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# Semi-classical limit of the Levy–Lieb functional in Density Functional Theory

*Limite semi-classique de la fonctionnelle de Levy–Lieb en théorie de la fonctionnelle de densité*

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## ABSTRACT

In a recent work, Bindini and De Pascale have introduced a regularization of  $N$ -particle symmetric probabilities that preserves their one-particle marginals. In this short note, we extend their construction to mixed quantum fermionic states. This enables us to prove the convergence of the Levy–Lieb functional in Density Functional Theory, to the corresponding multi-marginal optimal transport in the semi-classical limit. Our result holds for mixed states of any particle number  $N$ , with or without spin.

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## R É S U M É

Dans un travail récent, Bindini et de Pascale ont introduit une régularisation des probabilités symétriques décrivant  $N$  particules indiscernables, qui préserve la densité à une particule. Nous étendons ici leur construction aux états quantiques mixtes de fermions. Ceci nous permet de démontrer la convergence de la fonctionnelle de Levy–Lieb, objet central de la théorie de la fonctionnelle de densité (DFT), vers le transport optimal multi-marges associé, à la limite semi-classique. Notre résultat est valable pour les états mixtes de n'importe quel nombre de particules  $N$ , avec ou sans spin.

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## 1. Extending the Bindini–De Pascale construction

Let  $\mathbb{P}$  be a symmetric  $N$ -particle probability measure over  $(\mathbb{R}^d)^N$  and let

$$\rho_{\mathbb{P}}(x_1) = \int_{(\mathbb{R}^d)^{N-1}} d\mathbb{P}(x_1, x_2, \dots, x_N)$$

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be its one-particle marginal. We assume that  $\sqrt{\rho_{\mathbb{P}}} \in H^1(\mathbb{R}^d)$ , as is appropriate in Density Functional Theory (DFT) [11,14]. An interesting question, important for applications in DFT, is to approximate  $\mathbb{P}$  by a regular  $N$ -particle probability density  $\mathbb{P}_\varepsilon$  with the same density  $\rho_{\mathbb{P}_\varepsilon} = \rho_{\mathbb{P}}$ . A more challenging problem, considered first in [6,1] for  $N = 2, 3$  and solved for all  $N \geq 2$  in this note, is to find a fermionic quantum state  $\Gamma_\varepsilon$  with the same density  $\rho_{\Gamma_\varepsilon} = \rho_{\mathbb{P}}$  and a controlled kinetic energy.

Consider a radial function  $\chi \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$  with support in the unit ball of  $\mathbb{R}^d$ , such that  $\int_{\mathbb{R}^d} \chi^2 = 1$ , and denote  $\chi_\varepsilon(x) = \varepsilon^{-d/2} \chi(x/\varepsilon)$ . In [1], Bindini and De Pascale have introduced the following elegant regularization

$$\mathbb{P}_\varepsilon(x_1, \dots, x_N) = \iint_{\mathbb{R}^{2dN}} \prod_{k=1}^N \frac{\rho_{\mathbb{P}}(x_k) \chi_\varepsilon(x_k - z_k)^2 \chi_\varepsilon(z_k - y_k)^2}{\rho_{\mathbb{P}} * \chi_\varepsilon^2(z_k)} d\mathbb{P}(y_1, \dots, y_N) dz_1 \cdots dz_N. \tag{1.1}$$

We assume in the following that  $\mathbb{P}$  has its support in

$$D_\alpha^c := \left\{ X = (x_1, \dots, x_N) \in (\mathbb{R}^d)^N : |x_i - x_j| \geq \alpha, \forall i \neq j \right\},$$

for some  $\alpha > 0$ , a condition which is satisfied for minimizers of the Coulomb  $N$ -particle energy [2]. Since  $\chi$  is supported in the unit ball,  $\mathbb{P}_\varepsilon$  is then supported on the set  $D_{\alpha-4\varepsilon}^c$ , where all the particles are at a distance  $\alpha - 4\varepsilon$  to each other. In the following, we always assume that  $\varepsilon < \alpha/4$ . The purpose of the integration over the  $z_k$ 's in (1.1) is to regularize the probability  $\mathbb{P}$ , since the convolution

