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## A novel signal extraction approach for filtering and forecasting noisy exponential series



*Une nouvelle approche dans l'extraction de signal pour le filtrage et la prévision par des séries exponentielles avec bruit*

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### ABSTRACT

The coefficients of Linear Recurrent Relations (LRR) play a pivotal role in many forecasting techniques. Precise and closed form of the LRR coefficients enables one to achieve more accurate forecasts. On account to the fact that, in real-world situations, a time series data is contaminated with noise, extracting the noiseless series is of great importance. This paper seeks to obtain a closed form, with less noise level, of LRR coefficients for noisy exponential time series. Improving the filtering performance through employing noiseless eigenvectors of the covariance matrix is another novelty of this study. Our simulation results confirm that the proposed approach enhances filtering and forecasting results.

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### RÉSUMÉ

Les coefficients des relations récurrentes linéaires (RRL) jouent un rôle central dans beaucoup de techniques de prévision. Une formule exacte et close des coefficients d'une RRL permet d'obtenir des prévisions plus précises. Prenant en compte le fait que, dans la réalité, une suite temporelle de données est contaminée par du bruit, il est très important de pouvoir en extraire la série sans bruit. Ce texte vise à obtenir une forme close, avec un niveau de bruit moindre, des coefficients d'une RRL, pour les suites en temps exponentiel avec bruit. Une autre nouveauté de notre approche est l'amélioration de l'efficacité du filtrage par l'utilisation de vecteurs propres sans bruit de la matrice de covariance. Les résultats des simulations confirment que l'approche proposée améliore le filtrage et les prévisions.

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## 1. Introduction

A time series  $Y_N = \{y_1, y_2, \dots, y_N\}$  of length  $N$  is generated by a *linear recurrence relation* (LRR) of order  $d > 0$ , if there exists the coefficients  $\alpha_1, \dots, \alpha_d$  such that:

$$y_{i+d} = \sum_{j=1}^d \alpha_j y_{i+d-j}, \quad 1 \leq i \leq N-d, \quad \alpha_d \neq 0, \quad d < N. \quad (1)$$

The coefficients  $\alpha_1, \dots, \alpha_d$  are called the coefficients of the LRR or  $\alpha$ -coefficients. The class of time series governed by LRRs are rather wide and important for practical applications (see, for example, [1–3,5,7–18,22,23]). The Singular Spectrum Analysis (SSA) technique is one of the powerful and non-parametric techniques with capability of both forecasting and filtering where LRR is used for forecasting new data points [6,17].

Suppose  $L$  be an integer called *Window Length* such that  $2 \leq L \leq N/2$ . The starting point of SSA is to construct the trajectory matrix  $\mathbf{X} = [X_1 : \dots : X_K]$  via vectors  $X_i = (y_i, \dots, y_{i+L-1})^T \in \mathbf{R}^L$ ,  $i = 1, \dots, K$ , called *lagged vectors*, where  $K = N - L + 1$ . The trajectory matrix  $\mathbf{X}$  is a *Hankel* matrix in the sense that all elements on anti-diagonals  $i + j = \text{const.}$  are equal. The eigenvalues of  $\mathbf{X}\mathbf{X}^T$  are denoted by  $\lambda_1, \dots, \lambda_L$  in decreasing order of magnitude ( $\lambda_1 \geq \dots \geq \lambda_L \geq 0$ ), and the eigenvectors of  $\mathbf{X}\mathbf{X}^T$  corresponding to these eigenvalues are denoted by  $U_1, \dots, U_L$ . It is assumed that the eigenvectors have unit length, i.e.  $\|U_i\| = 1$ , where  $\|\cdot\|$  is the Euclidean norm.

The eigenvectors of  $\mathbf{X}\mathbf{X}^T$  play a very pivotal role in the reconstruction stage of SSA. Let  $I$  be the chosen set of eigentriples attained at the grouping step of SSA and  $U_i \in \mathbf{R}^L$ ,  $i \in I$ , be the corresponding eigenvectors. Denote by  $\mathcal{L} \subset \mathbf{R}^L$  the linear space spanned by the vectors  $U_i$ ,  $i \in I$ ; i.e.  $\mathcal{L} = \text{span}\{U_i, i \in I\}$ . Note that the set  $\{U_i, i \in I\}$  is an orthonormal basis in  $\mathcal{L}$ . To reconstruct the time series  $Y_N$  by set  $I$ , all lagged vectors  $X_i$  are first orthogonally projected onto  $\mathcal{L}$  through  $\hat{\mathbf{X}} = \sum_{j \in I} U_j U_j^T \mathbf{X}$ , where  $\mathbf{X}$  is the trajectory matrix of series  $Y_N$ , and where the matrix  $\hat{\mathbf{X}}$  consists of column vectors  $\hat{X}_i$ ,  $\hat{X}_i = \sum_{j \in I} U_j U_j^T X_i$ . Then the matrix  $\hat{\mathbf{X}}$  is diagonally averaged to get the reconstructed series  $\tilde{Y}_N = \{\tilde{y}_1, \dots, \tilde{y}_N\}$ .

