Complex analysis

A counterexample of a normality criterion for families of meromorphic functions

Un contre-exemple au critère de normalité pour les familles de fonctions méromorphes

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Abstract

Let $A > 1$ be a constant, and let $\mathcal{F}$ be a family of meromorphic functions in a domain $D$. If, for every function $f \in \mathcal{F}$, $f$ has only zeros of multiplicity at least 2 and satisfies the following conditions: (1) $f(z) = 0 \Rightarrow |f''(z)| \leq A|z|$, (2) $f''(z) \neq z$, (3) all poles of $f$ have multiplicity at least 4, then $\mathcal{F}$ is normal in $D$. In this paper, we first give an example to show that condition (3) is sharp, and prove that our counterexample is unique in some sense.

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Résumé

Soit $A > 1$ une constante et $\mathcal{F}$ une famille de fonctions méromorphes dans un domaine $D$. Si toute fonction $f \in \mathcal{F}$ n’a que des zéros de multiplicité au moins 2 et satisfait les conditions suivantes: (1) $f(z) = 0 \Rightarrow |f''(z)| \leq A|z|$, (2) $f''(z) \neq z$, (3) tous les pôles de $f$ ont multiplicité au moins 4, alors $\mathcal{F}$ est normale dans $D$. Dans cette Note, nous donnons un exemple montrant que la condition (3) est précise. Nous montrons ensuite que notre exemple est, en quelque sorte, unique.

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1. Introduction and main results

Let $D \subseteq \mathbb{C}$ be a domain, and $\mathcal{F}$ be a family of meromorphic functions defined on $D$. $\mathcal{F}$ is said to be normal on $D$, in the sense of Montel, if for each sequence $\{f_n\} \subseteq \mathcal{F}$ there exists a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}\}$ converges spherically locally uniformly on $D$ to a meromorphic function or $\infty$ (see [1, 3, 5]).

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**Theorem A.** Let \( k \geq 2 \) be a positive integer. Let \( \mathcal{F} \) be a family of meromorphic functions defined on a domain \( D \), all of whose zeros have multiplicity at least \( k + 1 \) and whose poles are multiple. Let \( h(z) \neq 0 \) be a holomorphic function on \( D \). If, for each \( f \in \mathcal{F} \), \( f^{(k)}(z) \neq h(z) \), then \( \mathcal{F} \) is normal in \( D \).

They [6] indicated that the multiplicity \( k + 1 \) of the zeros of functions in \( \mathcal{F} \) can not be reduced to \( k \), by considering the following example.

**Example 1.** (see [6]) Let \( \Delta = \{ z : |z| < 1 \} \), \( h(z) = z \), and let

\[
\mathcal{F} = \left\{ f_n(z) = n^k z \right\}.
\]

Clearly, all zeros of \( f_n \) are of multiplicity \( k \), and \( f_n^{(k)}(z) = nk! \neq z \) on \( \Delta \). However, \( \mathcal{F} \) fails to be equicontinuous at 0, and then \( \mathcal{F} \) is not normal in \( \Delta \).

Recently, Xu [4] proved that the multiplicity of the zeros of functions in \( \mathcal{F} \) can be reduced from \( k + 1 \) to \( k \) for the case \( h(z) = z \), but restricting the values \( f^{(k)} \) can take at the zeros of \( f \), as follows.

**Theorem B.** Let \( k \geq 4 \) be a positive integer, \( A > 1 \) be a constant. Let \( \mathcal{F} \) be a family of meromorphic functions in a domain \( D \). If, for every function \( f \in \mathcal{F} \), \( f \) has only zeros of multiplicity at least \( k \) and satisfies the following conditions:

- (a) \( f(z) = 0 \Rightarrow |f^{(k)}(z)| \leq A|z| \),
- (b) \( f^{(k)}(z) \neq z \),
- (c) all poles of \( f \) are multiple,

then \( \mathcal{F} \) is normal in \( D \).

**Theorem C.** Let \( k = 2 \) or \( 3 \), \( A > 1 \) be a constant. Let \( \mathcal{F} \) be a family of meromorphic functions in a domain \( D \). If, for every function \( f \in \mathcal{F} \), \( f \) has only zeros of multiplicity at least \( k \) and satisfies the following conditions:

- (a) \( f(z) = 0 \Rightarrow |f^{(k)}(z)| \leq A|z| \),
- (b) \( f^{(k)}(z) \neq z \),
- (c) all poles of \( f \) have multiplicity at least 3,

then \( \mathcal{F} \) is normal in \( D \).