$$\mathbb{Q}_\varepsilon(y_1, \dots, y_N) = \int_{(\mathbb{R}^d)^N} \prod_{k=1}^N \chi_\varepsilon(z_k - y_k)^2 d\mathbb{P}(y_1, \dots, y_N)$$

is now  $C^\infty$ . However, its density is  $\rho_{\mathbb{Q}_\varepsilon} = \rho_{\mathbb{P}} * \chi^2$ , and the purpose of the integration over the  $y_k$ 's is to map back the density to  $\rho_{\mathbb{P}}$ . Indeed, integrating (1.1) over  $x_2, \dots, x_N$ , we get  $\prod_{k=2}^N \rho_{\mathbb{P}} * \chi_\varepsilon^2(y_k)$  in the numerator, which cancels with the denominator. The corresponding integrals over  $z_2, \dots, z_N$  give  $(\int \chi_\varepsilon^2)^{N-1} = 1$  and we end up with

$$\rho_{\mathbb{P}_\varepsilon}(x_1) = \rho_{\mathbb{P}}(x_1) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\chi_\varepsilon(x_1 - z_1)^2 \chi_\varepsilon(z_1 - y_1)^2}{\rho_{\mathbb{P}} * \chi_\varepsilon^2(z_1)} \rho_{\mathbb{P}}(y_1) dy_1 dz_1 = \rho_{\mathbb{P}}(x_1).$$

A somewhat different method was introduced in [6] by Cotar, Friesecke and Kluppelberg.

In [1], Bindini and De Pascale prove that

$$\int_{\mathbb{R}^{dN}} |\nabla \sqrt{\mathbb{P}_\varepsilon}|^2 \leq N \left( \int_{\mathbb{R}^d} |\nabla \sqrt{\rho_{\mathbb{P}}}|^2 + \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} |\nabla \chi|^2 \right) \tag{1.2}$$

and use this to get some information on the semi-classical limit of the Levy–Lieb functional (to be discussed later in Section 2). Unfortunately, the probability (1.1) is really a classical object. Although for bosons one can use the symmetric wavefunction  $\Psi_\varepsilon = \sqrt{\mathbb{P}_\varepsilon}$ , for fermions the wavefunction must be anti-symmetric. In [6,1], the authors use the spin variable to antisymmetrize  $\sqrt{\mathbb{P}_\varepsilon}$  but, in dimension  $d = 3$ , this has so far limited the results to  $N = 2$  and  $N = 3$ . Note that in dimension  $d = 1$  and  $d = 2$ , one can use the multiplication by

$$\prod_{1 \leq j < k \leq N} \frac{x_j - x_k}{|x_j - x_k|}$$

which maps bosons onto fermions and conversely, whatever the value of  $N$  (if  $d = 2$  we identify  $\mathbb{R}^2$  with  $\mathbb{C}$ ). By the Borsuk–Ulam theorem, there is no such simple map in three space dimensions.

In this short note, we propose the following simple quantum extension of (1.1), for fermions:

$$\begin{aligned} \Gamma_\varepsilon = & \iint_{\mathbb{R}^{dN} \times \mathbb{R}^{dN}} \sqrt{\rho_{\mathbb{P}}^{\otimes N}} |\chi_{\varepsilon, z_1} \wedge \cdots \wedge \chi_{\varepsilon, z_N}\rangle \langle \chi_{\varepsilon, z_1} \wedge \cdots \wedge \chi_{\varepsilon, z_N}| \sqrt{\rho_{\mathbb{P}}^{\otimes N}} \\ & \times \prod_{k=1}^N \frac{\chi_\varepsilon(z_k - y_k)^2}{\rho_{\mathbb{P}} * \chi_\varepsilon^2(z_k)} d\mathbb{P}(y_1, \dots, y_N) dz_1 \cdots dz_N. \end{aligned} \tag{1.3}$$

Here  $\Gamma_\varepsilon$  is a non-negative self-adjoint operator that acts on the  $N$ -particle fermionic space  $L_a^2((\mathbb{R}^d)^N) = \bigwedge_1^N L^2(\mathbb{R}^d)$ . For simplicity, we do not consider spin here. It is easy to extend the trial state (1.3) to the case of particles with  $q$  spin states, for instance by putting all the particles in the same spin state. One can also construct any desired eigenvector of the total spin operator by adding an appropriate spin state to each function  $\chi_{\varepsilon, z_k}$  in (1.3).

