# Reverses of operator Aczél inequality 

## Inégalités inverses aux inégalités d'Aczél pour les opérateurs

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#### Abstract

In this paper, we present some inequalities involving operator decreasing functions and operator means. These inequalities provide some reverses of the operator Aczél inequality


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## R É S U M É

Nous présentons dans cette Note des inégalités faisant intervenir des fonctions décroissantes sur les opérateurs et des moyennes d'opérateurs. Ces inégalités fournissent des inverses aux inégalités d'Aczél pour les opérateurs dans le cas des moyennes géométriques pondérées.
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## 1. Introduction

Let $B(\mathcal{H})$ denote the $C^{*}$-algebra of all bounded linear operators on a Hilbert space $(\mathcal{H},\langle\cdot, \cdot\rangle)$. An operator $A \in B(\mathcal{H})$ is called positive if $\langle A x, x\rangle \geq 0$ for every $x \in \mathcal{H}$ and then we write $A \geq 0$. For self-adjoint operators $A, B \in B(\mathcal{H})$, we say $A \leq B$ if $B-A \geq 0$. Also we say that $A$ is strictly positive and we write $A>0$ if $\langle A x, x\rangle>0$ for every $x \in \mathcal{H}$ with $x \neq 0$. Let $f$ be a continuous real function on $(0, \infty)$. Then $f$ is said to be operator monotone (more precisely, operator monotone increasing) if $A \geq B$ implies $f(A) \geq f(B)$ for strictly positive operators $A, B$, and operator monotone decreasing if $-f$ is operator monotone or $A \geq B$ implies $f(A) \leq f(B)$.

Also, $f$ is said to be operator convex if $f(\alpha A+(1-\alpha) B) \leq \alpha f(A)+(1-\alpha) f(B)$ for all strictly positive operators $A, B$ and $\alpha \in[0,1]$, and operator concave if $-f$ is operator convex.

In 1956, Aczél [1] proved that if $a_{i}, b_{i}(1 \leq i \leq n)$ are positive real numbers such that $a_{1}^{2}-\sum_{i=2}^{n} a_{i}^{2}>0$ and $b_{1}^{2}-$ $\sum_{i=2}^{n} b_{i}^{2}>0$, then

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$$
\left(a_{1}^{2}-\sum_{i=2}^{n} a_{i}^{2}\right)\left(b_{1}^{2}-\sum_{i=2}^{n} b_{i}^{2}\right) \leq\left(a_{1} b_{1}-\sum_{i=2}^{n} a_{i} b_{i}\right)^{2} .
$$

Aczél's inequality has important applications in the theory of functional equations in non-Euclidean geometry [1,16] and references therein. In recent years, considerable attention has been given to this inequality involving its generalizations, variations, and applications. See [5,6,14] and references therein. Popoviciu [14] first presented an exponential extension of Aczél's inequality as follows:

Theorem A. Let $p>0, q>0, \frac{1}{p}+\frac{1}{q}=1, a_{1}^{p}-\sum_{i=2}^{n} a_{i}^{p}>0$, and $b_{1}^{q}-\sum_{i=2}^{n} b_{i}^{q}>0$. Then

$$
\left(a_{1}^{p}-\sum_{i=2}^{n} a_{i}^{p}\right)^{\frac{1}{p}}\left(b_{1}^{q}-\sum_{i=2}^{n} b_{i}^{q}\right)^{\frac{1}{q}} \leq a_{1} b_{1}-\sum_{i=2}^{n} a_{i} b_{i} .
$$

Aczél's and Popoviciu's inequalities were sharpened and a variant of Aczél's inequality in inner product spaces was given by Dragomir [6] by establishing the following theorem.

