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Number theory

CM fields with a reciprocal unit-primitive element

Corps CM ayant une unité reciproque primitive

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ABSTRACT

Let K be a noncyclotomic CM field. We show that the field $K \cap \mathbb{R}$ has a reciprocal unit-primitive element when K does. Also, we prove some related conditions that make the converse of this assertion true.

RÉSUMÉ

Soit K un corps CM non cyclotomique. On montre que, si K admet une unité réciproque primitive, il en est de même pour le corps $K \cap \mathbb{R}$. On prouve également des conditions qui rendent vraie l'inverse de cette proposition.

1. Introduction

Several authors studied the question, initiated by Lalande, in [5], of whether a number field admits a reciprocal unit-primitive element (see, for instance, [4], [7], [8] and the references therein). An algebraic number ϑ is said to be reciprocal if $1/\vartheta$ is a conjugate of ϑ. When such a reciprocal number is an algebraic integer, it will of course even be a unit. In this case, we prefer to call ϑ a reciprocal unit rather than a reciprocal algebraic integer.

As is [2], we are also interested in primitive elements for number fields as extensions of $\mathbb{Q}$. We need some shorthand here. A PE (primitive element) of a number field K is an element $\vartheta \in K$ such that $K = \mathbb{Q}(\vartheta)$. If ϑ is a unit as well, then it will be called a UPE (unit-primitive element) of K. When some PE (resp. some UPE) is also reciprocal, we will simply speak about reciprocal PE's (resp. reciprocal UPE's). The word "reciprocal" will always be spelled out for the sake of clarity.

The main question studied in this note is: which number fields admit a reciprocal UPE? In this context, we give another characterization of number fields K having a reciprocal PE; the first one is due to Lalande [5], in terms of the Galois group of the normal closure of K, and a simple alternative proof of this last-mentioned result was recently given in [8]. For the main problem, our focus will be on CM fields, and concomitantly we also have to look at totally real fields.

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From [7], we have the following result for a noncyclotomic CM field K.

(1) Assume that K has no roots of unity other than ± 1. Then, K admits a reciprocal UPE if and only if K^+ has a reciprocal totally positive UPE η, where $K^+ = K \cap \mathbb{R}$, and $K = \mathbb{Q}(\sqrt{-\eta})$. (We will reprove this here for the sake of completeness.)

(2) Let ϑ be a reciprocal UPE of K. Then, $[K:\mathbb{Q}]/2 = [K^+:\mathbb{Q}]$ is even and $\vartheta^2 = \zeta\eta$, where $\eta \in K^+$, ζ is a root of unity, η is a totally positive reciprocal unit and $\zeta \neq 1$.

It is worth noting that some explicit conditions which imply that a nonCM totally complex field, with degree not a multiple of 4, has a reciprocal UPE, have also been shown in [7]. It is our goal, here, to give general conditions that characterize CM fields which have a reciprocal UPE; this is treated in the last section. In the next one, we prove a new characterization of number fields (resp. of totally real number fields) having a reciprocal PE (resp. a reciprocal UPE).

2. Preliminaries

To begin with, let us clarify that an involution σ of a field K will always be understood to be an automorphism of K with order 2, that is $\sigma^2 = I_K \neq \sigma$, where I_K is the identity of K. If K is equipped with an involution, then $[K:\mathbb{Q}]$ is even.

Proposition 2.1. *Let K be any number field larger than $\mathbb{Q}$. Then, K has a reciprocal PE if and only K admits an involution.*

Proof. Let ϑ be a reciprocal PE of K, and let σ denote the map from K to $\mathbb{C}$, sending ϑ to its inverse. Then, σ maps K to itself, $\sigma^2(\vartheta) = \vartheta \neq \sigma(\vartheta)$, since $\vartheta \neq \pm 1$, and so σ is an involution of K.