In (1.3), we use the notation  $\chi_{\varepsilon,z}(x) = \chi_\varepsilon(x - z)$  and recall that  $\varepsilon < \alpha/4$  such that the functions  $\chi_{\varepsilon,z_1}, \dots, \chi_{\varepsilon,z_N}$  have disjoint supports. We use the ket-bra notation  $|\Psi\rangle\langle\Psi|$  for the orthogonal projection on  $\Psi$ , defined by  $|\Psi\rangle\langle\Psi|\varphi = \langle\Psi, \varphi\rangle\Psi$ . We call

$$\varphi_1 \wedge \dots \wedge \varphi_N(x_1, \dots, x_N) = \frac{1}{\sqrt{N!}} \sum_{\sigma \in \mathfrak{S}_N} \varepsilon(\sigma) \varphi_{\sigma(1)}(x_1) \dots \varphi_{\sigma(N)}(x_N)$$

the Slater determinant. Finally, the  $N$ -fold tensor product is defined by

$$\sqrt{\rho_{\mathbb{P}}^{\otimes N}}(x_1, \dots, x_N) = \prod_{j=1}^N \sqrt{\rho_{\mathbb{P}}}(x_j).$$

In (1.3),  $\sqrt{\rho_{\mathbb{P}}^{\otimes N}}$  is understood as a multiplication operator on  $L^2_a((\mathbb{R}^d)^N)$ . In particular, we have

$$\begin{aligned} \sqrt{\rho_{\mathbb{P}}^{\otimes N}} |\chi_{\varepsilon,z_1} \wedge \dots \wedge \chi_{\varepsilon,z_N}\rangle \langle \chi_{\varepsilon,z_1} \wedge \dots \wedge \chi_{\varepsilon,z_N} | \sqrt{\rho_{\mathbb{P}}^{\otimes N}} \\ = |\sqrt{\rho_{\mathbb{P}}}\chi_{\varepsilon,z_1} \wedge \dots \wedge \sqrt{\rho_{\mathbb{P}}}\chi_{\varepsilon,z_N}\rangle \langle \sqrt{\rho_{\mathbb{P}}}\chi_{\varepsilon,z_1} \wedge \dots \wedge \sqrt{\rho_{\mathbb{P}}}\chi_{\varepsilon,z_N} |. \end{aligned}$$

The integral kernel of  $\Gamma_\varepsilon$  is given by

$$\begin{aligned} \Gamma_\varepsilon(x_1, \dots, x_N; x'_1, \dots, x'_N) &= \frac{1}{N!} \iint_{\mathbb{R}^{dN} \times \mathbb{R}^{dN}} \det(\chi_{\varepsilon,z_i}(x_j)) \det(\chi_{\varepsilon,z_i}(x'_j)) \\ &\quad \times \prod_{k=1}^N \frac{\sqrt{\rho_{\mathbb{P}}}(x_k) \sqrt{\rho_{\mathbb{P}}}(x'_k) \chi(z_k - y_k)^2}{\rho_{\mathbb{P}} * \chi^2(z_k)} d\mathbb{P}(y_1, \dots, y_N) dz_1 \dots dz_N. \end{aligned} \tag{1.4}$$

Since the  $\chi_{\varepsilon,z_k}$  have disjoint supports, we have

$$\left( \det(\chi_{\varepsilon,z_i}(x_j)) \right)^2 = \sum_{\sigma, \sigma' \in \mathfrak{S}_N} \varepsilon(\sigma) \varepsilon(\sigma') \prod_{k=1}^N \chi_{\varepsilon,z_{\sigma(k)}}(x_k) \chi_{\varepsilon,z_{\sigma'(k)}}(x_k) = \sum_{\sigma \in \mathfrak{S}_N} \prod_{k=1}^N \chi_\varepsilon(x_k - z_{\sigma(k)})^2. \tag{1.5}$$