We remark that for \( k = 2 \) condition (c) in **Theorem C** is insufficient. For the case \( k = 2 \), the multiplicities of poles of \( f \in \mathcal{F} \) need be larger.

**Theorem C’.** Let \( A > 1 \) be a constant. Let \( \mathcal{F} \) be a family of meromorphic functions in a domain \( D \). If, for every function \( f \in \mathcal{F} \), \( f \) has only zeros of multiplicity at least 2 and satisfies the following conditions:

- (a) \( f(z) = 0 \Rightarrow |f^{(k)}(z)| \leq A|z| \),
- (b) \( f^{(k)}(z) \neq z \),
- (c’) all poles of \( f \) have multiplicity at least 4,

then \( \mathcal{F} \) is normal in \( D \).

In fact, case (a) in the proof (case 1) of Lemma 9 in [4, p. 478] can not be ruled out, since \( c_1, c_2, c_3 \) are complex numbers, so that \( f \) has another possible form

\[
f(z) = \frac{(z - c_1)^2(z - c_2)^2(z - c_3)^2}{6(z - b)^3}
\]

for \( k = 2 \), where \( c_1, c_2, c_3, \) and \( b \) are distinct constants. Now since the multiplicities of poles of \( f \in \mathcal{F} \) are at least 4 for \( k = 2 \), as the proof of Theorem 1 in [4, p. 483], we can also have the form (17) in [4], and hence **Theorem C’** holds (for details, see [4]).
Remark. For $k = 1$, the above theorems are no longer true, even if the multiplicities of poles of $f \in \mathcal{F}$ are large enough, which is shown by Example 2 in [4]. The following example shows that the number “4” in condition (c') of Theorem C' is sharp.

Example 2. Let $\Delta = \{z : |z| < 1\}$, and let
$$
\mathcal{F} = \left\{ f_n(z) = \frac{(z - 1/n)^2(z - e^{2\pi i}/n)^2(z - e^{4\pi i}/n)^2}{6z^3} \right\}.
$$
Clearly,
$$
f_n''(z) = z + \frac{2}{n^6 z^5} \neq z.
$$
For each $n$, $f_n$ has three zeros $z_1 = 1/n, z_2 = e^{2\pi i}/n$, and $z_3 = e^{4\pi i}/n$ of multiplicity 2,
$$
|f_n''(z_i)| = \frac{3}{n} \leq 3|z_i|, \ (i = 1, 2, 3).
$$
Since $f_n(1/n) = 0$ and $f_n(0) = \infty$, $\mathcal{F}$ fails to be equicontinuous at 0, and then $\mathcal{F}$ is not normal at 0.

Furthermore, we prove the following result, which illustrates that the above counterexample is unique in some sense.

**Theorem 1.** Let $A > 1$ be a constant, and let $\mathcal{F}$ be a family of meromorphic functions defined in $D$, all of whose zeros are multiple and whose poles all have multiplicity at least 3, such that for every $f \in \mathcal{F}$, $f(z) = 0 \Rightarrow |f''(z)| \leq A|z|$, and $f''(z) \neq z$. If $\mathcal{F}$ is not normal at $z_0 \in D$, then $z_0 = 0$, and there exist $r > 0$ and $\{f_n\} \subset \mathcal{F}$ such that
$$
f_n(z) = \frac{\prod_{i=1}^{3}(z - z_n)^2}{(n - \eta_n)^3} \hat{f}_n(z)
$$
on $\Delta_r = \{z : |z| < r\}$, where $\eta_n/\rho_n \to c_i \ (i = 1, 2, 3)$ and $\eta_n/\rho_n \to (c_1 + c_2 + c_3)/3$ for some sequence of positive numbers $\rho_n \to 0$ and distinct constants $c_1, c_2, c_3$. Moreover, $\hat{f}_n(z)$ is holomorphic and non-vanishing on $\Delta_r$, so that $\hat{f}_n(z) \to \hat{f}(z) \equiv 1/6$ locally uniformly on $\Delta_r$.

