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Combinatorics

Lah numbers and Lindström's lemma

Nombres de Lah et lemme de Lindström

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A B S T R A C T

We provide a combinatorial interpretation of Lah numbers by means of planar networks. Henceforth, as a consequence of Lindström's lemma, we conclude that the related Lah matrix possesses a remarkable property of total non-negativity.

R É S U M É

Nous donnons une interprétation combinatoire des nombres de Lah en termes de réseaux plans. Puis, comme conséquence du lemme de Lidström, nous en déduisons que la matrice de Lah associée possède la propriété remarquable d'être totalement non négative.

1. Introduction

The Lah numbers were introduced by Ivan Lah in 1952 and since then they are the subject of many prominent researches. For $n, k \in \mathbb{N}_0$, we define $L_{n.k}$ as the number of ways to partition the set $[n] = \{1, 2, \dots, n\}$ into k nonempty tuples (i.e. linearly ordered sets). We let $L_{0,0} := 1$. Define the *Lah matrix* $\mathrm{LM}_m = [L_{i,j}]$ as the matrix of dimension $m \times m$, whose element in the i-th row and j-th column is $L_{i,j}$. Note that LM_m is a low-triangular matrix. For the first column of LM, it holds $L_{m,1} = m!$ since we have to put all labeled "balls" into a sole "box" – where we distinguish the order of balls, meaning that we deal with permutations of n. Further consideration of this partitioning shows that the Lah numbers are recursive in nature, and more precisely

$$L_{n+1,k} = L_{n,k-1} + (n+k)L_{n,k}.$$

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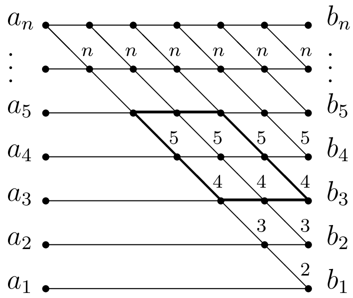

Fig. 1. The Lah numbers count weighted paths in a directed acyclic planar graph.

Other contexts where the Lah numbers appears include noncrossing partitions, Dyck paths, q-analogues as well as falling and rising factorials, just to name a few. In particular, the number of Dyck n-paths with $n+1-k$ peaks labeled $1, 2, \ldots, n+1-k$ is equal to $L_{n,k}$. Lah numbers were originally introduced as coefficients in the polynomial identity

$$x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} L_{n,k}\, x(x-1)\cdots(x-k+1),$$

where $n, k, x \in \mathbb{N}_0$. An explicit formula is known for Lah numbers,

$$L_{m,k} = \binom{m-1}{k-1}\frac{m!}{k!}\,. \qquad (1)$$

Some natural generalizations are done by Wagner [4] as well as by Ramirez and Shattuck [3].

2. The main result

A matrix is *totally non-negative* (resp. *positive*) if each of its minors is non-negative (resp. positive) [1]. In a planar acyclic weighted directed graph with n sources a_i's and n sinks b_i's, one defines a *weight matrix* $W = [w_{i,j}]$ of dimension $n \times n$, where $w_{i,j}$ is the sum of the weights of paths from a_i to b_j. Such graphs are also called planar networks.

We let $\Delta_{I,J}(M)$ denote the minor of a matrix M with the row indices from set I and the column indices from set J.

Lemma 1 *(Lindström's lemma). A minor $\Delta_{I,J}(W)$ of the weight matrix W of a planar network is equal to the sum of weights of all collections of vertex-disjoint paths that connect the sources labeled by I with the sinks labeled by J.*

We define a planar network N_n by the figure below (Fig. 1). Note that with the same network, but with unit weights, we obtain the "Pascal triangle" as the related weight matrix.

Theorem 1. *For $m, k \le n$, the Lah number $L_{m,k}$ corresponds to the number of weighted paths in the network N_n from vertex a_m to the vertex b_k.*

Proof. This obviously holds for $m < k$, so assume $m \ge k$. Notice that every directed path from a_m to b_k passes through the rectangular grid, which is of size $(m-k) \times (k-1)$ (e.g., for a_5 and b_3 it is marked in the figure). Thus the number of these paths is

$$\binom{m-k+k-1}{k-1} = \binom{m-1}{k-1}.$$

Every such path is of length $m-1$ consisting of $k-1$ "horizontal" edges and $m-k$ "diagonal" edges. Horizontal edges are all of weight 1 and regarding the diagonal edges, when moving from a_m to b_k, they have weights

$$m, m-1, \ldots, k+1,$$

respectively. So, each such path has weight $\frac{m!}{k!}$. This gives us that the total weight of the paths from a_m to b_k is

$$\binom{m-1}{k-1}\frac{m!}{k!},$$

which is the Lah number $L_{m,k}$ by (1). □

As an easy consequence from Lindström's lemma, we obtain the following.

Corollary 1. *The Lah triangular matrix* LM_m *is totally non-negative.*

Totally positive matrices, and in particular their eigenvalues, are related with the *variation-decreasing* vectors. Let $u = (u_1, u_2, \ldots, u_n)$ be a vector in $\mathbb{R}^n$. A *sign change* in u is a pair of indices (i, j) such that, for $i < j \le n$:

i) $u_k = 0$ for all k (if there are any), $i < k < j$, and
ii) $u_i u_j < 0$.

The *weak variation* $\mathrm{Var}^-(u)$ is the number of sign changes in u. For example, $\mathrm{Var}^-(2, -2, 0, 1, -3, 0, 0, 1) = 4$. Now, an $n \times m$ matrix M with real entries is *variation-decreasing* if, for all nonzero vectors $x \in \mathbb{R}^m$,

$$\mathrm{Var}^-(Mx) \le \mathrm{Var}^-(x). \tag{2}$$

We point out Motzkin's theorem that relates the notion of variation-decreasing matrices with total positivity (see J. Kung, G. Rota, and C. Yan [2]).

Theorem 2 *(Motzkin). A totally non-negative matrix is variation-decreasing.*

Apparently, once having known that the Lah triangular matrix LM_m is totally non-negative, we have that LM_m satisfies property (2).

Corollary 2. *The Lah triangular matrix* LM_m *is variation-decreasing.*

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