Using the symmetry of  $\mathbb{P}$ , we have

$$\int_{\mathbb{R}^{dN}} \prod_{k=1}^N \chi_\varepsilon(z_k - y_k)^2 d\mathbb{P}(y_1, \dots, y_N) = \int_{\mathbb{R}^{dN}} \prod_{k=1}^N \chi_\varepsilon(z_{\sigma(k)} - y_k)^2 d\mathbb{P}(y_1, \dots, y_N)$$

for every  $\sigma \in \mathfrak{S}_N$ . Therefore, in (1.5) the terms in the sum over the permutations  $\sigma \in \mathfrak{S}_N$  all contribute the same amount. We thus find that the diagonal of  $\Gamma_\varepsilon$  coincides with the Bindini–De Pascale probability density:

$$\Gamma_\varepsilon(x_1, \dots, x_N; x_1, \dots, x_N) = \mathbb{P}_\varepsilon(x_1, \dots, x_N).$$

From this, we conclude that

$$\text{Tr}(\Gamma_\varepsilon) = \int_{(\mathbb{R}^d)^N} \Gamma_\varepsilon(x_1, \dots, x_N; x_1, \dots, x_N) dx_1 \dots dx_N = \int_{\mathbb{R}^{dN}} d\mathbb{P}_\varepsilon(x_1, \dots, x_N) = 1,$$

and  $\Gamma_\varepsilon$  is a proper fermionic (mixed) state. We recall that the one-particle density of  $\Gamma_\varepsilon$  is defined by duality, requiring that  $\text{Tr}(\varphi(x_1)\Gamma_\varepsilon) = \int_{\mathbb{R}^d} \varphi \rho_{\Gamma_\varepsilon}$  for every  $\varphi \in L^\infty(\mathbb{R}^d)$ . For a continuous kernel such as  $\Gamma_\varepsilon$ , we have

$$\rho_{\Gamma_\varepsilon}(x_1) = \int_{\mathbb{R}^{dN}} \Gamma_\varepsilon(x_1, \dots, x_N; x_1, \dots, x_N) dx_2 \dots dx_N = \rho_{\mathbb{P}_\varepsilon}(x_1) = \rho_{\mathbb{P}}(x_1).$$

From the Cauchy–Schwarz inequality, we have

$$\int_{(\mathbb{R}^d)^N} |\nabla \sqrt{\mathbb{P}_\varepsilon}|^2 \leq \text{Tr}(-\Delta)\Gamma_\varepsilon,$$

which is in the wrong direction to conclude anything about the kinetic energy of  $\Gamma_\varepsilon$ . But we can prove the following theorem, which implies the Bindini–De Pascale estimate (1.2).

**Theorem 1** (Estimates on  $\Gamma_\varepsilon$ ). Let  $\mathbb{P}$  be a symmetric  $N$ -particle density with support in  $D_\alpha^c$  for some  $\alpha > 0$  and such that  $\sqrt{\rho_{\mathbb{P}}} \in H^1(\mathbb{R}^d)$ . Let  $\Gamma_\varepsilon$  be defined by (1.3). Then, for  $\varepsilon < \alpha/4$ , we have

$$\text{Tr}(-\Delta)\Gamma_\varepsilon = N \left( \int_{\mathbb{R}^d} |\nabla \sqrt{\rho_{\mathbb{P}}}|^2 + \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} |\nabla \chi|^2 \right). \tag{1.6}$$

In addition, for every symmetric function  $\Phi \in C^2(D_{\alpha-4\varepsilon}^c)$ , we have

$$\begin{aligned} \left| \text{Tr}(\Phi \Gamma_\varepsilon) - \int_{\mathbb{R}^{dN}} \Phi \, d\mathbb{P} \right| &= \left| \int_{\mathbb{R}^{dN}} \Phi \, d\mathbb{P}_\varepsilon - \int_{\mathbb{R}^{dN}} \Phi \, d\mathbb{P} \right| \\ &\leq \varepsilon^2 \left\{ \sum_{j=1}^N \|\nabla_j \Phi\|_{L^\infty(D_{\alpha-4\varepsilon}^c)} \left( \int_{\mathbb{R}^d} |\nabla \rho_{\mathbb{P}}| \right) \left( \int_{\mathbb{R}^d} |u|^2 \chi(u)^2 \, du \right) + 2 \sum_{j,k=1}^N \|\nabla_j \nabla_k \Phi\|_{L^\infty(D_{\alpha-4\varepsilon}^c)} \right\}. \end{aligned} \tag{1.7}$$

