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A variational principle for problems with a hint of convexity ☆

Un principe variationnel pour des problèmes avec une certaine convexité

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A R T I C L E   I N F O

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A B S T R A C T

A variational principle is introduced to provide a new formulation and resolution for several boundary value problems with a variational structure. This principle allows one to deal with problems well beyond the weakly compact structure. As a result, we study several super-critical semilinear Elliptic problems.

R É S U M É

Un principe variationnel est introduit pour fournir une nouvelle formulation et résolution de nombreux problèmes aux limites avec structure variationnelle. Ce principe permet de considérer des problèmes bien au-delà de la structure faiblement compacte. Ainsi, nous étudions de nombreux probèmes elliptiques semilinéaires supercritiques.

1. Introduction

Let V be a real Banach space and V^* its topological dual and let $\langle .,. \rangle$ be the pairing between V and V^*. Let $\Psi : V \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semi continuous function and let K be a convex and weakly closed subset of V. Assume that Ψ is Gâteaux differentiable on K and denote by $D\Psi$ the Gâteaux derivative of Ψ. Let $\Phi \in C^1(V, \mathbb{R})$ and consider the following problem,

$$\text{Find } u_0 \in K \text{ such that } D\Psi(u_0) = D\Phi(u_0). \tag{1}$$

The restriction of Ψ to K is denoted by Ψ_K and defined by

$$\Psi_K(u) = \begin{cases} \Psi(u), & u \in K, \\ +\infty, & u \notin K. \end{cases}$$

To find a solution for (1), we shall consider the critical points of the functional $I : V \to \mathbb{R} \cup \{+\infty\}$ defined by

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$$I(u) := \Psi_K(u) - \Phi(u).$$

According to Szulkin [15], we have the following definition for critical points of I (see also the appendix).

Definition 1.1. A point $u \in V$ is said to be a critical point of I if $I(u) \in \mathbb{R}$ and it satisfies the inequality

$$\Psi_K(v) - \Psi_K(u) \geq \langle D\Phi(u), v - u\rangle, \qquad \forall v \in V. \tag{2}$$

Note that a function u satisfying (2) is indeed a solution to the inclusion $D\Phi(u) \in \partial\Psi_K(u)$. Therefore, it is not necessarily a solution to (1) unless $D = V$. There is a well-developed theory to find critical points of functionals of the form I. We refer the interested reader to [15,12]. Here is our main result in this paper.

Theorem 1.2 *(Variational principle). Let $\Psi : V \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semi continuous function and let K be a convex and weakly closed subset of V. Assume that Ψ is Gâteaux differentiable on K and $\Phi \in C^1(V, \mathbb{R})$. If the following two assertions hold:*

(i) the functional $I : V \to \mathbb{R} \cup \{+\infty\}$ defined by $I(u) = \Psi_K(u) - \Phi(u)$ has a critical point $u_0 \in V$,
(ii) there exists $v_0 \in K$ such that $D\Psi(v_0) = D\Phi(u_0)$,

then $u_0 \in K$ is a solution to (1), that is,

$$D\Psi(u_0) = D\Phi(u_0).$$

The above theorem has many interesting applications in partial differential equations. We shall briefly recall some of them and refer the interested reader to [11], where some more general versions of Theorem 1.2 are established, and several applications in the fixed point theory and PDEs are provided. It is also worth noting that Theorem 1.2 extends some of variational principles established by the author in [10,9].

We shall now proceed with some applications.

1.1. A concave-convex nonlinearity

We consider the problem

$$\begin{cases} -\Delta u = |u|^{p-2}u + \mu|u|^{q-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^2-boundary and $1 < q \leq 2 < p$. This problem was studied by Ambrosetti et al. in [1], and Bartsch and Willem in [3]. Our plan is to show that, for positive μ and p bigger that the critical exponent $2^* = 2n/(n-2)$, problem (3) has a strong solution in $H^2(\Omega)$.

Let $V = H^2(\Omega) \cap H^1_0(\Omega)$, and let $I : V \to \mathbb{R}$ be the Euler–Lagrange functional corresponding to (3)

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{1}{p}\int_\Omega |u|^p \, dx - \frac{\mu}{q}\int_\Omega |u|^q \, dx.$$

For $r > 0$, define the convex set $K(r)$ by

$$K(r) = \Big\{u \in H^2(\Omega) \cap H^1_0(\Omega);\ \|u\|_{H^2(\Omega)} \leq r\Big\}.$$

We have the following result.

