mathbb{Q}_1$ -groupes métabéliens. © 2017 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

#### 1. Introduction

If  $\chi$  is an irreducible complex character of a finite group *G*, then it is well known that  $\chi(g)$  for any  $g \in G$  is an algebraic number. Let  $\mathbb{Q}(\chi)$  be the field generated by all  $\chi(g)$  when *g* runs over *G*. If  $\mathbb{Q}(\chi) = \mathbb{Q}$ , then  $\chi$  is called a rational character of *G*, and if each irreducible character of *G* is rational, then *G* is called a rational group or a  $\mathbb{Q}$ -group. Examples of rational groups are the Weyl groups of the complex Lie algebras [3]. Classification of finite  $\mathbb{Q}$ -groups is still an open problem, but in [5] it is shown that the only non-abelian simple  $\mathbb{Q}$ -groups are  $Sp_6(2)$  and  $O_8^+(2)$ . In [2], a generalization of  $\mathbb{Q}$ -groups is formulated as follows: a finite group *G* is called a  $\mathbb{Q}_1$ -group if all of its non-linear irreducible characters are rational. It is clear that every  $\mathbb{Q}$ -group is a  $\mathbb{Q}_1$ -group. The elementary properties of  $\mathbb{Q}_1$ -groups can be found in [4]. In [1], it is shown that if *G* is a metabelian  $\mathbb{Q}_1$ -group, then the exponent of the commutator subgroup *G'* is either a prime number or divides 16, 24, or 40. In this paper, using [1], we give the general structure of a metabelian  $\mathbb{Q}_1$ -group.

Throughout the paper, we consider finite solvable groups, and we employ the following notation and terminology: the semi-direct product of a group *K* with a group *H* is denoted by K : H. The symbol  $\mathbb{Z}_n$  denotes a cyclic group of order *n*. For a prime *p* and a non-negative integer *n*, the symbol  $E(p^n)$  denotes the elementary abelian *p*-group of order  $p^n$ .

Let us mention some important consequences of rational groups and  $\mathbb{Q}_1$ -groups. Let *G* be a finite group. Let nl(G) denote the set of non-linear irreducible characters of *G*.

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An element  $x \in G$  is called rational if  $\chi(x) \in \mathbb{Q}$  for every  $\chi \in Irr(G)$ ; otherwise, it is called an irrational element. Also,  $\chi \in Irr(G)$  is called a rational character if  $\chi(x) \in \mathbb{Q}$  for every  $x \in G$ .

**Lemma 1.1.** ([7, p. 11] and [6, p. 31]) A finite group G is a  $\mathbb{Q}$ -group if and only if for every  $x \in G$  of order n the elements x and  $x^m$  are conjugate in G, whenever (m, n) = 1. Equivalently,  $N_G(\langle x \rangle)/C_G(\langle x \rangle) \cong Aut(\langle x \rangle)$  for each  $x \in G$ .

The detailed proofs of Theorems 1.2 and 1.4 can be found in [4].

**Theorem 1.2.** Let G be a non-abelian  $\mathbb{Q}_1$ -group. Then the following are true:

(1) |G| is even;
(2) a quotient of G is a Q<sub>1</sub>-group.

**Definition 1.3.** Let *G* be a non-abelian finite group. The vanishing-off subgroup of *G* is defined as follows:

 $V(G) = \langle g \in G \mid \exists \chi \in nl(G) : \chi(g) \neq 0 \rangle.$ 

Notice that V(G) is a characteristic subgroup of G and V(G) is the smallest subgroup  $V \le G$  such that every character in nl(G) vanishes on G - V(G). Note that the exponent of a finite group denoted by exp(G) is the least common divisor of the orders of its elements.

**Theorem 1.4.** Let G be a non-abelian finite group. Then G is a  $\mathbb{Q}_1$ -group if and only if every element of V(G) is a rational element.

The main result of this paper is as follows.