In particular,  $\mathbb{P}_\varepsilon \rightarrow \mathbb{P}$ , as was proved in [1].

**Proof.** We have

$$\text{Tr}(-\Delta)\Gamma_\varepsilon = \sum_{\ell=1}^N \iint_{\mathbb{R}^{dN} \times \mathbb{R}^{dN}} \left( \int_{\mathbb{R}^{dN}} |\nabla_{x_\ell} (\sqrt{\rho_{\mathbb{P}}}^{\otimes N} \chi_{\varepsilon,z_1} \wedge \dots \wedge \chi_{\varepsilon,z_N})|^2 \right) \prod_{k=1}^N \frac{\chi_\varepsilon(z_k - y_k)^2}{\rho_{\mathbb{P}} * \chi_\varepsilon^2(z_k)} \, d\mathbb{P}(y_1, \dots, y_N) \, dz_1 \dots dz_N.$$

Similarly as in (1.5),

$$|\nabla_{x_\ell} (\sqrt{\rho_{\mathbb{P}}} \chi_{\varepsilon,z_1} \wedge \dots \wedge \sqrt{\rho_{\mathbb{P}}} \chi_{\varepsilon,z_N})|^2 = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} |\nabla(\sqrt{\rho_{\mathbb{P}}} \chi_{\varepsilon,z_{\sigma(\ell)}})(x_\ell)|^2 \prod_{\substack{k=1, \dots, N \\ k \neq \ell}} \rho_{\mathbb{P}}(x_k) \chi_{\varepsilon,z_{\sigma(k)}}(x_k)^2.$$

Integrating over  $x_1, \dots, x_N$ , we obtain

$$\sum_{\ell=1}^N \int_{\mathbb{R}^{dN}} |\nabla_{x_\ell} (\sqrt{\rho_{\mathbb{P}}}^{\otimes N} \chi_{\varepsilon,z_1} \wedge \dots \wedge \chi_{\varepsilon,z_N})|^2 = \sum_{j=1}^N \left( \int_{\mathbb{R}^d} |\nabla(\sqrt{\rho_{\mathbb{P}}} \chi_{\varepsilon,z_j})(x)|^2 \, dx \right) \prod_{\substack{k=1, \dots, N \\ k \neq j}} \rho_{\mathbb{P}} * \chi_\varepsilon^2(z_j).$$

Finally, integrating over the  $y_k$ 's and  $z_k$ 's, we conclude that

$$\text{Tr}(-\Delta)\Gamma_\varepsilon = N \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\nabla(\sqrt{\rho_{\mathbb{P}}} \chi_{\varepsilon,z})(x)|^2 \, dx \, dz = N \left( \int_{\mathbb{R}^d} |\nabla \sqrt{\rho_{\mathbb{P}}}(x)|^2 \, dx + \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} |\nabla \chi(x)|^2 \, dx \right).$$

To prove (1.7), we remark that

$$\iint_{\mathbb{R}^{2dN}} \prod_{k=1}^N \frac{\rho_{\mathbb{P}}(x_k) \chi_\varepsilon(x_k - z_k)^2 \chi_\varepsilon(z_k - y_k)^2}{\rho_{\mathbb{P}} * \chi_\varepsilon^2(z_k)} \, dx_1 \dots dx_N \, dz_1 \dots dz_N = 1.$$

