Mathematical analysis/Complex analysis

Improved version of Bohr’s inequality

Version améliorée de l’inégalité de Bohr

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1. Introduction and main results

The classical theorem of Bohr [3] (after subsequent improvements due to M. Riesz, I. Schur and F. Wiener) states that if \( f \) is a bounded analytic function on the unit disk \( D := \{ z \in \mathbb{C} : |z| < 1 \} \), with the Taylor expansion \( \sum_{k=0}^{\infty} a_k z^k \), and \( \| f \|_\infty := \sup_{z \in D} |f(z)| < \infty \), then

\[
M_f(r) := \sum_{n=0}^{\infty} |a_n|r^n \leq \| f \|_\infty \quad \text{for } 0 \leq r \leq 1/3
\]

and the constant \( 1/3 \) is sharp. There are a number of articles that deal with Bohr’s phenomenon. See, for example, [2,10], the recent survey on this topic by Abu-Muhanna et al. [1] and the references therein. Bombieri [4] considered the function \( m(r) \) defined by \( m(r) = \sup \left[ M_f(r)/\| f \|_\infty \right] \), where the supremum is taken over all nonzero bounded analytic functions, and proved that

\[
m(r) = \frac{3 - \sqrt{8(1 - r^2)}}{r} \quad \text{for } 1/3 \leq r \leq 1/\sqrt{2}.
\]
Later Bombieri and Bourgain [5] studied the behaviour of \( m(r) \) as \( r \to 1 \) (see also [6]) and proved the following result, which validated a question raised in [11, Remark 1] in the affirmative.

**Theorem A.** ([5, Theorem 1]) If \( r > 1/\sqrt{2} \), then \( m(r) < 1/\sqrt{1-r^2} \). With \( \alpha = 1/\sqrt{2} \), the function \( \varphi_\alpha(z) = (\alpha - z)/(1 - \alpha z) \) is extremal, giving \( m(1/\sqrt{2}) = \sqrt{2} \).

A lower estimate for \( m(r) \) as \( r \to 1 \) is also obtained in [5, Theorem 2]. We are now ready to state several different improved versions of the classical Bohr inequality (1).

**Theorem 1.** Suppose that \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) is analytic in \( \mathbb{D} \), \( |f(z)| \leq 1 \) in \( \mathbb{D} \), and \( S_r \) denotes the area of the Riemann surface of the function \( f^{-1} \) defined on the image of the subdisk \( |z| < r \) under the mapping \( f \). Then

\[
B_1(r) := \sum_{k=0}^{\infty} |a_k| r^k + \frac{16}{9} \left( \frac{S_r}{\pi} \right) \leq 1 \text{ for } r \leq \frac{1}{3}
\]

and the numbers \( 1/3 \) and \( 16/9 \) cannot be improved. Moreover,

\[
B_2(r) := |a_0|^2 + \sum_{k=1}^{\infty} |a_k| r^k + \frac{9}{8} \left( \frac{S_r}{\pi} \right) \leq 1 \text{ for } r \leq \frac{1}{2}
\]

and the constants \( 1/2 \) and \( 9/8 \) cannot be improved.

**Remark 1.** Let us remark that if \( f \) is a univalent function then \( S_r \) is the area of the image of the subdisk \( |z| < r \) under the mapping \( f \). In the case of multivalent function, \( S_r \) is greater than the area of the image of the subdisk \( |z| < r \). This fact could be shown by noting that

\[
S_r = \int_{f(\mathbb{D}_r)} |f'(z)|^2 \, dA(w) = \int_{f(\mathbb{D}_r)} \nu_f(w) \, dA(w) \geq \int_{f(\mathbb{D}_r)} dA(w) = \text{Area}(f(\mathbb{D}_r)),
\]

where \( \mathbb{D}_r = \{ z \in \mathbb{C} : |z| < r \} \) and \( \nu_f(w) = \sum_{f(z)=w} 1 \) denotes the counting function of \( f \).

**Theorem 2.** Suppose that \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) is analytic in \( \mathbb{D} \) and \( |f(z)| \leq 1 \) in \( \mathbb{D} \). Then

\[
|a_0| + \sum_{k=1}^{\infty} \left( |a_k| + \frac{1}{2} |a_k|^2 \right) r^k \leq 1 \text{ for } r \leq \frac{1}{3}
\]

and the numbers \( 1/3 \) and \( 1/2 \) cannot be improved.

**Theorem 3.** Suppose that \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) is analytic in \( \mathbb{D} \) and \( |f(z)| \leq 1 \) in \( \mathbb{D} \). Then

\[
\sum_{k=0}^{\infty} |a_k| r^k + |f(z) - a_0| \leq 1 \text{ for } r \leq \frac{1}{3}
\]

and the number \( 1/3 \) cannot be improved.

