Number theory

Geometric sequences and zero-free region of the zeta function

Suites géométriques et région sans zéro de la fonction zêta

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A R T I C L E   I N F O

Article history:
Received 18 August 2017
Accepted after revision 20 November 2017
Available online 5 January 2018
Presented by the Editorial Board

A B S T R A C T

Let \( \mathcal{N} \) be the linear space of functions \( \sum_{k=1}^{n} a_k \rho(\theta_k/x) \) with a condition \( \sum_{k=1}^{n} a_k \theta_k = 0 \) for \( 0 < \theta_k \leq 1 \). Here \( \rho(x) \) denotes the fractional part of \( x \). Beurling pointed out that the problem of how well a constant function can be approximated by functions in \( \mathcal{N} \) is closely related to the zero-free region of the Riemann zeta function. More precisely, Báez-Duarte gave a zero-free region related to a \( L^p \)-norm estimation of a constant function by using the Dirichlet series for the zeta function. In this paper, we consider the \( L^\infty \)-norm estimation of a constant function and give a wider zero-free region than that of the Báez-Duarte result.

R É S U M É

Soit \( \mathcal{N} \) l’espace vectoriel de fonctions \( \sum_{k=1}^{n} a_k \rho(\theta_k/x) \) satisfaisant la condition \( \sum_{k=1}^{n} a_k \theta_k = 0 \) pour \( 0 < \theta_k \leq 1 \), où \( \rho(x) \) désigne la partie fractionnaire de \( x \). Beurling a indiqué que le problème d’approximation d’une fonction constante par fonctions dans \( \mathcal{N} \) est étroitement lié à la région sans zéro de la fonction zêta de Riemann. Plus précisément, Báez-Duarte a donné une région sans zéro liée à une estimation de la norme \( L^p \) d’une fonction constante en utilisant les séries de Dirichlet pour la fonction zêta. Dans cet article, nous considérons une estimation de la norme \( L^\infty \) d’une fonction constante et donnons une région sans zéro plus large que celle du résultat de Báez-Duarte.

1. Introduction

Let \( \rho(x) \) be the fractional part of \( x \). The Nyman space \( \mathcal{N} \) consists of all functions of the form

\[
\sum_{k=1}^{n} a_k \rho \left( \frac{\theta_k}{x} \right)
\]

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https://doi.org/10.1016/j.crma.2017.11.021
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for any natural number $n$, which satisfies the condition $\sum_{k=1}^{n} a_k \theta_k = 0$ for $0 < \theta_k \leq 1$. In many approaches to solve the Riemann hypothesis, Beurling [5] and Nyman [8] found a connection between the existence of the nontrivial zeros of the Riemann zeta function and a density of a function space $\mathcal{N}$ in $L^p(0, 1)$. More precisely, the fact that $\mathcal{N}$ is dense in $L^p(0, 1)$ is equivalent to that the zeta function is zero-free on the Re $s > 1/p$. In his paper [5], Beurling also pointed out that the problem of how well a function $\chi$ can be approached by functions in $\mathcal{N}$ is closely related to the distribution of the primes even in case $\zeta$ has zeros close to the line Re $s = 1$. Here $\chi$ denotes the characteristic function on $(0, 1)$.

In [1], Báez-Duarte gave an explicit result about Beurling’s remark.

**Theorem 1.1.** If $f \in \mathcal{N}$, $1 < p \leq 2$, and $\epsilon = \|\chi - f\|_p$, then $\zeta$ does not vanish in the closed triangle with vertices at the points $(1/p, 1, 1 + (i/2)\epsilon^{-1})$.

Though the Theorem 1.1 gives each $f \in \mathcal{N}$ to a zero-free region for $\zeta$, the region is angled towards the line Re $s = 1$. In this paper we give a different zero-free region using $L^\infty$-norm.

We first introduce function spaces to work on. For $0 \leq \delta < 1$, we define $\mathcal{X}_\delta$ by

$\mathcal{X}_\delta := \{ f \in \overline{\mathcal{N}} : f(x) = 1 \text{ for } \delta < x < 1 \},$

where $\overline{\mathcal{N}}$ is the closure of $\mathcal{N}$ in $L^2(0, 1)$. Concrete functions in $\mathcal{X}_\delta$ are presented in Section 3.

The following is our main theorem.

**Theorem 1.2.** For $0 < \delta < 0.043$, suppose that $f \in \mathcal{X}_\delta$ and $\epsilon = \|\chi - f\|_\infty$. Then $\zeta(\sigma + it)$ does not vanish in a region given by

$$|t| < \frac{C}{\epsilon \delta^\sigma}$$

on the critical strip. Here $C = \pi/4 e^{2\pi}$.

As a consequence of Theorem 1.2, we see that the region

$$|t| < \frac{C}{\epsilon \delta}$$

is free from zero, which is more regular than Báez-Duarte’s result.