In this paper, we denote by $\Delta_r = \{z : |z| < r\}$ and $\Delta'_r = \{z : 0 < |z| < r\}$, and the number $r$ may be different in different places. When $r = 1$, we drop the subscript.

2. **Lemmas**

To prove our results, we need the following lemmas.

**Lemma 1.** ([2, Lemma 2]) Let $k$ be a positive integer and let $\mathcal{F}$ be a family of meromorphic functions in a domain $D$, all of whose zeros have multiplicity at least $k$, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0, f \in \mathcal{F}$. If $\mathcal{F}$ is not normal at $z_0 \in D$, then for each $\alpha, 0 \leq \alpha \leq k$, there exist a sequence of complex numbers $z_n \in D$, $z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0$, and a sequence of functions $f_n \in \mathcal{F}$ such that
$$
g_n(\xi) = \frac{f_n(z_n + \rho_n \xi)}{\rho_n^\alpha} \to g(\xi)
$$
locally uniformly with respect to the spherical metric, where $g$ is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least $k$, so that $g^k(\xi) \leq g^k(0) = kA + 1$. Moreover, $g(\xi)$ has order at most 2.

Here, as usual, $g^k(\xi) = |g'(\xi)|/(1 + |g(\xi)|^2)$ is the spherical derivative of $g$.

**Lemma 2.** ([4, Lemma 6]) Let $f$ be a transcendental meromorphic function of finite order $\rho$, and let $k \geq 2$ be a positive integer. If $f$ has only zeros of multiplicity at least $k$, and there exists $A > 1$ such that $f(z) = 0 \Rightarrow |f^{(k)}(z)| \leq A|z|$, then $f^{(k)}$ has infinitely many fix-points.

The next lemma is Lemma 9 in [4], but the form (4) is ruled out by mistake (since $c_1, c_2, c_3$ are complex numbers, $(c_1 - c_2)^2 + (c_1 - c_3)^2 + (c_2 - c_3)^2 = 0$ does not imply $c_1 = c_2 = c_3$. For details, see [4, p. 478]).

**Lemma 3.** (cf. [4, Lemma 9]) Let $f$ be a rational function, all of whose zeros are multiple. If $f''(z) \neq z$, then one of the following cases must occur:
Lemma 4. ([4, Lemma 11]) Let \( F \) be a family of meromorphic functions in a domain \( D, A > 1 \) be a constant. Suppose that, for every \( f \in F \), \( f \) has only zeros of multiplicity at least 2, and satisfies the following conditions:

(a) \( f(z) = 0 \Rightarrow |f''(z)| \leq A|z| \),
(b) \( f''(z) \neq z \),
(c) all poles of \( f \) are of multiplicity at least 3,

then \( F \) is normal in \( D \setminus \{0\} \).

3. Proof of Theorem 1

Since \( F \) is not normal at \( z_0 \), by Lemma 4, \( z_0 = 0 \). Without loss of generality, we assume \( D = \Delta = \{z : |z| < 1\} \). Again by Lemma 4, \( F \) is normal on \( \Delta' \).

Consider the family

\[ G = \left\{ g(z) = \frac{f(z)}{z} : f \in F \right\}. \]

We claim that \( f(0) \neq 0 \) for every \( f \in F \). Otherwise, if \( f(0) = 0 \), by the assumption of Theorem 1, \( |f''(0)| \leq 0 \), and then \( f''(0) = 0 \). But \( f''(z) \neq z \), which is a contradiction. Thus, for each \( g \in G \), \( g(0) = \infty \). Furthermore, all zeros of \( g(z) \) are multiple. On the other hand, by a simple calculation, we have:

\[ g''(z) = \frac{f''(z)}{z} - \frac{2g''(z)}{z}. \]

Since \( f(z) = 0 \Rightarrow |f''(z)| \leq A|z| \), we deduce that \( g(z) = 0 \Rightarrow |g''(z)| \leq A \).