The eigenvectors  $U_i$ ,  $i \in I$ , are also employed in the forecasting methods of SSA. Let  $\underline{U}_i \in \mathbf{R}^{L-1}$  be the vector consisting of the first  $L-1$  components of the vector  $U_i$ ,  $\pi_i$  be the last component of the vector  $U_i$  and  $v^2 = \sum_{i \in I} \pi_i^2$ . The last component  $z_L$  of any vector  $Z = (z_1, \dots, z_L)^T \in \mathcal{L}$  is a linear combination of the first components  $z_1, \dots, z_{L-1}$ , i.e.  $z_L = \alpha_1 z_{L-1} + \dots + \alpha_{L-1} z_1$ , (see [6]), where the vector  $A = (\alpha_{L-1}, \dots, \alpha_1)^T$  is obtained as follows:

$$A = \frac{1}{1 - v^2} \sum_{i \in I} \pi_i \underline{U}_i. \quad (2)$$

The  $\alpha$ -coefficients  $\{\alpha_j, j = 1, \dots, L-1\}$  in (2), which are made by eigenvectors  $U_i, i \in I$ , play a fundamental role in SSA forecasting. For example, in *Recurrent forecasting* (R-forecasting) if the time series  $Z_{N+h} = \{z_1, \dots, z_{N+h}\}$  is defined by:

$$z_i = \begin{cases} \tilde{y}_i & \text{for } i = 1, \dots, N, \\ \sum_{j=1}^{L-1} \alpha_j z_{i-j} & \text{for } i = N+1, \dots, N+h, \end{cases} \quad (3)$$

then the numbers  $z_{N+1}, \dots, z_{N+h}$  are the  $h$  step ahead recurrent forecasts. It is clear that R-forecasting is performed by the direct use of LRR in (1) and of  $\alpha$ -coefficients in (2).

The same approach can be used for *Vector forecasting* (V-forecasting). Consider the matrix  $\Pi = \underline{V}\underline{V}^T + (1 - v^2)AA^T$ , where the matrix  $\underline{V}$  consists of column vectors  $\underline{U}_i, i \in I$ . If the vectors  $W_i$  defined as:

$$W_i = \begin{cases} \hat{X}_i & \text{for } i = 1, \dots, K, \\ \mathcal{P}_{\text{Vec}} W_{i-1} & \text{for } i = K+1, \dots, K+h+L-1, \end{cases} \quad (4)$$

where  $\mathcal{P}_{\text{Vec}} W_{i-1} = \begin{pmatrix} \Pi \overline{W}_{i-1} \\ A^T \overline{W}_{i-1} \end{pmatrix}$  and  $\overline{W}_{i-1}$  is the vector consisting of the last  $L-1$  components of the vector  $W_{i-1}$ , then by constructing the matrix  $\mathbf{W} = [W_1 : \dots : W_{K+h+L-1}]$  and making its diagonal averaging the series  $\{z_1, \dots, z_{N+h+L-1}\}$  is obtained. The numbers  $z_{N+1}, \dots, z_{N+h}$  are the  $h$  step ahead vector forecasts. It is clear that  $\alpha$ -coefficients have also key role in V-forecasting through the matrix  $\Pi$  and the linear operator  $\mathcal{P}_{\text{Vec}}$ .

It can be assumed that  $Y_N$  is the sum of a noise free series (signal) and noise, i.e.:

$$y_t = s_t + n_t, \quad t = 1, \dots, N, \quad (5)$$

where  $s_t$  and  $n_t$  represent the signal and noise components, respectively. Equation (5) can be expressed in the following matrix form:

$$\mathbf{X} = \mathbf{S} + \mathbf{N}, \quad (6)$$

where  $\mathbf{S}$  and  $\mathbf{N}$  represent  $L \times K$  trajectory matrices of the signal and noise components, respectively. According to (6), the trajectory matrix  $\mathbf{X}$  is not a noiseless matrix. Hence, the eigenvectors of  $\mathbf{X}\mathbf{X}^T$  are contaminated with noise. Therefore, it seems that the accuracy of the reconstruction stage of SSA is affected by the contribution of some noise to eigenvectors  $U_i, i \in I$ , resulting in a reduction of reconstruction and forecasting performance.

The effect of noise level can also be traced in  $\alpha$ -coefficients. In theory, the  $\alpha$ -coefficients can get their exact or noise-free values if time series  $Y_N$  is not contaminated with noise; i.e.  $y_t = s_t$ . However, in reality, the  $\alpha$ -coefficients that are computed based on noisy time series are not noise free. This is because, as per (2), these coefficients are built on noisy eigenvectors  $U_i, i \in I$ . Furthermore, as Equation (2) indicates, the  $\alpha_i$  coefficient changes when the values of  $L$  and  $U_i$  change. However, this study proposes a closed form of the exact  $\alpha$ -coefficients for an exponential time series family. To the best of our knowledge, this investigation is carried out for the first time. Because of widespread applications of exponential series in various fields, ranging from population growth models, biology, non-linear time series analysis, and econometrics to financial time series [4,5,10,19–21], the proposed approaches can be used in many areas.