Theorem 1.3. *Assume that $1 < q < 2 < p < p^*$ where $p^* = (2n-4)/(n-4)$ for $n > 4$ and $p^* = \infty$ for $n \leq 4$. Then there exists $\mu^* > 0$ such that for each $\mu \in (0, \mu^*)$ problem (3) has a non-trivial solution. Indeed, for each $\mu \in (0, \mu^*)$, there exist positive numbers $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$ such that for each $r \in [r_1, r_2]$ the problem (3) has a solution $u \in K(r)$ with $I(u) < 0$.*

Proof. We apply Theorem 1.2, where

$$\Psi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx, \qquad \Phi(u) = \frac{1}{p}\int_\Omega |u|^p \, dx + \frac{\mu}{q}\int_\Omega |u|^q \, dx,$$

and $K := K(r)$ for some $r > 0$ to be determined. Note that the Sobolev space $H^2(\Omega)$ is compactly embedded in $L^t(\Omega)$ for $t < t^*$ where $t^* = 2n/(n-4)$ for $n > 4$, and $t^* = +\infty$ for $n \leq 4$. It then follows that the function Φ is continuously differentiable for $p < p^*$. By standard methods, there exists $u_0 \in K(r)$ such that

$$I(u_0) = \min_{u \in K(r)} I(u).$$

Since $1 < q < 2 < p$ and $\mu > 0$, it is easily seen that $I(u_0) < 0$ and therefore $u_0 \not\equiv 0$ is a critical point of I restricted to $K(r)$. To verify condition (ii) in Theorem 1.2, we show that there exists $v_0 \in K(r)$ such that $-\Delta v_0 = |u_0|^{p-2}u_0 + \mu|u_0|^{q-2}u_0$. The existence of such v_0 follows by standard arguments. We show that $v_0 \in K(r)$ for r small. It follows from the elliptic regularity theory (see Theorem 8.12 in [7]) that

$$\begin{aligned}\|v_0\|_{H^2(\Omega)} &\le C\Big(\big\||u_0|^{p-2}u_0\big\|_{L^2(\Omega)} + \mu\big\||u_0|^{q-2}u_0\big\|_{L^2(\Omega)}\Big)\\ &= C\Big(\|u_0\|^{p-1}_{L^{2(p-1)}(\Omega)} + \mu\|u_0\|^{q-1}_{L^{2(q-1)}(\Omega)}\Big),\end{aligned}$$

where C is a constant depending on Ω. Since $2(q-1) < 2(p-1) < t^*$, we obtain that

$$\begin{aligned}\|v_0\|_{H^2(\Omega)} &\le C_1\Big(\|u_0\|^{p-1}_{H^2(\Omega)} + \mu\|u_0\|^{q-1}_{H^2(\Omega)}\Big)\\ &\le C_1(r^{p-1} + \mu r^{q-1}),\end{aligned}$$

where C_1 is a constant in terms of p, q and Ω. Choose $\mu^* > 0$ small enough such that for each $\mu \in (0, \mu^*)$, there exist positive numbers $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$ such that $C_1(r^{p-1} + \mu r^{q-1}) \le r$ for all $r \in [r_1, r_2]$. It then follows that $v_0 \in K(r)$ provided $\mu \in (0, \mu^*)$ and $r \in [r_1, r_2]$. □

1.2. Non-homogeneous semilinear elliptic equations

Here we shall consider the problem

$$\begin{cases} -\Delta u = |u|^{p-2}u + f(x), & x \in \Omega,\\ u = 0, & x \in \partial\Omega,\end{cases} \tag{4}$$

where Ω is on open bounded domain in $\mathbb{R}^n$ with C^2-boundary. Problem (4) was treated in [2,14] for p less than the critical exponent 2^*. As an application of Theorem 1.2 together with elliptic regularity theory, we shall show that problem (4) has a solution for p beyond the critical Sobolev exponent. In this case, the standard variational methods fail to work. Note that our approach can be applied to more general nonlinearities (see [11]). We have the following theorem.

Theorem 1.4. *Let $2 < p < p^*$, where $p^* = (2n-4)/(n-4)$ for $n > 4$ and $p^* = \infty$ for $n \le 4$. There exists $\lambda > 0$ such that, for $\|f\|_{L^2(\Omega)} < \lambda$, problem (4) has a solution $u \in H^2(\Omega)$.*

Proof. Let $V = H^2(\Omega) \cap H^1_0(\Omega)$, and let $I : V \to \mathbb{R}$ be the Euler–Lagrange functional corresponding to (4),

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p\,\mathrm{d}x - \int_\Omega f u\,\mathrm{d}x.$$