Clearly, \( G \) is normal on \( \Delta' \). We claim that \( G \) is not normal at \( z = 0 \). Indeed, if \( G \) is normal at \( z = 0 \), then \( G \) is normal on the whole disk \( \Delta \) and hence equicontinuous on \( \Delta \) with respect to the spherical distance. On the other hand, \( g(0) = \infty \) for each \( g \in G \), so there exists \( \varepsilon > 0 \) such that for every \( g \in G \) and every \( z \in \Delta_\varepsilon \), \( |g(z)| \geq 1 \). Then \( f(z) \) is non-vanishing, and thus \( 1/f \) is holomorphic on \( \Delta_\varepsilon \) for all \( f \in F \). Since \( F \) is normal on \( \Delta' \) but not normal on \( \Delta \), the family \( F_1 = \{1/f, f \in F\} \) is holomorphic on \( \Delta_\varepsilon \) and normal on \( \Delta'_\varepsilon \), but it is not normal at \( z = 0 \). Therefore, there exists a sequence \( \{1/f_n\} \subset F_1 \) that converges locally uniformly on \( \Delta'_\varepsilon \), but not in \( \Delta_\varepsilon \). Hence, by the maximum modulus principle, \( 1/f_n \to \infty \) on \( \Delta_\varepsilon \). Thus \( f_n \to 0 \) converges locally uniformly on \( \Delta_\varepsilon \), and so does \( \{g_n\} \subset G \), where \( g_n = f_n/z \). But \( |g_n(z)| \geq 1 \) for \( z \in \Delta_\varepsilon \), which is a contradiction.

Then, by Lemma 1, there exist functions \( g_n \in G \), points \( z_n \to 0 \) and positive numbers \( \rho_n \to 0 \) such that

\[ G_n(\xi) = \frac{g_n(z_n + \rho_n \xi)}{\rho_n^2} \to G(\xi) \]

converges spherically uniformly on compact subsets of \( \mathbb{C} \), where \( G \) is a non-constant meromorphic function on \( \mathbb{C} \) and of finite order, all zeros of \( G \) are multiple, and \( G^\#(\xi) \leq G^\#(0) = 2A + 1 \) for all \( \xi \in \mathbb{C} \).

By [4, pages 481–482], we can assume that \( z_n/\rho_n \to \alpha \) (a finite complex number). Then

\[ \frac{g_n(\rho_n \xi)}{\rho_n^2} = \frac{g_n(z_n + \rho_n(\xi - z_n/\rho_n))}{\rho_n^2} = G_n(\xi - z_n/\rho_n) \to G(\xi - \alpha) = \tilde{G}(\xi) \]

spherically uniformly on compact subsets of \( \mathbb{C} \). Clearly, \( \tilde{G}(0) = \infty \).
Set
\[ H_n(\xi) = \frac{f_n(\rho_n \xi)}{\rho_n^3}. \]  
(5)

Then
\[ H_n(\xi) = \frac{f_n(\rho_n \xi)}{\rho_n^3} - \xi \frac{g_n(\rho_n \xi)}{\rho_n^2} \rightarrow \xi \hat{G}(\xi) = H(\xi) \]  
(6)
spherically uniformly on compact subsets of \( \mathbb{C} \), and
\[ H_n''(\xi) = \frac{f_n''(\rho_n \xi)}{\rho_n} \rightarrow H''(\xi) \]  
(7)
locally uniformly on \( \mathbb{C} \setminus H^{-1}(\infty) \). By the assumption of Theorem 1 and (6), all zeros of \( H \) are multiple, and all poles of \( H \) have multiplicity at least 3. Since \( \hat{G}(0) = \infty \), \( H(0) \neq 0 \).

Claim: (I) \( H(\xi) = 0 \Rightarrow |H''(\xi)| \leq A|\xi|; \) (II) \( H''(\xi) \notin \xi \).

If \( H(\xi_0) = 0 \), by Hurwitz’s theorem and (6), there exist \( \xi_n \rightarrow \xi_0 \) such that \( f_n(\rho_n \xi_n) = 0 \) for \( n \) sufficiently large. By the assumption, \( |f_n''(\rho_n \xi_n)| \leq A|\rho_n \xi_n| \). Then, it follows from (7) that \( |H''(\xi_0)| \leq A|\xi_0| \). Claim (I) is proved.