The remainder of this paper is organized as follows. Section 2 studies the closed form of exact  $\alpha$ -coefficients in exponential time series. In Section 3, the quality of reconstruction is evaluated using the results given in Section 2. This section presents the new versions of recurrent and vector forecasting methods to enhance the performance of forecasting. Section 4 provides the empirical results, and finally concluding remarks with a discussion and summary are presented in Section 5.

## 2. The exact formula of $\alpha$ -coefficients for an exponential series

It is worth mentioning that the number of  $\alpha$ -coefficients is equal to  $L - 1$  for any time series with the window length  $L$ . For example, there are three  $\alpha$ -coefficients  $\alpha_1, \alpha_2$ , and  $\alpha_3$  for  $L = 4$ . In order to better distinguish between the  $\alpha$ -coefficients for different values of  $L$ , we will use another notation afterwards, in which  $\alpha_{i,L}$  denotes the  $i$ th  $\alpha$ -coefficient for a given  $L$ , e.g.,  $\alpha_{3,5}$  is the third  $\alpha$ -coefficient for  $L = 5$ .

Consider now a noise free exponential series of length  $N$ :

$$y_t = s_t = \exp(\beta_0 + \beta_1 t), \quad t = 1, 2, \dots, N, \tag{7}$$

where  $\beta_0$  and  $\beta_1$  are parameters needed to be estimated. Let us first consider the structure of the matrix  $\mathbf{X}\mathbf{X}^T$  for an exponential series.

**Theorem 1.** For an exponential series  $y_t = \exp(\beta_0 + \beta_1 t)$ , the matrix  $\mathbf{X}\mathbf{X}^T$  has the following form:

$$\mathbf{X}\mathbf{X}^T = \gamma e^{2\beta_0} \mathbf{E}_L,$$

where,

$$\mathbf{E}_L = (e^{(i+j-2)\beta_1})_{i,j=1}^{L,L} = \begin{pmatrix} 1 & e^{\beta_1} & \dots & e^{(L-1)\beta_1} \\ e^{\beta_1} & e^{2\beta_1} & \dots & e^{L\beta_1} \\ \vdots & \vdots & \ddots & \vdots \\ e^{(L-1)\beta_1} & e^{L\beta_1} & \dots & e^{(2L-2)\beta_1} \end{pmatrix}_{L \times L}, \tag{8}$$

$$\gamma = \sum_{l=0}^{K-1} e^{2(1+l)\beta_1} \text{ and } K = N - L + 1.$$

**Proof.** If the  $i$ th row of the trajectory matrix  $\mathbf{X}$  is denoted by  $H_i$ , i.e.  $H_i = (y_i, \dots, y_{i+K-1})$ , then it can be concluded that the component of  $i$ th row and  $j$ th column of the matrix  $\mathbf{X}\mathbf{X}^T$  is given as follows:

$$\begin{aligned} H_i H_j^T &= \sum_{l=0}^{K-1} y_{i+l} y_{j+l} = \sum_{l=0}^{K-1} e^{\beta_0+(i+l)\beta_1} e^{\beta_0+(j+l)\beta_1} = e^{2\beta_0} \sum_{l=0}^{K-1} e^{(i+j+2l)\beta_1} \\ &= e^{2\beta_0} \sum_{l=0}^{K-1} e^{(i+j-2+2+2l)\beta_1} = e^{2\beta_0} \sum_{l=0}^{K-1} e^{(i+j-2)\beta_1} e^{(2+2l)\beta_1} \\ &= e^{2\beta_0} e^{(i+j-2)\beta_1} \sum_{l=0}^{K-1} e^{2(1+l)\beta_1} = \gamma e^{2\beta_0} e^{(i+j-2)\beta_1}. \end{aligned}$$

Therefore, by the definition of  $\mathbf{E}_L$  in (8), it can be concluded that  $\mathbf{X}\mathbf{X}^T = \gamma e^{2\beta_0} \mathbf{E}_L$ , where  $\gamma = \sum_{l=0}^{K-1} e^{2(1+l)\beta_1}$ .  $\square$

Theorem 1 shows that  $\mathbf{X}\mathbf{X}^T$  is a multiple of the matrix  $\mathbf{E}_L$ . Accordingly, the matrices  $\mathbf{X}\mathbf{X}^T$  and  $\mathbf{E}_L$  have similar eigenvectors (with different eigenvalues). Additionally, we have  $X_j = e^{\beta_1} X_{j-1} = e^{(j-1)\beta_1} X_1$ , where  $X_j$  is the  $j$ th column of the trajectory matrix  $\mathbf{X}$ . Accordingly, the following corollary can be obtained.

**Corollary 1.**  $\text{rank}(\mathbf{X}) = \text{rank}(\mathbf{X}\mathbf{X}^\top) = \text{rank}(\mathbf{E}_L) = 1$ .

**Theorem 2.** For an exponential series  $y_t = \exp(\beta_0 + \beta_1 t)$ , the eigenvector of matrices  $\mathbf{X}\mathbf{X}^\top$  and  $\mathbf{E}_L$  is  $\mathbf{e}_L$ , where  $\mathbf{e}_L = (1, e^{\beta_1}, \dots, e^{(L-1)\beta_1})^\top$ .