We apply Theorem 1.2, where

$$\Psi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x, \qquad \Phi(u) = \frac{1}{p}\int_\Omega |u|^p\,\mathrm{d}x + \int_\Omega f u\,\mathrm{d}x,$$

and

$$K := K(r) = \big\{u \in H^2(\Omega) \cap H^1_0(\Omega);\ \|u\|_{H^2(\Omega)} \le r\big\},$$

for some $r > 0$ to be determined. By standard methods, there exists $u_0 \in K(r)$ such that

$$I(u_0) = \min_{u \in K(r)} I(u).$$

To verify condition (ii) in Theorem 1.2, one needs to show that there exists $v_0 \in K(r)$ such that $-\Delta v_0 = |u_0|^{p-2}u_0 + f(x)$. The existence of $v_0 \in H^2(\Omega)$ is standard. The fact that $v_0 \in K(r)$ for $\|f\|_{L^2(\Omega)}$ small follows by the elliptic regularity theory and the argument made in the proof of Theorem 1.3. □

Remark 1.5. It is worth noting that the above theorem can also be proved using other arguments, such as the inverse mapping theorem or fixed point arguments.

1.3. Supercritical Neumann problems

We shall consider the existence of positive solutions to the Neumann problem

$$\begin{cases} -\Delta u + u = a(x)|u|^{p-2}u, & x \in B_1, \\ u > 0, & x \in B_1, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial B_1, \end{cases} \tag{5}$$

where B_1 is the unit ball centered at the origin in $\mathbb{R}^N$, $N \geq 3$, $p > 2$, and a is a radial function, i.e. $a(x) = a(r)$ where $r = |x|$.

Theorem 1.6. *Assume that $a \in L^\infty(0,1)$ is increasing, not constant and $a(r) > 0$ a.e. in $[0,1]$. Then problem (5) admits at least one radially increasing positive solution.*

Sketch of the proof. Let $V = L^p(\Omega) \cap H^1_r(\Omega)$, where H^1_r is the set of radial functions in $H^1(\Omega)$. We apply Theorem 1.2, where

$$\Psi(u) = \int_\Omega \frac{|\nabla u|^2 + u^2}{2}\,\mathrm{d}x, \qquad \Phi(u) = \frac{1}{p}\int_\Omega a(x)|u|^p\,\mathrm{d}x,$$

and

$$K = \big\{u \in V : u(r) \geq 0, u(r) \leq u(s), \forall r, s \in [0,1], r \leq s\big\}.$$

It can be easily deduced that $V \cap K$ is continuously embedded in $L^\infty(\Omega)$, from which one can apply Theorem 3.3 to show that $I = \Psi - \Phi$ restricted to K has a critical point $u_0 \in K$ of mountain pass type (see [5] for a detailed argument). It is also established in [5] that there exists $v_0 \in K$ satisfying $-\Delta v_0 + v_0 = a(|x|)|u_0|^{p-2}u_0$. Thus, by Theorem 1.2, u_0 is a non-negative and nontrivial solution to (5). It also follows from the maximum principle that u_0 is indeed positive. □

We remark that finding radially increasing solutions to problems of type (5) has been the subject of many studies in recent years, starting with the works [4,8,13].

2. Proof of the variational principle

In this section, we shall prove Theorem 1.2. We first recall some important definitions and results from convex analysis.

Let V be a real Banach space and V^* its topological dual and let $\langle .,. \rangle$ be the pairing between V and V^*. Let $\Psi : V \to \mathbb{R} \cup \{\infty\}$ be a proper convex function. The subdifferential $\partial\Psi$ of Ψ is defined to be the following set-valued operator: if $u \in Dom(\Psi) = \{v \in V;\ \Psi(v) < \infty\}$, set

$$\partial\Psi(u) = \big\{u^* \in V^*;\ \langle u^*, v - u\rangle + \Psi(u) \leq \Psi(v) \text{ for all } v \in V\big\}$$

and if $u \notin Dom(\Psi)$, set $\partial\Psi(u) = \varnothing$. If Ψ is Gâteaux differentiable at u, denote by $D\Psi(u)$ the derivative of Ψ at u. In this case $\partial\Psi(u) = \{D\Psi(u)\}$.

The Fenchel dual of an arbitrary function Ψ is denoted by Ψ^*, which is function on V^* and is defined by

$$\Psi^*(u^*) = \sup\{\langle u^*, u\rangle - \Psi(u);\ u \in V\}.$$

Clearly $\Psi^* : V^* \to \mathbb{R} \cup \{+\infty\}$ is convex and lower semi-continuous. The following standard result is crucial in the subsequent analysis (see Proposition 5.1 in [6] for a proof).