Suppose that there exists \( \xi_0 \) such that \( H''(\xi_0) = \xi_0 \). By (7),
\[ 0 \neq \frac{f_n''(\rho_n \xi) - \rho_n \xi_n}{\rho_n} = \frac{H_n''(\xi) - \xi}{H_n''(\xi) - \xi}, \]
uniformly on compact subsets of \( \mathbb{C} \setminus H^{-1}(\infty) \). Hurwitz’s theorem implies that \( H''(\xi) \equiv \xi \) on \( \mathbb{C} \setminus H^{-1}(\infty) \), and then on \( \mathbb{C} \). Hence \( H \) is a polynomial of degree 3. In view of the fact that all zeros of \( H \) are multiple, we know that \( H \) has only one zero, say \( \xi_1 \), with multiplicity 3, so that \( H''(\xi_1) = 0 \), and thus \( \xi_1 = 0 \) since \( H''(\xi) \equiv \xi \). But \( H(0) \neq 0 \), which is a contradiction. This proves claim (II).

Noting that \( H \) is of finite order, Lemma 2 implies that \( H \) must be a rational function. Since all poles of \( H \) have multiplicity at least 3, it follows from Lemma 3 that
\[ H(\xi) = \frac{1}{6}(\xi + c)^3 \]
or
\[ H(\xi) = \frac{\prod_{i=1}^{3}(\xi - c_i)^2}{6[\xi - (c_1 + c_2 + c_3)/3]^3}. \]
where \( c_1 \), \( c_2 \), and \( c_3 \) are distinct constants, and \( c \) is a nonzero constant. The former case can be ruled out as the form (17) in \([4, \text{pp. } 483-485]\). So this, together with (5) and (6), gives that
\[ \frac{f_n(\rho_n \xi)}{\rho_n^3} \rightarrow \frac{\prod_{i=1}^{3}(\xi - c_i)^2}{6[\xi - (c_1 + c_2 + c_3)/3]^3}. \]  
(8)
Noting that all zeros of \( f_n \) are multiple, there exist \( \xi_{ni} \rightarrow c_i (i = 1, 2, 3) \) and \( \lambda_n \rightarrow (c_1 + c_2 + c_3)/3 \) such that \( \xi_{ni} = \rho_n \xi_n (i = 1, 2, 3) \) are zeros of \( f_n \) with exact multiplicity 2, and \( \eta_n = \rho_n \lambda_n \) is the pole of \( f_n \) with exact multiplicity 3.

Now write
\[ f_n(z) = \frac{\prod_{i=1}^{3}(z - \xi_{ni})^2}{(z - \eta_n)^3} f_n(z). \]  
(9)
Then, by (8) and (9), we obtain
\[ \tilde{f}_n(\rho_n \xi) \rightarrow \frac{1}{6} \]  
(10)
on \( \mathbb{C} \).

Next we complete our proof in three steps.

**Step 1.** We first prove that there exists \( \delta > 0 \) such that \( \tilde{f}_n(z) \neq 0 \) on \( \Delta_\delta \).

Suppose not, taking a sequence and renumbering if necessary, that \( \tilde{f}_n \) has zeros tending to 0. Assume that \( \tilde{z}_n \rightarrow 0 \) is the zero of \( \tilde{f}_n \) with the smallest modulus. Then, by (10), it is easy to see that \( \tilde{z}_n/\rho_n \rightarrow \infty \).

Set
\[ \tilde{f}_n^*(z) = \tilde{f}_n(\tilde{z}_n z). \]  
(11)
Clearly, \( \tilde{f}_n^*(z) \) is well defined on \( \mathbb{C} \) and not vanishing on \( \Delta \). Moreover, \( \tilde{f}_n^*(1) = 0 \).
Now let
\[ M_n(z) = \frac{\prod_{i=1}^{3}(z - \xi_i/\zeta_i)}{(z - \eta_i/\zeta_i)^3} f_n^*(z). \] (12)

It follows from (9), (11), and (12) that
\[ M_n(z) = \frac{\prod_{i=1}^{3}(z^2_n - \xi_i)}{(z^2_n - \eta_i)^3} \frac{f_n(\hat{z}_n z)}{\hat{z}_n^3} = f_n(\hat{z}_n z). \]

Obviously, all zeros of \( M_n(z) \) have multiplicity at least 2 and all poles of \( M_n(z) \) have multiplicity at least 3. Since \( f_n(z) = 0 \Rightarrow |f_n(z)| \leq A|z| \), it follows that \( M_n(z) = 0 \Rightarrow |M_n''(z)| \leq A|z| \). In view of \( f_n''(z) \neq z \), we have
\[ M_n''(z) = z = \frac{f_n''(\hat{z}_n z) - 2\hat{z}_n z}{\hat{z}_n^3} \neq 0. \] (13)

Thus Lemma 4 implies that \( \{M_n(z)\} \) is normal on \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \).

