Algebra/Group theory

## Metabelian $\mathbb{Q}_{1}$-groups

## Les $\mathbb{Q}_{1}$-groupes métabéliens

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## A R T I C L E I N F O

## Article history:

Received 9 March 2017
Accepted after revision 27 October 2017
Available online 12 January 2018
Presented by the Editorial Board


#### Abstract

A finite group $G$ is called a $\mathbb{Q}_{1}$-group if all of its non-linear irreducible characters are rational valued. In this paper, we will find the general structure of a metabelian $\mathbb{Q}_{1}$-group. © 2017 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.


## R É S U M É

Un groupe fini $G$ est appelé un $\mathbb{Q}_{1}$-groupe si les valeurs des caractères non linéaires sont rationnelles. Dans cet article, nous déterminons la structure des $\mathbb{Q}_{1}$-groupes métabéliens.
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## 1. Introduction

If $\chi$ is an irreducible complex character of a finite group $G$, then it is well known that $\chi(g)$ for any $g \in G$ is an algebraic number. Let $\mathbb{Q}(\chi)$ be the field generated by all $\chi(g)$ when $g$ runs over $G$. If $\mathbb{Q}(\chi)=\mathbb{Q}$, then $\chi$ is called a rational character of $G$, and if each irreducible character of $G$ is rational, then $G$ is called a rational group or a $\mathbb{Q}$-group. Examples of rational groups are the Weyl groups of the complex Lie algebras [3]. Classification of finite $\mathbb{Q}$-groups is still an open problem, but in [5] it is shown that the only non-abelian simple $\mathbb{Q}$-groups are $S p_{6}(2)$ and $O_{8}^{+}(2)$. In [2], a generalization of $\mathbb{Q}$-groups is formulated as follows: a finite group $G$ is called a $\mathbb{Q}_{1}$-group if all of its non-linear irreducible characters are rational. It is clear that every $\mathbb{Q}$-group is a $\mathbb{Q}_{1}$-group. The elementary properties of $\mathbb{Q}_{1}$-groups can be found in [4]. In [1], it is shown that if $G$ is a metabelian $\mathbb{Q}_{1}$-group, then the exponent of the commutator subgroup $G^{\prime}$ is either a prime number or divides 16,24 , or 40 . In this paper, using [1], we give the general structure of a metabelian $\mathbb{Q}_{1}$-group.

Throughout the paper, we consider finite solvable groups, and we employ the following notation and terminology: the semi-direct product of a group $K$ with a group $H$ is denoted by $K: H$. The symbol $\mathbb{Z}_{n}$ denotes a cyclic group of order $n$. For a prime $p$ and a non-negative integer $n$, the symbol $E\left(p^{n}\right)$ denotes the elementary abelian $p$-group of order $p^{n}$.

Let us mention some important consequences of rational groups and $\mathbb{Q}_{1}$-groups. Let $G$ be a finite group. Let $n l(G)$ denote the set of non-linear irreducible characters of $G$.

[^0]An element $x \in G$ is called rational if $\chi(x) \in \mathbb{Q}$ for every $\chi \in \operatorname{Irr}(G)$; otherwise, it is called an irrational element. Also, $\chi \in \operatorname{Irr}(G)$ is called a rational character if $\chi(x) \in \mathbb{Q}$ for every $\chi \in G$.

Lemma 1.1. ([7, p. 11] and [6, p. 31]) A finite group $G$ is $a \mathbb{Q}$-group if and only if for every $x \in G$ of order $n$ the elements $x$ and $x^{m}$ are conjugate in $G$, whenever $(m, n)=1$. Equivalently, $N_{G}(\langle x\rangle) / C_{G}(\langle x\rangle) \cong$ Aut $(\langle x\rangle)$ for each $x \in G$.

The detailed proofs of Theorems 1.2 and 1.4 can be found in [4].

Theorem 1.2. Let $G$ be a non-abelian $\mathbb{Q}_{1}$-group. Then the following are true:
(1) $|G|$ is even;
(2) a quotient of $G$ is a $\mathbb{Q}_{1}$-group.

Definition 1.3. Let $G$ be a non-abelian finite group. The vanishing-off subgroup of $G$ is defined as follows:

$$
V(G)=\langle g \in G \mid \exists \chi \in n l(G): \chi(g) \neq 0\rangle .
$$

Notice that $V(G)$ is a characteristic subgroup of $G$ and $V(G)$ is the smallest subgroup $V \leq G$ such that every character in $n l(G)$ vanishes on $G-V(G)$. Note that the exponent of a finite group denoted by $\exp (G)$ is the least common divisor of the orders of its elements.

Theorem 1.4. Let $G$ be a non-abelian finite group. Then $G$ is $a \mathbb{Q}_{1}$-group if and only if every element of $V(G)$ is a rational element.

The main result of this paper is as follows.