For the converse, consider an involution σ of K, and take any primitive element a of K. Inspired by the proof of Theorem 2 of [3], we put $a_n := (\sigma(a)+n)/(a+n)$ for $n \in \mathbb{N}$. By construction, all a_n are reciprocal. If none of them is a primitive element of K, we find a proper subfield K' of K containing infinitely many of them. Hence, there is an embedding φ of K into $\mathbb{C}$, which is different from I_K, and fixes two elements a_r and a_s with $r < s$. Replacing a by $a+r$ and s by $s-r$, we can assume $r = 0$. This leads to

$$\sigma(a)\varphi(a) = a\varphi(\sigma(a)), \tag{*}$$

and

$$\sigma(a)\varphi(a) + s\varphi(a) + s\sigma(a) = a\varphi(\sigma(a)) + s\varphi(\sigma(a)) + sa.$$

Therefore, $a - \sigma(a) = \varphi(a) - \varphi(\sigma(a)) = (a-\sigma(a))\varphi(a)/a$, where the last equality uses again (*). Then, either $a - \sigma(a)$ must be zero or $\varphi(a)/a$ must be 1. Neither possibility can hold, since a is a PE of K, $\sigma \neq I_K$ and $\varphi \neq I_K$. Hence, some a_n is a reciprocal PE of K. □

Now, we characterize totally real number fields L with a reciprocal UPE, in terms of involutions. By the previous result, a necessary condition for L to have a reciprocal UPE is that L has an involution. In fact, this is already sufficient.

Proposition 2.2. *Let L denote any totally real number field equipped with an involution ρ. Then, L has a reciprocal UPE η such that $\eta\rho(\eta) = 1$ and η^n is a reciprocal UPE of L for any positive integer n.*

Proof. For every subfield E of L, we define $V_E = \mathbb{Q} \otimes_{\mathbb{Z}} O_E^*$, where O_E^* denotes (as usual) the unit group of E, and we think of this as a $\mathbb{Q}$-vector space (so multiplication is now considered as addition). Then, Dirichlet's theorem gives that V_E has dimension $[E:\mathbb{Q}] - 1$.

By the functoriality of the tensor product, the automorphism ρ induces an automorphism $\rho' = I_{\mathbb{Q}} \otimes \rho$ on the vector space V_L. Explicitly, $\rho'(r \otimes x) = r \otimes \rho(x)$ for all $(r, x) \in \mathbb{Q} \times O_E^*$. Under this involution ρ', the space V_L decomposes as $V_L = W^+ \oplus W^-$, where $W^{\pm}$ is the ± 1-eigenspace of ρ. It is easy to see that W^+ identifies with V_F, where $F \subset L$ is the fixed field of ρ. Indeed, the image of V_F in V_L is clearly fixed under ρ'; conversely, if $z = r \otimes x \in V_L$ is fixed by this way, it equals $\frac{1+r}{2}z = r \otimes y$ with $y = x\rho(x) \in F$, so it comes from V_F. Hence the dimension of W^- is $[L:\mathbb{Q}]/2$.

For any proper subfield E of L, we have that $\dim V_E = [E:\mathbb{Q}] - 1 \leq [L:\mathbb{Q}]/2 - 1 < \dim W^-$. Hence, the union U of all spaces V_E, with E running through the proper subfields of L, cannot contain W^-; even better, there is a one-dimensional $\mathbb{Q}$-vector space $T \subset W^-$ such that $T \setminus \{0\}$ is disjoint from U.

Now, pick a unit η of L, such that $1 \otimes \eta$ generates T. Then, all powers η^n, where $n > 0$, are primitive, since all elements $n \cdot (1 \otimes \eta) = 1 \otimes \eta^n \notin U$. Moreover, as $T \subset W^-$, we have $\rho'(1 \otimes \eta) = -1 \otimes \eta = 1 \otimes \eta^{-1}$. This means that $\rho(\eta)$ differs from η^{-1} at most by a sign. Replacing η by its square, we obtain the desired result. □