Since \( \xi_i/\rho_n = \xi_i \rightarrow c_i \) for \( i = 1, 2, 3 \), \( \eta_i/\rho_n = \lambda_i \rightarrow (c_1 + c_2 + c_3)/3 \) and \( \hat{z}_n/\rho_n \rightarrow \infty \), we have
\[ \frac{\xi_i}{\hat{z}_n} = \frac{\xi_i}{\rho_n} \frac{\rho_n}{\hat{z}_n} \rightarrow 0 \quad \text{for} \quad i = 1, 2, 3; \quad \frac{\eta_i}{\hat{z}_n} = \frac{\eta_i}{\rho_n} \frac{\rho_n}{\hat{z}_n} \rightarrow 0. \]

We now see from (12) that \( \{f_n^*\} \) is also normal on \( \mathbb{C}^* \).

Then, by taking a subsequence, we assume that \( f_n^* \rightarrow \hat{f}^* \) spherically locally uniformly on \( \mathbb{C}^* \). Moreover, since \( f_n^*(1) = \hat{f}_n^*(\hat{z}_n) = 0 \), we know that \( \hat{f}^*(1) = 0 \) with multiplicity at least 2.

Set
\[ L_n(z) = M_n''(z) - z. \] (14)

From (13), we have \( L_n \neq 0 \).

Now we show that \( \hat{f}^*(z) \neq 0 \). Otherwise \( \hat{f}_n^*(z) \rightarrow 0 \), thus \( L_n(z) \rightarrow -z \) and \( L_n'(z) \rightarrow -1 \) locally uniformly on \( \mathbb{C}^* \). By the argument principle, we get
\[ \left| n(1, L_n) - n(1, \frac{1}{L_n}) \right| = \frac{1}{2\pi} \left| \int_{|z|=1} \frac{L_n'}{L_n} \, dz \right| \rightarrow \frac{1}{2\pi} \left| \int_{|z|=1} \frac{1}{z} \, dz \right| = 1, \]

where \( n(r, f) \) denotes the number of poles of \( f \) in \( \Delta_r \), counting multiplicity. It follows that \( n(1, L_n) = 1 \). On the other hand, the poles of \( L_n(z) = M''_n(z) - z \) have multiplicity at least 5, which is a contradiction.

Then \( 1/f_n^* \rightarrow 1/\hat{f}^* \neq \infty \) spherically locally uniformly on \( \mathbb{C}^* \). Recalling that \( \hat{f}_n^* \neq 0 \) on \( \Delta \), then \( 1/\hat{f}_n^* \) is holomorphic on \( \Delta \). The maximum modulus principle implies that \( 1/f_n^* \rightarrow 1/\hat{f}^* \), and then \( f_n^* \rightarrow \hat{f}^* \) on \( \Delta \). Hence, \( f_n^* \rightarrow \hat{f}^* \) spherically locally uniformly on \( \mathbb{C} \). In particular, \( f_n^*(0) = \hat{f}_n^*(0) \rightarrow 1/6 = \hat{f}^*(0) \).

Then, we obtain from (12) and (14) that
\[ L_n(z) \rightarrow L(z) = (z^2 \hat{f}^*(z))'' - z \]
locally uniformly on \( \mathbb{C}^* \setminus \{(\hat{f}^*)^{-1}(\infty)\} \). Note that \( L_n(z) \neq 0 \), then each \( 1/L_n(z) \) is holomorphic on \( \mathbb{C} \), and thus \( 1/L_n(z) \rightarrow 1/L(z) \) locally uniformly on \( \mathbb{C}^* \setminus \{(\hat{f}^*)^{-1}(\infty)\} \). By Hurwitz’s theorem, \( 1/L(z) \equiv \infty \) or \( 1/L(z) \) is holomorphic on \( \mathbb{C}^* \setminus \{(\hat{f}^*)^{-1}(\infty)\} \). If \( 1/L(z) \equiv \infty \), then \( L(z) \equiv 0 \) on \( \mathbb{C}^* \setminus \{(\hat{f}^*)^{-1}(\infty)\} \), and hence on \( \mathbb{C} \), that is,
\[ (z^2 \hat{f}^*(z))'' - z \equiv 0. \]