Hence, using the shorter notation  $X = (x_1, \dots, x_N)$ ,  $Y = (y_1, \dots, y_N)$  and  $Z = (z_1, \dots, z_N)$ , we have

$$\int_{\mathbb{R}^{dN}} \Phi \, d\mathbb{P}_\varepsilon - \int_{\mathbb{R}^{dN}} \Phi \, d\mathbb{P} = \iint_{\mathbb{R}^{3dN}} (\Phi(X) - \Phi(Y)) \prod_{k=1}^N \frac{\rho_{\mathbb{P}}(x_k) \chi_\varepsilon(x_k - z_k)^2 \chi_\varepsilon(z_k - y_k)^2}{\rho_{\mathbb{P}} * \chi_\varepsilon^2(z_k)} \, d\mathbb{P}(Y) \, dX \, dZ.$$

Since  $|x_j - y_j| \leq 2\varepsilon$  due to the support of  $\chi$ , we conclude from the fundamental theorem of calculus that

$$\begin{aligned} \left| \int_{\mathbb{R}^{dN}} \Phi \, d\mathbb{P}_\varepsilon - \int_{\mathbb{R}^{dN}} \Phi \, d\mathbb{P} - \iint_{\mathbb{R}^{3dN}} d\mathbb{P}(Y) \, dX \, dZ \left( \sum_{\ell=1}^N \nabla_\ell \Phi(Y) \cdot (x_\ell - y_\ell) \right) \right| \\ \times \prod_{k=1}^N \frac{\rho_{\mathbb{P}}(x_k) \chi_\varepsilon(x_k - z_k)^2 \chi_\varepsilon(z_k - y_k)^2}{\rho_{\mathbb{P}} * \chi_\varepsilon^2(z_k)} \leq 2\varepsilon^2 \sum_{\ell,m=1}^N \|\nabla_{x_\ell} \nabla_{x_m} \Phi\|_{L^\infty(D_{\alpha-4\varepsilon}^c)}. \end{aligned}$$

It remains to estimate the term involving  $\nabla_\ell \Phi$ . By symmetry, it is sufficient to look at the case  $\ell = 1$ , which can be expressed in the form

$$\begin{aligned} \iint_{\mathbb{R}^{3dN}} \nabla_1 \Phi(Y) \cdot (x_1 - y_1) \prod_{k=1}^N \frac{\rho_{\mathbb{P}}(x_k) \chi_\varepsilon(x_k - z_k)^2 \chi_\varepsilon(z_k - y_k)^2}{\rho_{\mathbb{P}} * \chi_\varepsilon^2(z_k)} d\mathbb{P}(Y) dX dZ \\ = \iint_{\mathbb{R}^{dN+2d}} \nabla_1 \Phi(Y) \cdot (x_1 - y_1) \frac{\rho_{\mathbb{P}}(x_1) \chi_\varepsilon(x_1 - z_1)^2 \chi_\varepsilon(z_1 - y_1)^2}{\rho_{\mathbb{P}} * \chi_\varepsilon^2(z_1)} d\mathbb{P}(Y) dx_1 dz_1. \end{aligned}$$

Here we can replace  $x_1 - y_1$  by  $x_1 - z_1$  since  $\int_{\mathbb{R}^d} (z_1 - y_1) \chi_\varepsilon(z_1 - y_1)^2 dz_1 = 0$ , due to the symmetry of  $\chi$ . We then estimate

$$\begin{aligned} \left| \iint_{\mathbb{R}^{3dN}} \nabla_1 \Phi(Y) \cdot (x_1 - y_1) \frac{\rho_{\mathbb{P}}(x_1) \chi_\varepsilon(x_1 - z_1)^2 \chi_\varepsilon(z_1 - y_1)^2}{\rho_{\mathbb{P}} * \chi_\varepsilon^2(z_1)} d\mathbb{P}(Y) dx_1 dz_1 \right| \\ \leq \|\nabla_1 \Phi\|_{L^\infty(D_{\alpha-4\varepsilon}^c)} \iint_{\mathbb{R}^{2d}} \left| \int_{\mathbb{R}^d} (x_1 - z_1) \rho_{\mathbb{P}}(x_1) \chi_\varepsilon(x_1 - z_1)^2 dx_1 \right| \frac{\chi_\varepsilon(z_1 - y_1)^2}{\rho_{\mathbb{P}} * \chi_\varepsilon^2(z_1)} \rho_{\mathbb{P}}(y_1) dz_1 dy_1 \\ = \|\nabla_1 \Phi\|_{L^\infty(D_{\alpha-4\varepsilon}^c)} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (x - z) \rho_{\mathbb{P}}(x) \chi_\varepsilon(x - z)^2 dx dz \right|. \end{aligned}$$