**Proof.** According to Corollary 1, the matrix  $\mathbf{E}_L$  has only one eigenvector. It can be easily shown that  $\mathbf{E}_L = \mathbf{e}_L \mathbf{e}_L^\top$ . We have:

$$\mathbf{E}_L \mathbf{e}_L = \mathbf{e}_L \mathbf{e}_L^\top \mathbf{e}_L = \|\mathbf{e}_L\|^2 \mathbf{e}_L. \quad (9)$$

Relation (9) indicates that  $\|\mathbf{e}_L\|^2$  and  $\mathbf{e}_L$  are the eigenvalue and the eigenvector of the matrix  $\mathbf{E}_L$ , respectively. Consequently, the eigenvalue and eigenvector of the matrix  $\mathbf{X}\mathbf{X}^\top$  are  $\gamma e^{2\beta_0} \|\mathbf{e}_L\|^2$  and  $\mathbf{e}_L$ , respectively.  $\square$

Since the rank of the trajectory matrix for the exponential series (7) is equal to one for different values of  $\beta_0$  and  $\beta_1$ , we then choose  $I = \{1\}$  for reconstruction and forecasting. Now, having the eigenvector of  $\mathbf{X}\mathbf{X}^\top$ , it is possible to drive a formula to determine the exact values of the  $\alpha$ -coefficients.

**Theorem 3.** The closed form of  $\alpha$ -coefficients for an exponential series  $y_t = \exp(\beta_0 + \beta_1 t)$  is as follows:

$$\alpha_{i,L} = \frac{e^{(2L-i-2)\beta_1}}{\|\mathbf{e}_{L-1}\|^2}, \quad i = 1, 2, \dots, L-1. \quad (10)$$

**Proof.** Recall from Section 1 that for any time series with trajectory matrix  $\mathbf{X}$ ,  $U_1 = (u_1, u_2, \dots, u_L)^\top$  is the first eigenvector of  $\mathbf{X}\mathbf{X}^\top$  with unit length, i.e.  $\|U_1\| = 1$ . Thus, by applying Theorem 2, we have  $U_1 = \frac{\mathbf{e}_L}{\|\mathbf{e}_L\|}$  and,

$$\begin{aligned} U_1 &= (u_1, u_2, \dots, u_L)^\top = \frac{1}{\|\mathbf{e}_L\|} (1, e^{\beta_1}, \dots, e^{(L-1)\beta_1})^\top, \\ \underline{U}_1 &= (u_1, u_2, \dots, u_{L-1})^\top = \frac{1}{\|\mathbf{e}_L\|} (1, e^{\beta_1}, \dots, e^{(L-2)\beta_1})^\top = \frac{\mathbf{e}_{L-1}}{\|\mathbf{e}_L\|}, \\ v^2 &= \sum_{i \in I} \pi_i^2 = \pi_1^2 = u_L^2 = \frac{e^{2(L-1)\beta_1}}{\|\mathbf{e}_L\|^2}. \end{aligned}$$

From (2),

$$\begin{aligned} A &= (\alpha_{L-1,L}, \alpha_{L-2,L}, \dots, \alpha_{1,L})^\top = \frac{1}{1-v^2} \sum_{i \in I} \pi_i U_i = \frac{u_L}{1-u_L^2} U_1 \\ &= \frac{\frac{e^{(L-1)\beta_1}}{\|\mathbf{e}_L\|}}{1 - \frac{e^{2(L-1)\beta_1}}{\|\mathbf{e}_L\|^2}} \times \frac{\mathbf{e}_{L-1}}{\|\mathbf{e}_L\|} \\ &= \frac{e^{(L-1)\beta_1}}{\|\mathbf{e}_L\|^2 - e^{2(L-1)\beta_1}} \times \mathbf{e}_{L-1} \\ &= \frac{e^{(L-1)\beta_1}}{1 + e^{2\beta_1} + \dots + e^{2(L-1)\beta_1} - e^{2(L-1)\beta_1}} \mathbf{e}_{L-1} \\ &= \frac{e^{(L-1)\beta_1}}{1 + e^{2\beta_1} + \dots + e^{2(L-2)\beta_1}} \mathbf{e}_{L-1} \\ &= \frac{e^{(L-1)\beta_1}}{\|\mathbf{e}_{L-1}\|^2} \mathbf{e}_{L-1}. \end{aligned} \quad (11)$$

Therefore,

$$\alpha_{i,L} = \frac{e^{(2L-i-2)\beta_1}}{\|\mathbf{e}_{L-1}\|^2}, \quad i = 1, 2, \dots, L-1. \quad \square$$

**Corollary 2.** For an exponential series  $y_t = \exp(\beta_0 + \beta_1 t)$ , we have the following.

1. For each  $i$ ,  $\alpha_{i,L} > 0$ .
2.  $\alpha_{i+1,L} = e^{-\beta_1} \alpha_{i,L}$ ,  $i = 1, 2, \dots, L-1$ .
3. For each  $i$ ,  $\alpha_{i,L}$  is independent of  $N$  and  $\beta_0$ . In other words, the value of  $N$  and  $\beta_0$  have no effect on the  $\alpha$ -coefficients.
4. If  $\beta_1 > 0$ , the sequence  $\{\alpha_{i,L}\}_{i=1}^{L-1}$  is decreasing and, if  $\beta_1 < 0$ , it is a increasing sequence.