**2. Proof of the theorem**

For $f \in \mathcal{N}$ as

$$f(x) = \sum_{k=1}^{n} a_k \rho \left( \frac{\theta_k}{x} \right)$$

with a condition $\sum_{k=1}^{n} a_k \theta_k = 0$ for $0 < \theta_k \leq 1$, we get

$$\text{Re } f(x) = \sum_{k=1}^{n} \text{Re}(a_k) \rho \left( \frac{\theta_k}{x} \right) \quad \text{and} \quad \text{Im } f(x) = \sum_{k=1}^{n} \text{Im}(a_k) \rho \left( \frac{\theta_k}{x} \right).$$

Since $\sum_{k=1}^{n} \text{Re}(a_k) \theta_k = \sum_{k=1}^{n} \text{Im}(a_k) \theta_k = 0$, both Re $f$ and Im $f$ also belong to $\mathcal{N}$. So we may assume that $f$ is a real-valued function without loss of generality. Moreover note that

$$f(x) = 0 \quad \text{for } \max \theta_k \leq x.$$

Thus, a contraction operator $T_v$ defined by

$$T_v f(x) := \begin{cases} f(x/v), & 0 < x \leq v \\ 0, & v < x < 1 \end{cases}$$

for $0 < v < 1$, is closed on $\mathcal{N}$. As a result, $T_v$ is closed on $\overline{\mathcal{N}}$.

In [4], Bercovici and Foias obtained the following equivalent form for $\overline{\mathcal{N}}$ using the Mellin transform;

$$\overline{\mathcal{N}} = \left\{ f \in L^2(0, 1) : \frac{Mf(s)}{\zeta(s)} \text{ is analytic on } \text{Re } s > \frac{1}{2} \right\}. \quad (2.1)$$
Here $Mf$ is the Mellin transform defined by
\[
Mf(s) := \frac{1}{\sqrt{2\pi}} \int_0^1 f(x) x^{s-1} \, dx \]
for $f \in L^2(0, 1)$. By considering orthogonals in (2.1), Balazard and Saias pointed out that the Bercovici–Foias theorem gives a complete characterization for the complement space of $\mathcal{N}$ in $L^2(0, 1)$. More precisely, we have the following theorem.

**Theorem 2.1.** Let $\mathcal{N}^\perp$ be the orthogonal complement of $\mathcal{N}$ in $L^2(0, 1)$. Then we have
\[
\mathcal{N}^\perp = \text{span}\{x \rightarrow x^{s-1}\log^k x, \xi(s) = 0 \text{ with } \Re s > 1/2\},
\]
where $0 \leq k \leq \text{multiplicity of } s$.

See [2,3,11,12] for more results of $\mathcal{N}$ and $\mathcal{N}^\perp$. In (2.2), we put
\[
\varphi_3(x) := \text{Im}(x^{s-1}).
\]
Clearly we have
\[
\varphi_3(x) = x^{\sigma-1} \sin(t \log x),
\]
where $s = \sigma + it$. The graph of $\varphi_3$ rapidly oscillate near the origin. So $\varphi_3$ has infinitely many zeros near the origin. The zeros of $\varphi_3$ on $(0, 1]$, listed in decreasing order, are
\[
\rho_n := r^n \quad \text{with} \quad r := e^{-\pi/t}
\]
for $n = 0, 1, \ldots$. For each natural number $n$, the area of $\varphi_3$ on $[r_n, r_{n-1}]$ is given by
\[
A_n := \int_{r_n}^{r_{n-1}} |\varphi_3(x)| \, dx = \int_{r_n}^{r_{n-1}} |\text{Im}(x^{s-1})| \, dx
\]
\[
= \left| \text{Im} \left( \int_{r_n}^{r_{n-1}} x^{s-1} \, dx \right) \right| = \frac{t}{\sigma^2 + t^2} \frac{1 + r^\sigma}{r^\sigma} (\sigma)^n.
\]
Consequently, we obtain two geometric sequences $r_n$ and $A_n$ for each $\varphi_3$, which are crucial in the proof of our main theorem. The following lemma can be easily proved by elementary calculus.

**Lemma 2.2.** For $0 < x \leq 1$ we have
\[
\frac{e^{2\pi x}}{e^{\pi x} - 1} \leq \frac{c}{x},
\]
where $c = e^{2\pi}/\pi$.

Now we prove our main result.

**Proof.** Let $f \in \mathcal{X}$ for a sufficiently small $\delta > 0$ and $\epsilon = \|X - f\|_\infty$. We borrow the well-known fact that $\xi(s) \neq 0$ with $|t| < 1$ in the critical strip. Assume that there is a zero $s_0 = \sigma_0 + it_0$, with
\[
1 < t_0 < \frac{C}{\epsilon \delta^{\sigma_0}} \quad \text{and} \quad \sigma_0 > 1/2,
\]
where $C = \pi/4 e^{2\pi}$. We will complete the proof by deriving a contradiction.