It follows that
\[ \hat{f}^*(z) = \frac{z^2 + c_1 z + c_2}{6z^2}, \]
where \( c_1, c_2 \) are constants. Since \( \hat{f}^*(1) = 0 \) and all zeros of \( \hat{f}^* \) are multiple, we have
\[ \hat{f}^*(z) = \frac{(z - 1)^3}{6z^2}, \]
which is impossible, since \( z^2 + c_1 z + c_2 \neq (z - 1)^3 \). Thus \( 1/L(z) \) is holomorphic on \( \mathbb{C}^* \setminus \{(\hat{f}^*)^{-1}(\infty)\} \). The maximum modulus principle implies that \( L_n(z) \rightarrow L(z) \) locally uniformly on \( \mathbb{C} \). Since \( L_n(z) \neq 0 \), we have \( L(z) \neq 0 \) or \( L(z) \equiv 0 \). As before, \( L(z) \equiv 0 \) is impossible. Then we have \( L(z) \neq 0 \). But \( L(0) = 0 \) since \( \hat{f}^*(0) = 1/6 \), which is a contradiction. Thus our claim is proved.
**Step 2.** We now show that \( \hat{f}_n \to \hat{f} \) spherically locally uniformly on \( \Delta \), and each \( \hat{f}_n(z) \) is holomorphic on \( \Delta_{\delta'} \) for some \( \delta' > 0 \).

Since \( \{f_n\} \), and hence \( \{\hat{f}_n\} \) is normal on \( \Delta' \), taking a subsequence and renumbering, we have \( \hat{f}_n \to \hat{f} \) spherically locally uniformly on \( \Delta' \).

We claim that \( \hat{f}(z) \neq 0 \) on \( \Delta' \). Otherwise, we have \( f_n''(z) \to 0 \) and \( f_n'''(z) \to 0 \) locally uniformly on \( \Delta' \). Then the argument principle yields that

\[
\left| n\left( \frac{1}{2}, f_n'' - z \right) - n\left( \frac{1}{2}, f_n''' - z \right) \right| = \frac{1}{2\pi} \left| \int_{|z| = \frac{1}{2}} \frac{f_n''' - 1}{f_n'' - z} \, dz \right| \to \frac{1}{2\pi} \left| \int_{|z| = \frac{1}{2}} \frac{1}{z} \, dz \right| = 1.
\]

Now that \( f_n'''(z) \neq z \), it follows that \( n\left( \frac{1}{2}, f_n'' \right) = n\left( \frac{1}{2}, f_n''' \right) = 1 \), which is impossible.

Recalling that \( \hat{f}_n(z) \neq 0 \), as before, the maximum modulus principle implies that \( \hat{f}_n \to \hat{f} \) spherically locally uniformly on \( \Delta \). Since \( \hat{f}_n(0) \to 1/6 \), we have \( \hat{f}(0) = 1/6 \). Hence \( \hat{f} \) is holomorphic at 0. Moreover, there exists a positive number \( \delta' \) such that each \( \hat{f}_n \) is holomorphic on \( \Delta_{\delta'} \).

**Step 3.** Finally, we prove that \( \hat{f}(z) \equiv 1/6 \).

By (9), we get \( f_n(z) \to z^3 \hat{f}(z) \) on \( \Delta' \). Thus

\[
f_n'''(z) - z = [z^3 \hat{f}(z)]''' - z,
\]

on \( \Delta' \setminus \{\hat{f}^{-1}(\infty)\} \). If \( [z^3 \hat{f}(z)]''' - z \neq 0 \), noting that \( f_n'''(z) \neq z \), the maximum modulus principle implies that (15) still holds on \( \Delta \). Then, Hurwitz’s theorem yields that \( [z^3 \hat{f}(z)]''' - z \neq 0 \), violating the fact that \( ([z^3 \hat{f}(z)]''' - z)_{z=0} = 0 \). Hence, \( [z^3 \hat{f}(z)]''' - z \equiv 0 \). This, together with \( \hat{f}(0) = 1/6 \), gives \( \hat{f}(z) \equiv 1/6 \).

Letting \( r = \min(\delta, \delta') \), the proof of **Theorem 1** is completed. ☐

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**References**