Proposition 2.1. *Let $\Psi : V \to \mathbb{R} \cup \{+\infty\}$ be convex and lower-semi continuous. Then the following holds:*

$$\Psi(u) + \Psi^*(u^*) = \langle u, u^*\rangle \quad \Longleftrightarrow \quad u^* \in \partial\Psi(u).$$

Proof of Theorem 1.2. Since u_0 is a critical point of $I(u) = \Psi_K(u) - \Phi(u)$, it follows from Definition 1.1 that

$$\Psi_K(v) - \Psi_K(u_0) \geq \langle D\Phi(u_0), v - u_0\rangle, \quad \forall v \in V. \tag{6}$$

It follows from (i) and (ii) in the theorem that $u_0, v_0 \in K$ and $D\Psi(v_0) = D\Phi(u_0)$. Thus, it follows from inequality (6) with $v = v_0$ that

$$\Psi(v_0) - \Psi(u_0) \geq \langle D\Psi(v_0), v_0 - u_0\rangle. \tag{7}$$

Since Ψ is Gâteaux differentiable at $v_0 \in K$, it follows that $\partial\Psi(v_0) = \{D\Psi(v_0)\}$, which together with the convexity of Ψ allows one to obtain that

$$\Psi(u_0) - \Psi(v_0) \geq \langle D\Psi(v_0), u_0 - v_0\rangle. \tag{8}$$

It follows from (7) and (8) that

$$\Psi(v_0) - \Psi(u_0) = \langle D\Psi(v_0), v_0 - u_0\rangle. \tag{9}$$

We now claim that $D\Psi(v_0) = D\Psi(u_0)$, from which the desired result follows,

$$D\Psi(u_0) = D\Psi(v_0) = D\Phi(u_0).$$

Proof of the claim. Let $w^* = D\Psi(v_0)$. Since Ψ is convex and lower semi continuous it follows from Proposition 2.1 that

$$\Psi(v_0) + \Psi^*(w^*) = \langle w^*, v_0\rangle. \tag{10}$$

It now follows from (9) and (10) that

$$\langle w^*, u_0\rangle - \Psi(u_0) = \langle w^*, v_0\rangle - \Psi(v_0) = \Psi^*(w^*),$$

from which one obtains

$$\Psi(u_0) + \Psi^*(w^*) = \langle w^*, u_0\rangle.$$

This indeed implies that $w^* \in \partial\Psi(u_0)$ by virtue of Proposition 2.1. Since Ψ is Gâteaux differentiable at u_0, we have that $\partial\Psi(u_0) = \{D\Psi(u_0)\}$. Therefore,

$$D\Psi(u_0) = w^* = D\Psi(v_0),$$

as claimed. □

3. Appendix

We shall now recall some notations and results for the minimax principles of the lower semi-continuous functions used throughout the paper.

Definition 3.1. Let V be a real Banach space, $\Phi \in C^1(V, \mathbb{R})$ and $\Psi : V \to (-\infty, +\infty]$ be proper (i.e. $Dom(\Psi) \neq \emptyset$), convex and lower semi-continuous. A point $u \in V$ is said to be a critical point of

$$I := \Psi - \Phi, \tag{11}$$

if $u \in Dom(\Psi)$ and if it satisfies the inequality

$$< D\Phi(u), u - v > + \Psi(v) - \Psi(u) \geq 0, \qquad \forall v \in V. \tag{12}$$

Definition 3.2. We say that I satisfies the Palais–Smale compactness condition (PS) if every sequence $\{u_n\}$ such that $I(u_n) \to c \in \mathbb{R}$, and

$$< D\Phi(u_n), u_n - v > + \Psi(v) - \Psi(u_n) \geq -\epsilon_n \|v - u_n\|, \qquad \forall v \in V,$$

where $\epsilon_n \to 0$, then $\{u_n\}$ possesses a convergent subsequence.

The following is proved in [15].

Theorem 3.3. *(Mountain Pass Theorem). Suppose that $I : V \to (-\infty, +\infty]$ is of the form (11) and satisfies the Palais–Smale condition and the Mountain Pass Geometry (MPG):*

1) $I(0) = 0$, and there exists $e \in V$ such that $I(e) \leq 0$,
2) there exists some ρ such that $0 < \rho < \|e\|$ and, for every $u \in V$ with $\|u\| = \rho$, one has $I(u) > 0$.

Then I has a critical value $c \geq \rho$, which is characterized by

$$c = \inf_{g\in\Gamma} \sup_{t\in[0,1]} I[g(t)],$$

where $\Gamma = \{g \in C([0, 1], V) : g(0) = 0, g(1) = e\}$.

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