### 3. Reconstruction and forecasting formula for exponential series

#### 3.1. Reconstruction

The reconstruction step of SSA for any time series includes the following two steps.

1. The orthogonal projection of the lagged vector  $X_i$  on  $\mathcal{L} = \text{span}\{U_i, i \in I\}$ , denoted by  $\widehat{X}_i$ , that is,  $\widehat{X}_i = \sum_{j \in I} U_j U_j^T X_i$ . Projecting all lagged vectors can be done by  $\widehat{\mathbf{X}} = \sum_{j \in I} U_j U_j^T \mathbf{X}$ , where  $\mathbf{X}$  is the trajectory matrix of the original series  $Y_N$ , and the matrix  $\widehat{\mathbf{X}}$  consists of the column vectors  $\widehat{X}_i$ .
2. The Hankelization of the matrix  $\widehat{\mathbf{X}}$  via diagonal averaging to reconstruct the series  $\widetilde{Y}_N = \{\widetilde{y}_1, \dots, \widetilde{y}_N\}$ . In other words,  $\widetilde{\mathbf{X}} = \mathcal{H}(\widehat{\mathbf{X}})$ , where  $\widetilde{\mathbf{X}}$  is the trajectory matrix of reconstructed series and  $\mathcal{H}(\cdot)$  is the Hankelization operator.

It is clear that eigenvectors  $U_i, i \in I$ , have key role in the reconstruction step of SSA. However, in reality, these eigenvectors are contaminated with noise, which, according to (6), may result in a low accuracy of the reconstruction step. Therefore, utilizing noise-free eigenvectors can improve the performance of the reconstruction step. In this subsection, a new reconstruction procedure for exponential series based on the noise-free eigenvector  $U_1 = \frac{\mathbf{e}_L}{\|\mathbf{e}_L\|}$  is provided. Since  $\mathbf{E}_L = \mathbf{e}_L \mathbf{e}_L^T$ , we have

$$U_1 U_1^T = \frac{\mathbf{e}_L}{\|\mathbf{e}_L\|} \frac{\mathbf{e}_L^T}{\|\mathbf{e}_L\|} = \frac{\mathbf{e}_L \mathbf{e}_L^T}{\|\mathbf{e}_L\|^2} = \frac{1}{\|\mathbf{e}_L\|^2} \mathbf{E}_L. \tag{12}$$

Consequently, two steps of proposed reconstruction steps for exponential series are as follows:

- 1) orthogonal projection of the lagged vectors on  $\mathcal{L} = \text{span}\{\frac{\mathbf{e}_L}{\|\mathbf{e}_L\|}\}$  by  $\widehat{\mathbf{X}} = \frac{1}{\|\mathbf{e}_L\|^2} \mathbf{E}_L \mathbf{X}$ ;
- 2) Hankelization of the matrix  $\widehat{\mathbf{X}}$  via diagonal averaging to get the reconstructed series  $\widetilde{Y}_N = \{\widetilde{y}_1, \dots, \widetilde{y}_N\}$ , i.e.  $\widetilde{\mathbf{X}} = \mathcal{H}(\widehat{\mathbf{X}})$ .

#### 3.2. New R-forecasting

Considering the proposed reconstruction of Exponential series, the new R-forecasting algorithm for Exponential series can be presented as follows:

- 1) the lagged vectors are projected on  $\mathcal{L} = \text{span}\{\frac{\mathbf{e}_L}{\|\mathbf{e}_L\|}\}$  by  $\widehat{\mathbf{X}} = \frac{1}{\|\mathbf{e}_L\|^2} \mathbf{E}_L \mathbf{X}$ ;
- 2) the reconstructed time series  $\widetilde{Y}_N = \{\widetilde{y}_1, \dots, \widetilde{y}_N\}$  is obtained by diagonal averaging of  $\widehat{\mathbf{X}}$ ; i.e.  $\widetilde{\mathbf{X}} = \mathcal{H}(\widehat{\mathbf{X}})$ ;
- 3) the time series  $Z_{N+h} = \{z_1, \dots, z_{N+h}\}$  is defined by:

$$z_i = \begin{cases} \widetilde{y}_i & \text{for } i = 1, \dots, N, \\ \sum_{j=1}^{L-1} \alpha_{j,L} z_{i-j} & \text{for } i = N + 1, \dots, N + h, \end{cases}$$

where  $\alpha_{j,L} = \frac{e^{(2L-j-2)\beta_1}}{\|\mathbf{e}_{L-1}\|^2}$  and  $z_{N+1}, \dots, z_{N+h}$  form the  $h$  step ahead recurrent forecasts for exponential series.