Remark 2.3. Prompted by a question of the referee, let us point out that it is easy in practice to find such UPE's. Let for example L be the totally real quartic field with defining polynomial $a^4 - 34a^2 + 17$. It has a visible involution ρ, sending a to $-a$. By PARI [1], one can get three independent fundamental units u_k, $k = 1, 2, 3$. We simply look at $\eta_k = u_k/\rho(u_k)$ (up to a factor 2, η_k is the projection of u_k into the minus part W^-). All η_k are now reciprocal units under ρ, and using the function *modreverse* one may quickly check which of them is primitive for L. When the authors did this, already

$\eta_1 = (3a^3 + a^2 - 95a - 89)/16$ came out primitive, hence a reciprocal UPE. (It is not guaranteed that every run of PARI will produce the same set of fundamental units.) Roughly speaking, the content of the preceding proposition is that this naive non-deterministic algorithm for finding a reciprocal UPE will work.

3. CM fields

We now turn to CM fields K. Let always Ω_K denote the group of roots of unity in K, and $M_K = |\Omega_K|$ its order. Let τ denote the complex conjugation on K. Then, τ is an involution of K, K^+ is the fixed field of τ, and τ commutes with all embeddings of K into $\mathbb{C}$. If K is cyclotomic, say $K = \mathbb{Q}(\zeta)$, for some root of unity ζ, then K has a reciprocal UPE, namely ζ, with a corresponding involution τ. Conversely, if the CM field K has a reciprocal UPE ϑ and the corresponding involution is τ, then Kronecker's theorem gives immediately that ϑ is a root of unity and so K is cyclotomic.

Henceforth, we explicitly exclude cyclotomic fields from our study; that is, we assume without further mention that the CM field K is not cyclotomic. In particular, we have the following result.

Proposition 3.1. *If the CM field K has an involution σ, then the restriction of σ to K^+ is an involution of K^+. Hence, K^+ has a reciprocal UPE when K does.*

Proof. From the facts that $\sigma(K^+) \subset K^+$ (recall that K^+ is totally real) and K^+ is the fixed subfield of K by τ, we obtain the first assertion in Proposition 3.1. The second one follows then immediately from Proposition 2.1 and Proposition 2.2. □

To study the converse of the second assertion in Proposition 3.1, we shall consider two main cases, and let us call them (A) and (B): (A) stands for $M_K = 2$ and (B) means $M_K > 2$ (equivalently, $K = K^+(\Omega_K)$). The subcase of (A) where the unit index Q_K equals one is the simplest of all. Here every unit of K is already in K^+, and K does not even have a UPE. Recall that Q_K is the index of the group $O^*_{K^+}\,\Omega_K$ in O^*_K, and $Q_K \in \{1, 2\}$ (see for example [6, page 56]). Next in simplicity is the case (A) with $Q_K = 2$. This means that there is a unit $\beta \in K$ which is not in K^+, but necessarily $b := \beta^2$ is in K^+. Then, $K = K^+(\sqrt{b})$, and from the hypothesis that K is CM we can infer that b is totally negative. To simplify the notation in what follows, let us introduce the following terminology.

Definition 3.2. A totally negative unit $b \in K^+$ which has a square root in K is called a "typical unit" for the extension K/K^+.

It is easy to see that $u^2 b$ is a typical unit for K/K^+, $\forall\ u \in O^*_{K^+}$, when b is so, and all typical units for K/K^+ are of this last-mentioned form. Notice also, by the above, that typical units exist if $(Q_K, M_K) = (2, 2)$, and do not exist for $(Q_K, M_K) = (1, 2)$. Moreover, we have when $Q_K = 1$: K contains typical units if and only if $i = \sqrt{-1} \in K$, and we shall see in the proof of Theorem 3.4 below that typical units exist when $M_K > 2 = Q_K$, and that M_K is not divisible by 4.