Finally, we also prove the following sharp inequality.

**Theorem 4.** Suppose that \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) is analytic in \( \mathbb{D} \) and \( |f(z)| \leq 1 \) in \( \mathbb{D} \). Then

\[
|f(z)|^2 + \sum_{k=1}^{\infty} |a_k|^2 r^{2k} \leq 1 \text{ for } r \leq \sqrt{\frac{11}{27}} = 0.63828 \ldots
\]

and this number cannot be improved.
2. Proofs of Theorems 1, 2, 3 and 4

If \( f \) and \( g \) are analytic in \( \mathbb{D} \), then \( g \) is subordinate to \( f \), written \( g < f \) or \( g(z) < f(z) \), if there exists a function \( \omega \) analytic in \( \mathbb{D} \) satisfying \( \omega(0) = 0 \), \( |\omega(z)| < 1 \) and \( g(z) = f(\omega(z)) \) for \( z \in \mathbb{D} \). If \( f \) is univalent in \( \mathbb{D} \), then \( g < f \) if and only if \( g(0) = f(0) \) and \( g(\mathbb{D}) \subset f(\mathbb{D}) \) (see [7, p. 190 and p. 253] and [1,8]).

For the proof of Theorem 1, we need the following lemma, especially when \( 0 < r \leq 1/2 \).

**Lemma 1.** Let \( |b_0| < 1 \) and \( 0 < r \leq 1/\sqrt{2} \). If \( g(z) = \sum_{k=0}^{\infty} b_k z^k \) is analytic and satisfies the inequality \( |g(z)| < 1 \) in \( \mathbb{D} \), then the following sharp inequality holds:

\[
\sum_{k=1}^{\infty} k|b_k|^2 r^{2k} \leq r^2 \frac{(1 - |b_0|^2)^2}{(1 - |b_0|^2 r^2)^2}.
\]

**Proof.** Let \( b_0 = a \). Then, it is easy to see that the condition on \( g \) can be rewritten in terms of subordination as

\[
g(z) = \sum_{k=0}^{\infty} b_k z^k \preceq \varphi_a(z) = a - \frac{a - z}{1 - az} = a - (1 - |a|^2) \sum_{k=1}^{\infty} (\overline{a})^{k-1} z^k, \quad z \in \mathbb{D},
\]

where \( \preceq \) denotes the subordination. Note that \( \varphi_a \) is analytic in \( \mathbb{D} \) and \( |\varphi_a(z)| < 1 \) for \( z \in \mathbb{D} \). The subordination relation (6) gives

\[
\sum_{k=1}^{\infty} k|b_k|^2 r^{2k} \leq (1 - |a|^2)^2 \sum_{k=1}^{\infty} k|a|^{2(k-1)} r^{2k} = r^2 \frac{(1 - |a|^2)^2}{(1 - |a|^2 r^2)^2}
\]

from which we arrive at the inequality (5), which proves Lemma 1. For \( 0 < r \leq 1/\sqrt{2} \), it is important to note here that the sequence \( \{kr^{2k}\} \) is non-increasing for all \( k \geq 1 \), so that we were able to apply the classical Goluzin’s inequality [8] (see also [7, Theorem 6.3]), which extends the classical Rogosinski inequality. \( \square \)

**Proof of Theorem 1.** Since the left-hand side of (2) is an increasing function of \( r \), it is enough to prove it for \( r = 1/3 \). Therefore, we set \( r = 1/3 \). Moreover, the present authors in the proof of Theorem 1 in [9] proved the following inequalities:

\[
\sum_{k=1}^{\infty} |a_k|r^k \leq \begin{cases} 
A(r) := \frac{r(1 - |a|^2)}{1 - r|a|} & \text{for } |a| \geq r \\
B(r) := \frac{r^2(1 - |a|^2)}{\sqrt{1 - r^2}} & \text{for } |a| < r.
\end{cases}
\]

(7)

Note that \( |a_k| \leq 1 - |a|^2 \) for \( k \geq 1 \) and, from the definition of \( S_r \), we see that

\[
S_r = \frac{1}{\pi} \int_{|z|<r} |f'(z)|^2 \, dx \, dy = \sum_{k=1}^{\infty} k|a_k|^2 r^{2k}
\]

\[
\leq (1 - |a|^2)^2 \sum_{k=1}^{\infty} kr^{2k} = (1 - |a|^2)^2 \frac{r^2}{(1 - r^2)^2}.
\]

(8)