Similar to common R-forecasting in SSA, the proposed R-forecasting algorithm for exponential series can be expressed in vector form. Let  $\mathcal{P}_{\text{Rec}} : \mathbf{R}^L \mapsto \mathbf{R}^L$  be the linear operator defined as:

$$\mathcal{P}_{\text{Rec}} Z = \begin{pmatrix} \overline{Z} \\ A^T \overline{Z} \end{pmatrix}, \tag{13}$$

where  $\overline{Z} \in \mathbf{R}^{L-1}$  is the vector consisting of the last  $L - 1$  components of the vector  $Z = (z_1, \dots, z_L)^T \in \mathbf{R}^L$  and  $A$  is the vector of  $\alpha$ -coefficients from Relation (11). The vector form of new R-forecasting algorithm is as follows:

$$W_i = \begin{cases} \widetilde{X}_i & \text{for } i = 1, \dots, K, \\ \mathcal{P}_{\text{Rec}} W_{i-1} & \text{for } i = K + 1, \dots, K + h, \end{cases} \tag{14}$$

where  $\widetilde{X}_i$  is the  $i$ th column of  $\widetilde{\mathbf{X}} = \mathcal{H}(\widehat{\mathbf{X}})$ . The matrix  $\mathbf{W} = [W_1 : \dots : W_{K+h}]$  is the trajectory matrix of the series  $Z_{N+h}$ . Consequently, the  $h$  step ahead recurrent forecasts  $z_{N+1}, \dots, z_{N+h}$  can be achieved.

#### 3.3. New V-forecasting

Consider the common V-forecasting reviewed in Section 1; at first, the matrix  $\Pi$  should be determined. From Theorem 3, we have:

$$U_1 = \frac{\mathbf{e}_L}{\|\mathbf{e}_L\|}, \quad \underline{V} = \underline{U}_1 = \frac{\mathbf{e}_{L-1}}{\|\mathbf{e}_L\|}, \quad \mathbf{v} = \frac{e^{(L-1)\beta_1}}{\|\mathbf{e}_L\|}, \quad A = \frac{e^{(L-1)\beta_1}}{\|\mathbf{e}_{L-1}\|^2} \mathbf{e}_{L-1}.$$

It can be easily shown that  $\mathbf{E}_{L-1} = \mathbf{e}_{L-1} \mathbf{e}_{L-1}^T$ . Therefore,

$$\begin{aligned} \underline{V} \underline{V}^T &= \frac{\mathbf{e}_{L-1} \mathbf{e}_{L-1}^T}{\|\mathbf{e}_L\| \|\mathbf{e}_L\|} = \frac{1}{\|\mathbf{e}_L\|^2} \mathbf{E}_{L-1}, \\ AA^T &= \frac{e^{(L-1)\beta_1}}{\|\mathbf{e}_{L-1}\|^2} \mathbf{e}_{L-1} \frac{e^{(L-1)\beta_1}}{\|\mathbf{e}_{L-1}\|^2} \mathbf{e}_{L-1}^T = \frac{e^{2(L-1)\beta_1}}{\|\mathbf{e}_{L-1}\|^4} \mathbf{E}_{L-1}. \end{aligned}$$

Thus,

$$\begin{aligned} \Pi &= \underline{V} \underline{V}^T + (1 - v^2) AA^T \\ &= \frac{1}{\|\mathbf{e}_L\|^2} \mathbf{E}_{L-1} + \left(1 - \frac{e^{2(L-1)\beta_1}}{\|\mathbf{e}_L\|^2}\right) \frac{e^{2(L-1)\beta_1}}{\|\mathbf{e}_{L-1}\|^4} \mathbf{E}_{L-1} \\ &= \left\{ \frac{1}{\|\mathbf{e}_L\|^2} + \left(1 - \frac{e^{2(L-1)\beta_1}}{\|\mathbf{e}_L\|^2}\right) \frac{e^{2(L-1)\beta_1}}{\|\mathbf{e}_{L-1}\|^4} \right\} \mathbf{E}_{L-1}. \end{aligned} \tag{15}$$

The new V-forecasting algorithm for exponential series can be presented as follows:

- 1) project the lagged vector  $X_i$  on  $\mathcal{L} = \text{span}\{\frac{\mathbf{e}_i}{\|\mathbf{e}_i\|}\}$  by  $\hat{X}_i = \frac{1}{\|\mathbf{e}_i\|^2} \mathbf{E}_L X_i$ ;
- 2) apply Relations (11) and (15), and define the linear operator  $\mathcal{P}_{\text{Vec}}: \mathbf{R}^L \mapsto \mathcal{L}$  as follows:

$$\mathcal{P}_{\text{Vec}} Z = \begin{pmatrix} \Pi \bar{Z} \\ A^T \bar{Z} \end{pmatrix}; \tag{16}$$

- 3) define the vectors  $W_i$  as:

$$W_i = \begin{cases} \hat{X}_i & \text{for } i = 1, \dots, K, \\ \mathcal{P}_{\text{Vec}} W_{i-1} & \text{for } i = K + 1, \dots, K + h + L - 1; \end{cases} \tag{17}$$

- 4) by constructing the matrix  $\mathbf{W} = [W_1 : \dots : W_{K+h+L-1}]$  and making its diagonal average, the series  $\{z_1, \dots, z_{N+h+L-1}\}$  is obtained;
- 5) the values  $z_{N+1}, \dots, z_{N+h}$  form the  $h$  step ahead vector forecasts for exponential series.