Theorem A. Suppose that $G$ is a metabelian $\mathbb{Q}_{1}$-group and let $P \in \operatorname{Syl}_{2}(G)$.
Then, one of the following occurs:
(1) $G$ is a 2 -group and $\exp \left(G^{\prime}\right)$ divides 16;
(2) $G \cong\left(E\left(3^{n}\right): P\right): \mathbb{Z}_{m}$ or $G \cong P: \mathbb{Z}_{m}$, where $m$ is a positive integer that is coprime to 6 . Also $P$ is a rational group, when $G \cong$ $\left(E\left(3^{n}\right): P\right): \mathbb{Z}_{m}$ also $E\left(3^{n}\right): P$ is a rational group, and $\exp \left(P^{\prime}\right)$ divides 8;
(3) $G \cong E\left(3^{n}\right): P$ or $G \cong E\left(5^{n}\right): P$, where $P$ is a nonabelian $\mathbb{Q}_{1}$-group that is metabelian. Moreover, $\exp \left(P^{\prime}\right)$ divides 8;
(4) $G \cong E\left(p^{n}\right):\left(\left(\mathbb{Z}_{m}\right) \times E\left(2^{n}\right)\right)$, where $p$ is an odd prime and $m$ is an odd positive integer.

## 2. Proof of Theorem $A$

Let $G$ be a metabelian $\mathbb{Q}_{1}$-group and $P \in S y l_{2}(G)$. First, suppose that $P \subseteq V(G)$. In this case, by [10], $G \cong V(G): \mathbb{Z}_{m}$, where $V(G)$ is a rational group and $m$ is an odd integer. $V(G)$ is metabelian, because $G$ is metabelian. So, we deduce from [1] that $V(G)$ is a $\{2,3\}$-group. Since $G^{\prime} \leq V(G)$, so $\exp \left(G^{\prime}\right)$ divides 16 or 24 . From the rationality of $V(G)$, we conclude that $V(G) / G^{\prime}$ is an elementary abelian 2-group. Therefore, if $\exp \left(G^{\prime}\right)$ divides 16 , then $V(G)$ is a rational 2-group. In this case, we show that $\exp \left(G^{\prime}\right) \neq 16$. Otherwise, there exists $g \in G^{\prime}$ of order 16 . Since $V(G)$ is rational, so $\frac{N_{V(G)}(\langle g\rangle)}{\left.C_{V(G)}(l g)\right)} \cong \mathbb{Z}_{2} \times \mathbb{Z}_{4}$. On the other hand, $\frac{N_{V(G)}(\langle g\rangle)}{C_{V(G)}(\langle g\rangle)}$ is an elementary abelian 2-group, because $G^{\prime} \leq C_{V(G)}(\langle g\rangle)$. Therefore, $\exp \left(G^{\prime}\right)$ can not be equal to 16 . Now, suppose that $\exp \left(G^{\prime}\right)$ divides 24 . The Sylow 3 -subgroup of $G^{\prime}$ is elementary abelian, because $G^{\prime}$ is abelian. Let $S \in S y l_{3}(G)$. Since $S$ is characteristic in $G^{\prime}$ and $G^{\prime}$ is normal in $V(G), S$ is normal in $V(G)$. If $S>1$, then $S=E\left(p^{n}\right)$, and then recall that $V(G)=S P=E\left(3^{n}\right): P$ is rational and $P \cong V(G) / S$ will be rational since it is the quotient of a rational group. If $S=1$, then $V(G)=P$ is rational. Also, it is not difficult to see that $\exp \left(P^{\prime}\right)$ will be the 2-part of $\exp \left(G^{\prime}\right)$, so $\exp \left(P^{\prime}\right)$ divides 8 . Hence, we get the case (2) of the main theorem.

If $P \nsubseteq V(G)$ and $P$ is non-abelian, then, by [9], $G \cong K: P$, where $K$ is a $\{3,5,7\}$-group. By [8, Lemma 3.3], we have $K \varsubsetneqq V(G)$, and since every element in $V(G)$ of odd order is contained in $G^{\prime}, K \subset G^{\prime}$. This shows that $\exp \left(G^{\prime}\right)$ can not be an odd prime greater than 5 . Now, if $\exp \left(G^{\prime}\right)$ divides 16 , then the case (1) of the main theorem follows. If $\exp \left(G^{\prime}\right)$ divides 24 , then similar to previous paragraph, the Sylow 3-subgroup of $G^{\prime}$ is elementary abelian and is normal in $G$. Similarly, if $\exp \left(G^{\prime}\right)$ divides 40 , then the Sylow 5 -subgroup of $G^{\prime}$ is elementary abelian and is normal in $G$. This leads to the case (3) of the main theorem.

For the case (4) of the main theorem, let $P \nsubseteq V(G)$ and $P$ is abelian. By [9, Theorem 2.8] and its proof, $G \cong G^{\prime}$ : $\left(\mathbb{Z}_{m} \times E\left(2^{n}\right)\right.$ ) and $G^{\prime}$ has odd order. Therefore, $\exp \left(G^{\prime}\right)$ can only be an odd prime. Thus $G^{\prime}$ is an elementary abelian $p$-group for some odd prime $p$. This completes the proof of the main theorem.

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