Theorem B. Let $a, b$ be real numbers and $x, y$ be vectors of an inner product space such that $a^{2}-\|x\|^{2}>0$ or $b^{2}-\|y\|^{2}>0$. Then

$$
\begin{equation*}
\left(a^{2}-\|x\|^{2}\right)\left(b^{2}-\|y\|^{2}\right) \leq(a b-\operatorname{Re}\langle x, y\rangle)^{2} . \tag{1}
\end{equation*}
$$

Moslehian in [13] proved an operator version of the classical Aczél inequality involving $\alpha$-geometric mean $A \not \sharp_{\alpha} B=$ $A^{1 / 2}\left(A^{-1 / 2} B A^{-1 / 2}\right)^{\alpha} A^{1 / 2}$, in the following form.

Theorem C. Let g be a non-negative operator decreasing and operator concave function on $(0, \infty), \frac{1}{p}+\frac{1}{q}=1, p, q>1$, and $A$ and $B$ be strictly positive operators. Then

$$
\begin{align*}
& g\left(A^{p}\right) \sharp_{\frac{1}{q}} g\left(B^{q}\right) \leq g\left(A^{p} \sharp_{\frac{1}{q}} B^{q}\right),  \tag{2}\\
& \left\langle g\left(A^{p}\right) \xi, \xi\right\rangle^{\frac{1}{p}}\left\langle g\left(B^{q}\right) \xi, \xi\right\rangle^{\frac{1}{q}} \leq\left\langle g\left(A^{p} \sharp_{\frac{1}{q}} B^{q}\right) \xi, \xi\right\rangle \tag{3}
\end{align*}
$$

for all $\xi \in \mathcal{H}$.
In this paper, we present some reverses of operator Aczél inequalities (2) and (3) by using several reverse Young's inequalities. In fact, we show some upper bounds for the inequalities in Theorem C. These results are proved for a nonnegative operator decreasing function $g$ and the condition of operator concavity has been omitted. So, we use less restrictive conditions on $g$. The statements are organized in two sections related to different coefficients.

## 2. Reverse inequalities via Kantorovich's constant

Let $A$ and $B$, be strictly positive operators. For each $\alpha \in[0,1]$ the $\alpha$-arithmetic mean is defined as $A \nabla_{\alpha} B:=(1-\alpha) A+$ $\alpha B$, and the $\alpha$-geometric mean is

$$
A \not \sharp_{\alpha} B=A^{1 / 2}\left(A^{-1 / 2} B A^{-1 / 2}\right)^{\alpha} A^{1 / 2} .
$$

Clearly, if $A B=B A$, then $A \not \sharp_{\alpha} B=A^{1-\alpha} B^{\alpha}$. Basic properties of the arithmetic and geometric means can be found in [10]. It is well-known as the Young inequality:

$$
A \sharp_{\alpha} B \leq(1-\alpha) A+\alpha B .
$$

The celebrated Kantorovich constant is defined by

$$
\begin{equation*}
K(t)=\frac{(t+1)^{2}}{4 t}, \quad t>0 . \tag{4}
\end{equation*}
$$

The function $K$ is decreasing on $(0,1)$ and increasing on $[1, \infty), K(t)=K\left(\frac{1}{t}\right)$, and $K(t) \geq 1$ for every $t>0$ [10].
The research on the Young inequality is interesting, and there are several multiplicative and additive reverses of this inequality [7,12]. One of this reverse inequalities is given by Liao et al. [12] using the Kantorovich constant as follows.

Lemma 1. [12, Theorem 3.1] Let $A, B$ be positive operators satisfying the following conditions $0<m I \leq A \leq m^{\prime} I \leq M^{\prime} I \leq B \leq M I$ or $0<m I \leq B \leq m^{\prime} I \leq M^{\prime} I \leq A \leq M I$, for some constants $m, m^{\prime}, M, M^{\prime}$. Then

$$
\begin{equation*}
(1-\alpha) A+\alpha B \leq K(h)^{R}\left(A \not \sharp_{\alpha} B\right) \tag{5}
\end{equation*}
$$

where $h=\frac{M}{m}, \alpha \in[0,1], R=\max \{1-\alpha, \alpha\}$ and $K(h)$ is the Kantorovich constant, defined as in (4).
In the following, we generalize Lemma 1 with the more general sandwich condition $0<s A \leq B \leq t A$. The sketch of the proof is similar to that of [17, Theorem 2.1].