### 4. Empirical results

#### 4.1. Reconstruction

In this subsection, the performance of the proposed reconstruction approach is evaluated using simulated noisy exponential series with various noise levels. Consider the noisy exponential series:

$$y_t = \exp(0.1 + 0.01t) + n_t, \quad t = 1, 2, \dots, 100,$$

where  $n_t$  is the normally distributed noise series with zero mean. The accuracy of the results' reconstruction is measured using the Root Mean Squared Error (RMSE). The following RMSE ratio is applied to compare the proposed and common reconstruction methods:

$$RRMSE = \frac{\left(\sum_{t=1}^N (y_t - \tilde{y}_t)^2\right)^{1/2}}{\left(\sum_{t=1}^N (y_t - \tilde{y}_t)^2\right)^{1/2}}, \tag{18}$$

where  $\tilde{y}_t$  and  $\tilde{y}_t$  are reconstructed series at time  $t$  achieved by the proposed and common reconstruction methods, respectively. If  $RRMSE < 1$ , then the proposed reconstruction outperforms the competitor method. Alternatively, when  $RRMSE > 1$ , it would indicate that the performance of the proposed reconstruction is worse than the latter.

Table 1 reports the RRMSE values for the reconstruction of exponential series. The window length ( $L$ ) ranges from 2 to 50, and different values of the standard deviation of noise series have been used. As can be seen in this table, the proposed reconstruction always outperforms the common reconstruction for each value of the standard deviation. The performance of the proposed reconstruction rises as  $L$  reaches greater values. In addition, the proposed reconstruction is much better than the other one as the standard deviation increases.

**Table 1**  
RRMSE for reconstruction.

Standard deviation	<i>L</i>																
	2	5	8	11	14	17	20	23	26	29	32	35	38	41	44	47	50
1	1.000	0.998	0.995	0.990	0.984	0.976	0.965	0.952	0.938	0.927	0.906	0.889	0.866	0.841	0.807	0.778	0.726
4	0.984	0.987	0.976	0.971	0.960	0.948	0.933	0.922	0.911	0.890	0.875	0.845	0.832	0.793	0.766	0.735	0.697
6	0.966	0.930	0.906	0.879	0.871	0.848	0.820	0.812	0.784	0.761	0.737	0.724	0.687	0.668	0.647	0.614	0.580

**Table 2**  
RRMSE values for recurrent forecasting.

Forecasting horizon	Standard deviation	<i>L</i>													
		2	5	8	11	14	17	20	23	26	29	32	35	38	
<i>h</i> = 1	1	0.960	0.922	0.885	0.851	0.819	0.775	0.761	0.732	0.685	0.662	0.625	0.593	0.580	
	4	0.870	0.863	0.817	0.774	0.726	0.683	0.648	0.616	0.615	0.599	0.544	0.512	0.485	
	6	0.854	0.842	0.758	0.736	0.675	0.635	0.597	0.553	0.532	0.505	0.487	0.448	0.437	
<i>h</i> = 3	1	0.909	0.882	0.853	0.823	0.791	0.747	0.732	0.698	0.671	0.633	0.595	0.583	0.547	
	4	0.817	0.786	0.754	0.742	0.714	0.671	0.629	0.612	0.573	0.569	0.513	0.511	0.460	
	6	0.756	0.732	0.719	0.663	0.629	0.604	0.575	0.566	0.537	0.500	0.474	0.453	0.414	
<i>h</i> = 6	1	0.823	0.818	0.794	0.764	0.745	0.715	0.673	0.664	0.631	0.592	0.562	0.544	0.523	
	4	0.724	0.678	0.663	0.647	0.644	0.612	0.603	0.535	0.535	0.526	0.486	0.461	0.446	
	6	0.695	0.658	0.645	0.627	0.603	0.572	0.536	0.514	0.494	0.485	0.458	0.423	0.424	
<i>h</i> = 12	1	0.757	0.733	0.703	0.681	0.650	0.629	0.606	0.584	0.562	0.541	0.533	0.527	0.519	
	4	0.702	0.687	0.653	0.622	0.580	0.573	0.544	0.532	0.529	0.517	0.501	0.493	0.481	
	6	0.669	0.647	0.633	0.612	0.573	0.557	0.532	0.523	0.517	0.495	0.488	0.476	0.467	
<i>h</i> = 24	1	0.572	0.525	0.513	0.499	0.481	0.457	0.441	0.438	0.414	0.402	0.364	0.354	0.336	
	4	0.475	0.455	0.423	0.395	0.381	0.378	0.361	0.353	0.334	0.324	0.311	0.306	0.286	
	6	0.445	0.431	0.409	0.387	0.372	0.368	0.356	0.332	0.321	0.319	0.307	0.287	0.264	