By the symmetry of  $\chi$  and the fundamental theorem of calculus, we have

$$\left| \int_{\mathbb{R}^d} (x - z) \rho_{\mathbb{P}}(x) \chi_\varepsilon(x - z)^2 dx \right| = \left| \int_{\mathbb{R}^d} (x - z) (\rho_{\mathbb{P}}(x) - \rho_{\mathbb{P}}(z)) \chi_\varepsilon(x - z)^2 dx \right| \leq \int_0^1 \int_{\mathbb{R}^d} |\nabla \rho_{\mathbb{P}}(z + tu)| |u|^2 \chi_\varepsilon(u)^2 du.$$

Integrating over  $z$ , we find the claimed estimate

$$\begin{aligned} \left| \iint_{\mathbb{R}^{3dN}} \nabla_1 \Phi(Y) \cdot (x_1 - y_1) \prod_{k=1}^N \frac{\rho_{\mathbb{P}}(x_k) \chi_\varepsilon(x_k - z_k)^2 \chi_\varepsilon(z_k - y_k)^2}{\rho_{\mathbb{P}} * \chi_\varepsilon^2(z_k)} d\mathbb{P}(Y) dX dZ \right| \\ \leq \varepsilon^2 \|\nabla_1 \Phi\|_{L^\infty(D_{\alpha-4\varepsilon}^c)} \left( \int_{\mathbb{R}^d} |\nabla \rho_{\mathbb{P}}| \right) \left( \int_{\mathbb{R}^d} |u|^2 \chi(u)^2 du \right). \quad \square \end{aligned}$$

## 2. Semiclassical limit of the Levy-Lieb functional

Here we restrict ourselves for simplicity to the physical space  $\mathbb{R}^3$  and the Coulomb potential. Density Functional Theory is based on the following functional [13,14,4]

$$\mathcal{E}(\rho) = \min_{\substack{\Gamma = \Gamma^* \geq 0 \\ \text{Tr}(\Gamma) = 1 \\ \rho_\Gamma = \rho}} \text{Tr} \left( - \sum_{j=1}^N \Delta_{x_j} + \sum_{1 \leq j < k \leq N} \frac{1}{|x_j - x_k|} \right) \Gamma, \tag{2.1}$$

of the density  $\rho$ , a given non-negative function such that  $\int_{\mathbb{R}^3} \rho = N$  and  $\sqrt{\rho} \in H^1(\mathbb{R}^3)$ . In the minimum  $\Gamma$  is an operator acting on the fermionic space  $\bigwedge_1^N L^2(\mathbb{R}^3)$ . Motivated by arguments of Hohenberg and Kohn [12], Levy introduced in [13] a functional similar to (2.1), but with the additional constraint that  $\Gamma = |\Psi\rangle\langle\Psi|$  is a rank-one orthogonal projection (pure state). The latter was rigorously studied by Lieb in [14], who proposed to extend the definition to mixed states, as in (2.1). The minimum over mixed states (2.1) has better mathematical properties than with pure states [14]. For instance,  $\mathcal{E}$  is convex and, by the linearity in  $\Gamma$ , we have the dual formulation

$$\mathcal{E}(\rho) = \sup \left\{ \int_{\mathbb{R}^3} \rho(x) V(x) dx : V \in L^{3/2}(\mathbb{R}^3, \mathbb{R}) + L^\infty(\mathbb{R}^3, \mathbb{R}), - \sum_{j=1}^N \Delta_{x_j} - \sum_{j=1}^N V(x_j) + \sum_{1 \leq j < k \leq N} \frac{1}{|x_j - x_k|} \geq 0 \right\},$$

which is the quantum equivalent of the Kantorovich duality used in optimal transport [16]. The last inequality is in the sense of self-adjoint operators.