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Theorem A. Suppose that G is a metabelian \mathbb{Q}_1-group and let P \in Syl_2(G). Then, one of the following occurs:
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- (1) *G* is a 2-group and  $\exp(G')$  divides 16;
- (2)  $G \cong (E(3^n) : P) : \mathbb{Z}_m$  or  $G \cong P : \mathbb{Z}_m$ , where *m* is a positive integer that is coprime to 6. Also *P* is a rational group, when  $G \cong (E(3^n) : P) : \mathbb{Z}_m$  also  $E(3^n) : P$  is a rational group, and  $\exp(P')$  divides 8;
- (3)  $G \cong E(3^n) : P$  or  $G \cong E(5^n) : P$ , where P is a nonabelian  $\mathbb{Q}_1$ -group that is metabelian. Moreover,  $\exp(P')$  divides 8;
- (4)  $G \cong E(p^n) : ((\mathbb{Z}_m) \times E(2^n))$ , where p is an odd prime and m is an odd positive integer.

#### 2. Proof of Theorem A

Let *G* be a metabelian  $\mathbb{Q}_1$ -group and  $P \in Syl_2(G)$ . First, suppose that  $P \subseteq V(G)$ . In this case, by [10],  $G \cong V(G) : \mathbb{Z}_m$ , where V(G) is a rational group and *m* is an odd integer. V(G) is metabelian, because *G* is metabelian. So, we deduce from [1] that V(G) is a {2,3}-group. Since  $G' \leq V(G)$ , so  $\exp(G')$  divides 16 or 24. From the rationality of V(G), we conclude that V(G)/G' is an elementary abelian 2-group. Therefore, if  $\exp(G')$  divides 16, then V(G) is a rational 2-group. In this case, we show that  $\exp(G') \neq 16$ . Otherwise, there exists  $g \in G'$  of order 16. Since V(G) is rational, so  $\frac{N_{V(G)}(\{g\})}{C_{V(G)}(\{g\})} \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ . On the other hand,  $\frac{N_{V(G)}(\{g\})}{C_{V(G)}(\{g\})}$  is an elementary abelian 2-group, because  $G' \leq C_{V(G)}(\{g\})$ . Therefore,  $\exp(G')$  can not be equal to 16. Now, suppose that  $\exp(G')$  divides 24. The Sylow 3-subgroup of G' is elementary abelian, because G' is abelian. Let  $S \in Syl_3(G)$ . Since *S* is characteristic in G' and G' is normal in V(G), *S* is normal in V(G). If S > 1, then  $S = E(p^n)$ , and then recall that  $V(G) = SP = E(3^n)$ : *P* is rational and  $P \cong V(G)/S$  will be rational since it is the quotient of a rational group. If S = 1, then V(G) = P is rational. Also, it is not difficult to see that  $\exp(P')$  will be the 2-part of  $\exp(G')$ , so  $\exp(P')$ divides 8. Hence, we get the case (2) of the main theorem.

If  $P \nsubseteq V(G)$  and P is non-abelian, then, by [9],  $G \cong K : P$ , where K is a {3, 5, 7}-group. By [8, Lemma 3.3], we have  $K \subsetneqq V(G)$ , and since every element in V(G) of odd order is contained in G',  $K \subset G'$ . This shows that  $\exp(G')$  can not be an odd prime greater than 5. Now, if  $\exp(G')$  divides 16, then the case (1) of the main theorem follows. If  $\exp(G')$  divides 24, then similar to previous paragraph, the Sylow 3-subgroup of G' is elementary abelian and is normal in G. Similarly, if  $\exp(G')$  divides 40, then the Sylow 5-subgroup of G' is elementary abelian and is normal in G. This leads to the case (3) of the main theorem.

For the case (4) of the main theorem, let  $P \nsubseteq V(G)$  and P is abelian. By [9, Theorem 2.8] and its proof,  $G \cong G'$ :  $(\mathbb{Z}_m \times E(2^n))$  and G' has odd order. Therefore,  $\exp(G')$  can only be an odd prime. Thus G' is an elementary abelian p-group for some odd prime p. This completes the proof of the main theorem.

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