Let $\rho_n$ and $A_n$ be the geometric sequences corresponding to $\varphi_{\rho_n}$. From $t_0 > 1$, we have
\[
r = e^{-\pi/t_0} > e^{-\pi} \approx 0.043,
\]
where $r$ is defined in (2.3) for $\varphi_{\rho_n}$. So we can choose the positive integer $N$ such that
\[
r^N \leq \delta < r^{N-1}.
\]
Then we consider a function \( f - T_r f \). From \( f \in \mathcal{X}_\delta \), we get
\[
(f - T_r f)(x) = \begin{cases} 
1, & r < x < 1 \\
0, & \delta < x \leq r \\
\text{absolute value} \leq 2\epsilon, & 0 < x \leq \delta.
\end{cases}
\]

By Theorem 2.1, we have
\[
0 = \int_0^1 (f - T_r f) \cdot \varphi_{\delta_0} = \int_0^\delta (f - T_r f) \cdot \varphi_{\delta_0} + \int_{\delta}^1 \varphi_{\delta_0}.
\]

Thus we get
\[
A_1 = \left| \int_0^\delta (f - T_r f) \cdot \varphi_{\delta_0} \right| \leq 2\epsilon \cdot \int_0^{r^{N-1}} |\varphi_{\delta_0}| = 2\epsilon \cdot \sum_{n=N}^\infty A_n.
\]

Moreover, we have
\[
1 = \frac{2\epsilon \cdot \sum_{n=N}^\infty A_n}{A_1} \leq \frac{2\epsilon r^{\sigma_0 N}}{r^{\sigma_0}(1 - r^{\sigma_0})} \leq \frac{2\epsilon \delta^{\sigma_0}}{r^{\sigma_0}(1 - r^{\sigma_0})}.
\]

the last inequality holds by (2.4). By Lemma 2.2,
\[
\frac{1}{r^{\sigma_0}(1 - r^{\sigma_0})} = \frac{e^{2\pi r^{\sigma_0}/t_0}}{e^{2\pi r^{\sigma_0}/t_0} - 1} \leq \frac{e^{2\pi}}{\pi} \cdot \frac{t_0}{\sigma_0} = \frac{e^{2\pi}}{\pi} \cdot t_0
\]

Consequently, we obtain that
\[
1 \leq \frac{2\epsilon \delta^{\sigma_0}}{r^{\sigma_0}(1 - r^{\sigma_0})} \leq \frac{4e^{2\pi}}{\pi} \cdot \epsilon \delta^{\sigma_0} t_0 < 1,
\]
which is impossible. Thus we finish the proof. \( \square \)

3. Remark and question

For an example function in \( \mathcal{X}_\delta \), we define the natural approximation \( f_n \) by
\[
f_n(x) = n g(n) \rho \left( \frac{1}{n x} \right) - \sum_{k=1}^n \mu(k) \rho \left( \frac{1}{k x} \right), \quad \text{where} \quad g(n) := \sum_{k=1}^n \frac{\mu(k)}{k}.
\]

Here \( \mu \) denotes the Möbius function. Then, the fact that \( f_n \) belongs to \( \mathcal{X}_{1/n} \) follows from the well-known one:
\[
\sum_{k=1}^\infty \mu(k) \left[ \frac{1}{k x} \right] = 1 \quad \text{for} \quad 0 < x \leq 1.
\]

See [1] for more results. As a summatory function of \( \mu \), let
\[
M(n) := \sum_{k=1}^n \mu(k).
\]

The properties of the functions \( \mu \) and \( M \) are central in the theory of prime numbers. There is an exhaustive list of results of \( \mu, M \). We refer the reader to [6,7,9,10] for related work.

The oscillating property of \( M \) is known by Pintz, see [10]. More precisely, \( M \) changes signs infinitely many times. In case of \( g \), it is known that
\[
\lim_{n \to \infty} g(n) = 0.
\]

However, the oscillating property of \( g \) is not known yet. If \( g \) also changes signs infinitely often, then we obtain
\[
|g(n)| \leq 1/n
\]
for infinitely many $n$’s. As a result, we have

$$\|ng(n)\rho\left(\frac{1}{nx}\right)\|_{\infty} \leq 1, \text{ for infinitely many } n \text{'s.}$$

So we only need to consider the second term of $f_n$ for $\|X - f_n\|_{\infty}$ on $(0, 1/n)$. Thus the following is an interesting question.

**Question 3.1.** Does the sequence

$$g(n) = \sum_{k=1}^{n} \frac{\mu(k)}{k}$$

has infinitely many sign-changing solutions?

**Acknowledgement**

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2017R1A6A3A11035180). The author would like to express his sincere thanks to the referee for careful reading of the manuscript and helpful comments.

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