Lemma 2. Let $0<s A \leq B \leq t A$ for some scalars $0<s \leq t$ and $\alpha \in[0,1]$. Then

$$
\begin{equation*}
(1-\alpha) A+\alpha B \leq \max \left\{K(s)^{R}, K(t)^{R}\right\}\left(A \not \sharp_{\alpha} B\right), \tag{6}
\end{equation*}
$$

where $R=\max \{\alpha, 1-\alpha\}$ and $K(t)$ is the Kantorovich constant, defined as in (4).
Proof. From [12, Corollary 2.2] if $x$ is a positive number and $\alpha \in[0,1]$, then

$$
(1-\alpha)+\alpha x \leq K(x)^{R} x^{\alpha}
$$

Thus for every strictly positive operator $0<s I \leq C \leq t I$, we have

$$
(1-\alpha)+\alpha C \leq \max _{s \leq x \leq t} K(x)^{R} C^{\alpha}
$$

Substituting $A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$ for $C$, we get

$$
(1-\alpha)+\alpha A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \leq \max _{s \leq x \leq t} K(x)^{R}\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\alpha}
$$

Multiplying $A^{\frac{1}{2}}$ to the both sides in the above inequality, and using the fact that $\max _{s \leq x \leq t} K(x)=\max \{K(s), K(t)\}$, the desired inequality is obtained.

Remark 1. We remark that Lemma 2 is a generalization of Lemma 1. Since, if $0<m I \leq A \leq m^{\prime} I \leq M^{\prime} I \leq B \leq M I$ or $0<m I \leq$ $B \leq m^{\prime} I \leq M^{\prime} I \leq A \leq M I$, then $\frac{m}{M} A \leq B \leq \frac{M}{m} A$. Now by letting $s=\frac{m}{M}$ and $t=\frac{M}{m}$ in Lemma 2, the inequality (5) is obtained. Note that $K(t)=K\left(\frac{1}{t}\right)$ for every $t>0$.

Lemma 3. Let $g$ be a non-negative operator monotone decreasing function on $(0, \infty)$ and $A$ be a strictly positive operator. Then, for every scalar $\lambda \geq 1$

$$
\frac{1}{\lambda} g(A) \leq g(\lambda A)
$$

Proof. First note that since $g$ is analytic on $(0, \infty)$, we may assume that $g(x)>0$ for all $x>0$; otherwise, $g$ is identically zero. Also, since $g$ is an operator monotone decreasing on $(0, \infty)$, so $f=1 / g$ is an operator monotone on $(0, \infty)$ and hence an operator concave function [3]. On the other hand, it is known that, for every non-negative concave function $f$ and $\lambda \geq 1$, $f(\lambda x) \leq \lambda f(x)$. Therefore, for every $\lambda \geq 1$, we have

$$
(g(\lambda A))^{-1} \leq \lambda(g(A))^{-1}
$$

Reversing this inequality, we obtain the result.
Proposition 1. Let $g$ be a non-negative operator monotone decreasing function on $(0, \infty)$ and $0<s A \leq B \leq t A$ for some constants $0<s \leq t$. Then, for all $\alpha \in[0,1]$

$$
\begin{equation*}
g\left(A \not \sharp_{\alpha} B\right) \leq \max \left\{K(s)^{R}, K(t)^{R}\right\}\left(g(A) \not \sharp_{\alpha} g(B)\right), \tag{7}
\end{equation*}
$$

where $R=\max \{\alpha, 1-\alpha\}$ and $K(t)$ is the Kantorovich constant, defined as in (4).
Proof. Since $0<s A \leq B \leq t A$, from Lemma 2, we have

$$
(1-\alpha) A+\alpha B \leq \lambda\left(A \not \sharp_{\alpha} B\right),
$$

where $\lambda=\max \left\{K(s)^{R}, K(t)^{R}\right\}$. We know that $\lambda \geq 1$. Also, the function $g$ is operator monotone decreasing and so

$$
g\left(\lambda\left(A \not \sharp_{\alpha} B\right)\right) \leq g((1-\alpha) A+\alpha B) .
$$