4.2. Forecasting

In this subsection, the performance of the proposed recurrent and vector forecasting algorithms are compared with common versions of them by applying simulated exponential series. The series is split into two sets: the training and the testing part. The accuracy of the results' forecasting is measured using the widely used metric, Root Mean Squared Error (RMSE):

$$RRMSE_h = \frac{\left(\sum_{t=m}^{m+n-h} (y_{t+h} - \hat{y}_{t+h|t})^2\right)^{1/2}}{\left(\sum_{t=m}^{m+n-h} (y_{t+h} - \hat{y}_{t+h|t}^c)^2\right)^{1/2}}, \tag{19}$$

where, *m* is the length of training sample, *n* is the length of test sample, *h* is the length of forecast horizon,  $\hat{y}_{t+h|t}$  is the *h*-step ahead forecast obtained by the proposed recurrent (or vector) forecasting and  $\hat{y}_{t+h|t}^c$  is the *h*-step ahead forecast taken through the common recurrent (or vector) forecasting. If  $RRMSE_h < 1$ , then the proposed recurrent (or vector) forecasting outperforms the competitor method at the horizon *h*. Alternatively, when  $RRMSE_h > 1$ , it would indicate that the performance of the proposed recurrent (or vector) forecasting is worse than the other one.

Let us now consider the following noisy exponential series:

$$y_t = \exp(0.01t) + n_t, \quad t = 1, 2, \dots, 100,$$

where  $n_t$  is the normally distributed noise series with zero mean. The first 70 observations were considered as training samples (*m* = 70) and the rest as test samples (*n* = 30).

Tables 2 and 3 report the RRMSE for recurrent and vector forecasting at *h* = 1, 3, 6, 12, and 24 forecast horizons using 1000 iterations. For each *h*, different values of the standard deviation for the noise series has been used. As can be seen from Tables 2 and 3, all RRMSEs are less than one, confirming that the proposed recurrent and vector forecasting approaches outperform the common basic recurrent and vector forecasting at all forecast horizons, for each value of the window length (*L*) and different values of the standard deviation of the noise series. It is noteworthy that the efficiency of the proposed forecasting methods increases as *L* increases. Additionally, for each *L* and *h*, the efficiency increases as the standard deviation intensifies.

**Table 3**  
RRMSE for vector forecasting.

Forecasting horizon	Standard deviation	$L$												
			2	5	8	11	14	17	20	23	26	29	32	35
$h = 1$	1	0.959	0.929	0.904	0.874	0.851	0.834	0.804	0.795	0.766	0.749	0.726	0.716	0.697
	4	0.906	0.899	0.848	0.804	0.779	0.748	0.733	0.722	0.700	0.687	0.666	0.653	0.630
	6	0.855	0.840	0.803	0.788	0.759	0.725	0.703	0.681	0.670	0.656	0.647	0.631	0.600
$h = 3$	1	0.908	0.891	0.864	0.840	0.815	0.787	0.767	0.749	0.718	0.703	0.673	0.653	0.630
	4	0.883	0.872	0.841	0.807	0.779	0.762	0.743	0.716	0.690	0.664	0.643	0.621	0.598
	6	0.858	0.836	0.808	0.781	0.764	0.745	0.731	0.704	0.678	0.650	0.635	0.596	0.568
$h = 6$	1	0.831	0.827	0.810	0.787	0.766	0.738	0.724	0.696	0.681	0.655	0.625	0.597	0.588
	4	0.815	0.808	0.789	0.770	0.740	0.718	0.677	0.664	0.629	0.616	0.604	0.567	0.520
	6	0.781	0.763	0.753	0.730	0.727	0.709	0.665	0.650	0.607	0.590	0.582	0.556	0.504
$h = 12$	1	0.782	0.746	0.719	0.701	0.679	0.659	0.639	0.620	0.597	0.568	0.557	0.537	0.508
	4	0.763	0.726	0.700	0.676	0.648	0.640	0.620	0.593	0.577	0.551	0.530	0.515	0.486
	6	0.735	0.711	0.688	0.654	0.639	0.625	0.618	0.583	0.564	0.537	0.526	0.507	0.476
$h = 24$	1	0.720	0.672	0.612	0.585	0.570	0.558	0.539	0.517	0.499	0.468	0.456	0.441	0.426
	4	0.681	0.661	0.607	0.576	0.547	0.539	0.518	0.489	0.465	0.451	0.436	0.407	0.402
	6	0.666	0.643	0.582	0.562	0.533	0.520	0.497	0.470	0.451	0.439	0.422	0.393	0.371

## 5. Conclusion

The current study proposes a closed form of filtered LRR coefficients (or exact  $\alpha$ -coefficients) for exponential time series. On account of the fact that the  $\alpha$ -coefficients play a fundamental role in both the recurrent and the vector forecasting approach within the SSA framework, the accuracy of filtering and forecasting results are enhanced using the filtered closed form of the  $\alpha$ -coefficients.

The simulation results confirmed the superiority of the proposed approach. The results also indicate that the length of the time series ( $N$ ) and  $\beta_0$  do not have any effect on the  $\alpha$ -coefficients.

Accordingly, employing the proposed approach is recommended for filtering and forecasting time series where there exists an exponential-like trend as part of the series component.

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