With this notation, our result for the case (A) may be stated as follows; this part is already on page 431 (Theorem 1.12 (iii)) of [7]:

Theorem 3.3. *Suppose $Q_K = M_K = 2$. Then, K has a reciprocal UPE if and only if there is a typical unit for K/K^+, which is a reciprocal PE for K^+. In particular, if K has a reciprocal UPE, then so does K^+.*

Proof. To show the direct implication, in the first assertion of Theorem 3.3, notice that the equation $M_K = 2 = Q_K$ yields $O^*_{K^+}\Omega_K = O^*_{K^+}\,\{\pm 1\} = O^*_{K^+}$ and $[O^*_K : O^*_{K^+}\Omega_K] = [O^*_K : O^*_{K^+}] = 2$. Now, let ϑ be any reciprocal UPE for K. Then, $\vartheta \in K$, $\vartheta \notin K^+$, $O^*_K = \vartheta\, O^*_{K^+} \cup O^*_{K^+}$, $b := \vartheta^2 \in K^+$ (as mentioned above, this holds for every unit in K which is not in K^+), and so b is totally negative, as each conjugate of the CM field K is nonreal; hence, b is a typical unit for K/K^+. Also, b is a reciprocal PE for K^+, since ϑ is so for K.

Conversely, if b is a typical unit for K/K^+ which is a reciprocal PE for K^+, corresponding to the involution ρ of K^+, then $\sqrt{b}$ is a reciprocal unit of K corresponding to a suitable extension of ρ to K. Moreover, if b is primitive for K^+, then $\mathbb{Q}(\sqrt{b}) = K$. □

We now turn to case (B), which has not been treated fully so far.

Theorem 3.4. *Assume that $M_K > 2$. Then, K has a reciprocal UPE if and only if one of the following conditions holds.*

(I) K^+ admits a reciprocal UPE and 4 divides M_K.

(II) K^+ has an involution ρ such that there is an odd prime p dividing M_K with the property that ρ restricts to the identity on $\mathbb{Q}(\zeta_p) \cap K^+ = \mathbb{Q}(\zeta_p)^+$, where $\zeta_p^p = 1$ and $\zeta_p \neq 1$.

(III) $Q_K = 2$ and there is a typical unit for K/K^+ which is a reciprocal PE for K^+.

Proof. Let us first show that (I) implies (IV), where (IV) stands for the assertion: K has a reciprocal UPE. Note that $i \in K$ when 4 divides M_K. By Proposition 2.2, we can take a primitive element η of K^+, reciprocal under the involution ρ, such that η^2 is still primitive for K^+. Then, $\vartheta := i\eta \in K \setminus K^+$, ϑ is a unit, ϑ^2 is a PE for K^+, and so ϑ is a UPE for K. Moreover, ϑ is reciprocal corresponding to the extension of ρ that sends i to $-i$.

Next, we assume (II) and we show that (IV) holds. Note that $\zeta_p \in K$, where p is as in (II). Again by Proposition 2.2, we can take a UPE $\eta \in K^+$ under the involution ρ such that η^p is still primitive for K^+. Now, we claim that $\vartheta := \zeta_p\eta$ is a reciprocal UPE for K. It is primitive (the argument being similar to the previous paragraph, looking at ϑ^p), so the point is whether it is again reciprocal. The task is to extend ρ to an automorphism σ of K which takes ϑ to its inverse. Let τ' be the complex conjugation on the field $\mathbb{Q}(\zeta_p)$. Clearly, the intersection $\mathbb{Q}(\zeta_p) \cap K^+$ is $\mathbb{Q}(\zeta_p)^+$. Also, the involutions ρ and τ' both restrict to the identity on $\mathbb{Q}(\zeta_p) \cap K^+$: for τ', this is tautological, and for ρ, this is our assumption. Hence, these two involutions "glue" to give an involution σ on the compositum field $K = \mathbb{Q}(\zeta_p)K^+$, and by construction it inverts ϑ.

The proof of the implication (III) $\Rightarrow$ (IV) is similar to the second one in the proof of Theorem 3.3, since this did not require the condition $M_K = 2$.

Now, suppose that (IV) holds. If (I) is not true, either K^+ has no reciprocal UPE or 4 does not divide M_K. Since by assumption, K has a reciprocal UPE, by Proposition 3.1, K^+ has also a reciprocal UPE. Hence, 4 does not divide M_K, and so there is an odd natural number $m > 1$ such that $M_K = 2m$. To show that (II) or (III) must hold, consider the subgroup, say Z, of the cyclic group Ω_K, with order m.