It is possible to introduce an effective semi-classical parameter  $\eta = \hbar^2$  by scaling the density  $\rho$ . Namely, for  $\rho_\eta(x) = \eta^3 \rho(\eta x)$ , we have

$$\frac{\mathcal{E}(\rho_\eta)}{\eta} = \min_{\substack{\Gamma = \Gamma^* \geq 0 \\ \text{Tr}(\Gamma) = 1 \\ \rho_\Gamma = \rho}} \text{Tr} \left( -\eta \sum_{j=1}^N \Delta_{x_j} + \sum_{1 \leq j < k \leq N} \frac{1}{|x_j - x_k|} \right) \Gamma. \quad (2.2)$$

In the limit  $\eta \rightarrow 0$ , we prove the convergence to the Coulomb multi-marginal optimal transport problem

$$\mathcal{E}_{\text{OT}}(\rho) = \min_{\substack{\mathbb{P} \text{ symmetric} \\ \text{probability on } (\mathbb{R}^3)^N (\mathbb{R}^3)^N \\ \rho_{\mathbb{P}} = \rho}} \int \sum_{1 \leq j < k \leq N} \frac{1}{|x_j - x_k|} d\mathbb{P}(x_1, \dots, x_N), \quad (2.3)$$

which has recently received a lot of attention [3,6,7,5,8,15]

**Theorem 2** (Semi-classical limit). *Let  $N \geq 2$  and let  $\rho \geq 0$  be such that  $\int_{\mathbb{R}^3} \rho = N$  and  $\sqrt{\rho} \in H^1(\mathbb{R}^3)$ . Then we have for a constant  $C$  (depending on  $N$  and  $\rho$ )*

$$\mathcal{E}_{\text{OT}}(\rho) \leq \frac{\mathcal{E}(\eta^3 \rho(\eta \cdot))}{\eta} \leq \mathcal{E}_{\text{OT}}(\rho) + C(\sqrt{\eta} + \eta). \quad (2.4)$$

In particular,

$$\lim_{\eta \rightarrow 0} \frac{\mathcal{E}(\eta^3 \rho(\eta \cdot))}{\eta} = \mathcal{E}_{\text{OT}}(\rho).$$

This theorem generalizes the results in [6,1] for  $N = 2, 3$  in the pure state case. Results for  $N \geq 3$  have been announced in [9, Ref. 7], but they were not yet available at the time this note was written. It would be interesting to extend our findings to pure states.

Semi-classical analysis suggests that the behavior in  $\sqrt{\eta}$  is optimal for small  $\eta$ . The next order (in  $\sqrt{\eta} = \hbar$ ) in the expansion of  $\mathcal{E}(\rho_\eta)/\eta$  was predicted in [10].

**Proof.** Let  $\mathbb{P}$  be an optimizer for  $\mathcal{E}_{\text{OT}}(\rho)$ . It has been shown in [2] that  $\mathbb{P}$  has its support on  $D_\alpha^c$  for some  $\alpha > 0$ . Taking then our  $\Gamma_\varepsilon$  as a trial state, we find, by Theorem 1

$$\frac{\mathcal{E}(\rho_\eta)}{\eta} \leq \eta N \left( \int_{\mathbb{R}^d} |\nabla \sqrt{\rho}|^2 + \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} |\nabla \chi|^2 \right) + \mathcal{E}_{\text{OT}}(\rho) + C\varepsilon^2 \left( \frac{N^3}{(\alpha - 4\varepsilon)^2} \int_{\mathbb{R}^d} |\nabla \rho_{\mathbb{P}}| \int_{\mathbb{R}^d} |u|^2 \chi(u)^2 du + \frac{N^4}{(\alpha - 4\varepsilon)^3} \right).$$

We have used here that  $\Phi(X) = \sum_{1 \leq j < k \leq N} |x_j - x_k|^{-1}$  is  $C^\infty$  on  $D_{\alpha-4\varepsilon}^c$ . Optimizing in  $\varepsilon$  gives the result.  $\square$

**Remark 1.** The convergence of states in the limit  $\eta \rightarrow 0$  is proved as in [1].

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