At first, we consider the case \( |a| \geq r = 1/3 \). In this case, using (7) and (8), we have

\[
B_1(r) = |a_0| + \sum_{k=1}^{\infty} |a_k|r^k + \frac{16}{9\pi} S_r \leq |a_0| + A(1/3) + \frac{16}{9\pi} S_{1/3}
\]

\[
\leq |a_0| + \frac{1}{3 - |a|} + \frac{(1 - |a|^2)^2}{4} = 1 - \frac{(1 - |a|^2)^3 (5 - |a|^2)}{4(3 - |a|)} \leq 1.
\]

Next we consider the case \( |a| < r = 1/3 \). Again, using (7) and (8), we deduce that

\[
B_1(r) = \sum_{k=0}^{\infty} |a_k|r^k + \frac{16}{9\pi} S_r \leq |a_0| + B(1/3) + \frac{16}{9\pi} S_{1/3}
\]
\[ \leq |a_0| + \frac{\sqrt{1 - |a_0|^2}}{\sqrt{8}} + \frac{(1 - |a_0|^2)^2}{4} \]
\[ \leq \frac{1}{3} + \frac{1}{\sqrt{8}} + \frac{1}{4} < 1 \quad \text{(since } |a_0| < 1/3) \]

and the desired inequality (2) follows.

To prove that the constant 16/(9π) is sharp, we consider the function \( f = \varphi_a \) given by

\[ \varphi_a(z) = \frac{a - z}{1 - az} = a - (1 - a^2) \sum_{k=1}^{\infty} a^{k-1} z^k, \quad z \in \mathbb{D}, \]

where \( a \in (0, 1) \). For this function, straightforward calculations show that

\[ \sum_{k=0}^{\infty} |a_k|^k + \frac{\lambda}{\pi} S_\pi = a + r \frac{1 - a^2}{1 - ra} + \lambda(1 - a^2)^2 \frac{r^2}{(1 - a^2 r^2)^2}; \]

In the case \( r = 1/3 \), the last expression becomes

\[ a + \frac{1 - a^2}{3 - a} + 9 \lambda \frac{(1 - a^2)^2}{(9 - a^2)^2} = 1 - \frac{2(1 - a)^3(19 + 12a + a^2)}{2(9 - a^2)^2} + \frac{9(1 - a^2)^2}{(9 - a^2)^2}. \]

which is obviously bigger than 1 in case \( \lambda > 16/9 \) and \( a \rightarrow 1 \). The proof of the first part of Theorem 1 is complete.

Let us now verify the inequality (3). To do it we will use the method presented above and Lemma 1 for \( r \leq 1/2 \). From Lemma 1, it follows that

\[ \frac{S_r}{\pi} \leq (1 - |a_0|^2)^2 \frac{r^2}{(1 - |a_0|^2 r^2)^2}, \quad r \leq 1/2. \]

Let \( r \leq 1/2 \) and we first consider the case \( |a_0| \geq 1/2 \). Then, using (7) and (9), we obtain that

\[ B_2(r) = |a_0|^2 + \sum_{k=1}^{\infty} |a_k|^k + \frac{9}{8\pi} S_r \leq |a_0|^2 + A(1/2) + \frac{9}{8\pi} S_{1/2} \]
\[ \leq |a_0|^2 + \frac{1 - |a_0|^2}{2 - |a_0|^2} + \frac{4(1 - |a_0|^2)^2}{4 - |a_0|^2} \]
\[ = 1 - \frac{(1 - |a_0|^2)^2(1 + |a_0|)(7 + 6|a_0| + 2|a_0|^2)}{2(4 - |a_0|^2)^2} \leq 1. \]

Now we consider the case \( |a_0| < 1/2 \). In this case, using (7) and (9), we have

\[ B_2(r) \leq |a_0|^2 + B(1/2) + \frac{9}{8\pi} S_{1/2} \]
\[ \leq |a_0|^2 + \frac{\sqrt{1 - |a_0|^2}}{\sqrt{3}} + \frac{4(1 - |a_0|^2)^2}{4 - |a_0|^2} \]
\[ \leq \frac{1}{\sqrt{3}} + |a_0|^2 + \frac{4(1 - |a_0|^2)^2}{4 - |a_0|^2} \]
\[ \leq \frac{1}{\sqrt{3}} + \frac{41}{100} - \frac{(1 - 4|a_0|^2)(256 - 104|a_0|^2 + 25|a_0|^4)}{100(|a_0|^2 - 4)^2} \]

which is less than 1. The sharpness of the constant 9/8 can be established as in the previous case and thus, we omit the details. The proof of the theorem is complete. \( \square \)