Because $-1 \in O^*_{K^+}$, we see that $Q_K = [O^*_K : O^*_{K^+}Z]$. Hence, O^*_K is a direct product of $O^*_{K^+}$ and Z, when $Q_K = 1$, since $O^*_{K^+} \cap Z = \{1\}$. For $Q_K = 2$ consider an element $\beta_0 \in O^*_K$ which is not in $O^*_{K^+}Z$, but $\beta_0^2 \in O^*_{K^+}Z$. Then, there is $z \in Z$ such that $\beta_0^2 z^2 \in O^*_{K^+}$, as every element of the group Z is a square in Z. If $\beta := \beta_0 z$, then $\beta \in O^*_K$, $\beta^2 \in O^*_{K^+}$ and $\beta \notin O^*_{K^+}$. Thus, $K = K^+(\beta)$, $\beta^2 < 0$, since otherwise $\beta \in K \cap \mathbb{R} = K^+$, and so β^2 is a typical unit for the extension K/K^+ (recall that all conjugates of K are nonreal). Moreover, the equalities $O^*_K = O^*_{K^+}Z \cup \beta O^*_{K^+}Z = (O^*_{K^+} \cup \beta O^*_{K^+})Z$ and $(O^*_{K^+} \cup \beta O^*_{K^+}) \cap Z = \{1\}$ yield that O^*_K is a direct product of the groups $O^*_{K^+} \cup \beta O^*_{K^+}$ and Z.

Consequently, if we set $U := O^*_{K^+}$ when $Q_K = 1$, and $U := O^*_{K^+} \cup \beta O^*_{K^+}$ for $Q_K = 2$, then every unit of K can uniquely be written as $u\zeta$, with $u \in U$ and $\zeta \in Z$.

Now, assume that ϑ is a reciprocal UPE for K under the involution σ of K, and write $\vartheta = u\zeta$ as above. If $\zeta = 1$, then $\vartheta = u \in \beta O^*_{K^+}$, $Q_K = 2$ and so ϑ^2 is a typical unit for the extension K/K^+. Hence, (III) holds, as ϑ^2 is a reciprocal UPE for K^+. We may therefore assume that $\zeta \neq 1$, and show that (II) must hold.

Recall that the reciprocal UPE ϑ is written $u\zeta$ with $u \in U$, $\zeta \in Z$, and ζ is a root of unity whose order, say n, is an odd integer greater than 1. When we apply σ to $u\zeta$ we get from the uniqueness of this product decomposition, that both factors u and ζ must be reciprocal under σ. Let p be any prime divisor of n. Then, p divides m, ζ_p is a power of ζ, and ζ_p still reciprocal under σ; thus, σ acts as a complex conjugation on the field $\mathbb{Q}(\zeta_p) \subset K$. Finally, if ρ designates the restriction of σ to K^+, then $\rho = I_{\mathbb{Q}(\zeta_p)^+}$, since complex conjugation and ρ agree on $\mathbb{Q}(\zeta_p) \cap K^+ = \mathbb{Q}(\zeta_p)^+$, and so (II) holds. $\square$

There is a nice corollary: if M_K is divisible by at least one prime number $p \equiv 3 (\mathrm{mod}\, 4)$, then the degree of the field $\mathbb{Q}(\zeta_p)^+$ is odd, and it is automatic that every involution on a bigger field restricts to the identity on $\mathbb{Q}(\zeta_p)^+$; thus, the condition (II) in Theorem 3.4, say no more and no less that "K^+ has an involution". Hence, by Proposition 2.2, we have the following corollary.