**Proof of Theorem 2.** Let \( A(r) \) and \( B(r) \) be defined as in (7). Furthermore, the present authors in [9] demonstrated the following inequality for the coefficients of \( f \):

\[ \sum_{k=1}^{\infty} |a_k|^k r^k \leq \frac{r(1 - |a_0|^2)^2}{1 - |a_0|^2 r}. \]  \( \text{(10)} \)

Also, it is worth pointing out that the inequality (10) for \( 0 < r \leq 1/\sqrt{2} \) follows from (5) by integrating it. As remarked in the proof of earlier theorems, it suffices to prove the inequality (4) for \( r = 1/3 \), and thus we may set \( r = 1/3 \) in the proof below. At first, we consider the case \( |a_0| \geq 1/3 \) so that, by (7) and (10),
\[
\sum_{k=0}^{\infty} |a_k|r^k + \frac{1}{2} \sum_{k=1}^{\infty} |a_k|^2 r^k \leq |a_0| + A(1/3) + \frac{(1 - |a_0|^2)^2}{6 - 2|a_0|^2} \\
= |a_0| + \frac{1 - |a_0|^2}{3 - |a_0|} + \frac{(1 - |a_0|^2)^2}{6 - 2|a_0|^2} \\
= 1 - \frac{(1 - |a_0|^2)^2}{2} \leq 1 \quad \text{(since } |a_0| \leq 1). 
\]

Similarly, for the case \(|a_0| < 1/3\), we have, by (7) and (10),
\[
\sum_{k=0}^{\infty} |a_k|r^k + \frac{1}{2} \sum_{k=1}^{\infty} |a_k|^2 r^k \leq |a_0| + B(1/3) + \frac{(1 - |a_0|^2)^2}{6 - 2|a_0|^2} \\
\leq |a_0| + \frac{\sqrt{1 - |a_0|^2}}{\sqrt{8}} + \frac{(1 - |a_0|^2)^2}{6 - 2|a_0|^2} \\
\leq \frac{1}{3} + \frac{1}{\sqrt{8}} + \frac{1}{6} < 1, 
\]
which concludes the proof of Theorem 2 since the proof of sharpness follows similarly. \(\square\)

**Proof of Theorem 3.** Let \(A(r)\) and \(B(r)\) be defined as in (7). Also, we may let \(r = 1/3\). Accordingly, we first consider the case \(|a_0| \geq 1/3\), so that
\[
\sum_{k=0}^{\infty} |a_k|r^k + |f(z) - a_0|^2 \leq |a_0| + A(1/3) + A(1/3)^2 \\
= |a_0| + \frac{1 - |a_0|^2}{3 - |a_0|} + \frac{(1 - |a_0|^2)^2}{(3 - |a_0|)^2} \\
= 1 - \frac{(1 - |a_0|)^3(5 + |a_0|)}{(3 - |a_0|)^2} \leq 1 \quad \text{(since } |a_0| \leq 1). 
\]

Next, we consider the case \(|a_0| < 1/3\) so that
\[
\sum_{k=0}^{\infty} |a_k|r^k + |f(z) - a_0|^2 \leq |a_0| + B(1/3) + B(1/3)^2 \\
= |a_0| + \frac{\sqrt{1 - |a_0|^2}}{\sqrt{8}} + \frac{1 - |a_0|^2}{8} \\
\leq \frac{1}{3} + \frac{1}{\sqrt{8}} + \frac{1}{8} < 1. 
\]
This concludes the proof of Theorem 2 and the sharpness follows similarly. \(\square\)

**Proof of Theorem 4.** Using (10) (see [9, Lemma 1]) and the classical inequality for \(|f(z)|\), we have
\[
|f(z)|^2 + \sum_{k=1}^{\infty} |a_k|^2 r^{2k} \leq \left(\frac{r + |a_0|}{1 + r|a_0|}\right)^2 + \frac{r^2(1 - |a_0|^2)^2}{1 - |a_0|^2r^2}. 
\]

For \(r = \sqrt{17/27}\), the last expression on the right gives
\[
1 - \frac{3(1 - |a_0|^2)}{(9 + \sqrt{33}|a_0|)^2(27 - 11|a_0|^2)}(135 - 66\sqrt{33}|a_0| + 66\sqrt{33}|a_0|^3 + 121|a_0|^4), 
\]
and straightforward calculations show that this expression is less than or equal to 1 for all \(|a_0| \leq 1\). The example
\[
f(z) = \frac{z + a}{1 + az} 
\]
with \(a = \sqrt{3/11}\) shows that \(r = \sqrt{17/27}\) is sharp. This completes the proof. \(\square\)
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