Corollary 3.5. *Suppose that M_K is divisible by 4, or by a prime number congruent to 3 modulo 4. Then, K has a reciprocal UPE if and only if K^+ does.*

Consider, for instance, any biquadratic field $K = \mathbb{Q}(\sqrt{d}, \mathrm{i})$ (resp. $K = \mathbb{Q}(\sqrt{d}, \mathrm{e}^{\mathrm{i}2\pi/3})$), where d is a squarefree integer greater than 3. Then, K is CM, $K^+ = \mathbb{Q}(\sqrt{d})$ and K is noncyclotomic (recall that the only biquadratic cyclotomic fields are $\mathbb{Q}(\sqrt{2}, i) = \mathbb{Q}(\mathrm{e}^{\mathrm{i}2\pi/8})$ and $\mathbb{Q}(\sqrt{3}, \mathrm{i}) = \mathbb{Q}(\mathrm{e}^{\mathrm{i}2\pi/12})$). Furthermore, since the nontrivial subfields of K are $\mathbb{Q}(\sqrt{d})$, $\mathbb{Q}(i)$ and $\mathbb{Q}(\mathrm{i}\sqrt{d})$ (resp. are $\mathbb{Q}(\sqrt{d})$, $\mathbb{Q}(\mathrm{e}^{\mathrm{i}2\pi/3})$ and $\mathbb{Q}(\mathrm{e}^{\mathrm{i}2\pi/3}\sqrt{d})$), we see that $\Omega_K = \{\pm 1, \pm \mathrm{i}\}$ and $M_K = 4$ (resp. $\Omega_K = \{\pm 1, \pm \mathrm{e}^{\mathrm{i}2\pi/3}, \pm \mathrm{e}^{-\mathrm{i}2\pi/3}\}$ and $M_K = 6$). It follows, by Corollary 3.5 (and Proposition 2.2), that K admits a reciprocal UPE, as the automorphism of $\mathbb{Q}(\sqrt{d})$, sending $\sqrt{d}$ to $-\sqrt{d}$, is an involution of K^+.

We have been negligent in discussing possible overlaps between the three assertions (I), (II), and (III). It is clearly possible to insert the extra condition that 4 does not divide M_K into (II) to make (I) and (II) mutually exclusive. But the interplay with (III) is not clear yet. Also, it is worth noting that the converse of the second assertion in Proposition 3.1 is not always true, as illustrated by the following examples.

Example 1. Let L be the quartic cyclic field of conductor 17 (note that this field is totally real and is abelian), and let K be the CM field obtained by composing L with an arbitrary imaginary quadratic field F, so we have $L = K^+$. Via PARI, one checks that the signature map from the units of K^+ to the four-dimensional $\mathbb{F}_2$-vector space $\{\pm 1\}^4$ is onto, in other words, all 16 possible signatures at the four (real) places of K^+ occur, and this implies that every totally positive unit of K^+ is a square in K^+. If $M_K = 2$, then Theorem 1.5 in [7] implies, immediately, that K has no reciprocal UPE. For $M_K > 2$ (which only occurs for $F = \mathbb{Q}(\sqrt{d})$ with $d = -1, -3, -17, -3 \cdot 17$), we get the same result, arguing as follows. Case (III) in Theorem 3.4 cannot hold (one would get $K = K^+(\mathrm{i})$), and visibly neither (I) nor (II) is true; thus, by Theorem 3.4, K has no reciprocal UPE. Finally, because K^+ admits an involution, Proposition 2.2 gives that K^+ has a reciprocal UPE.

Example 2. Let F be a totally real cubic field whose normal closure has Galois group S_3 and is unramified at 2 and 5. Let N be the real quadratic field with conductor 5. Finally, let $K = F(\zeta_5)$. Then, K is CM, $K^+ = FN$, and K^+ admits an involution (take the nontrivial automorphism of N and extend it as the identity on F to K^+). It is also easy to see that this is the only involution on K^+ (the main reason being that F has no automorphism other than the identity). Hence, condition (II) in Theorem 3.4 fails. For reasons of ramification, (I) fails too. Finally, the equality $Q_K = 2$ cannot hold, since otherwise K would arise from K^+ by adjoining the square root of a unit, and this produces ramification only above 2. Therefore, (III) is not fulfilled either. Hence, K has no reciprocal UPE, but K^+ has, and in this time we are in the case $M_K > 2$ (in fact $M_K = 10$). Here is one numerical instance: F is the cubic field with discriminant 229, and generating the polynomial $x^3 - 4x - 1$.

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