Now, we can write

$$
\frac{1}{\lambda} g\left(A \sharp_{\alpha} B\right) \leq g\left(\lambda\left(A \sharp_{\alpha} B\right)\right) \leq g((1-\alpha) A+\alpha B) \leq g(A) \sharp_{\alpha} g(B),
$$

where the first inequality follows from Lemma 3 and the last inequality follows from [2, Theorem 2.1].
Now, we can give the reverse of inequalities in Theorem $C$ as follows.
Theorem 1. Let $g$ be a non-negative operator monotone decreasing function on $(0, \infty), \frac{1}{p}+\frac{1}{q}=1, p, q>1$, and $0<s A^{p} \leq B^{q} \leq t A^{p}$ for some constants $s, t$. Then, for all $\xi \in \mathcal{H}$

$$
\begin{align*}
& g\left(A^{\left.{ }_{H_{\frac{1}{q}}} B^{q}\right) \leq \max \left\{K(s)^{R}, K(t)^{R}\right\}\left(g\left(A^{p}\right) \sharp_{\frac{1}{q}} g\left(B^{q}\right)\right),}\right.  \tag{8}\\
& \left\langle g\left(A^{p} \sharp_{\frac{1}{q}} B^{q}\right) \xi, \xi\right\rangle \leq \max \left\{K(s)^{R}, K(t)^{R}\right\}\left\langle g\left(A^{p}\right) \xi, \xi\right\rangle^{\frac{1}{p}}\left\langle g\left(B^{q}\right) \xi, \xi\right\rangle^{\frac{1}{q}}, \tag{9}
\end{align*}
$$

where $R=\max \left\{\frac{1}{p}, \frac{1}{q}\right\}$, and $K(t)$ is the Kantorovich constant defined as in (4).
Proof. Letting $\alpha=\frac{1}{q}$ and replacing $A^{p}$ and $B^{q}$ with $A$ and $B$ in Proposition 1, we reach the inequality (8). To prove the inequality (9), first note that under the condition $0<s A^{p} \leq B^{q} \leq t A^{p}$ from Lemma 2, we have

$$
A^{p} \nabla_{\alpha} B^{q} \leq \max \left\{K(s)^{R}, K(t)^{R}\right\}\left(A^{\left.p_{\sharp_{\alpha}} B^{q}\right) . ~}\right.
$$

For convenience, set $\lambda=\max \left\{K(s)^{R}, K(t)^{R}\right\}$. So, for the operator monotone decreasing functions $g$ and $\alpha=\frac{1}{q}$

$$
\begin{equation*}
g\left(\lambda\left(A^{p}{\underset{\frac{1}{q}}{ }} B^{q}\right)\right) \leq g\left(A^{p} \nabla_{\frac{1}{q}} B^{q}\right) . \tag{10}
\end{equation*}
$$

Now compute

$$
\begin{aligned}
\left\langle g\left(A^{p} \sharp_{\frac{1}{q}} B^{q}\right) \xi, \xi\right\rangle & \leq \lambda\left\langleg \left(\lambda \left( A^{\left.\left.\left.p_{H_{\frac{1}{q}}} B^{q}\right)\right) \xi, \xi\right\rangle}\right.\right.\right. \\
& \left.\leq \lambda\left\langle g\left(A^{p} \nabla_{\frac{1}{q}} B^{q}\right)\right) \xi, \xi\right\rangle \\
& \leq \lambda\left\langle g\left(A^{p}\right) \sharp_{\frac{1}{q}} g\left(B^{q}\right) \xi, \xi\right\rangle \\
& \leq \lambda\left\langle g\left(A^{p}\right) \xi, \xi\right\rangle^{\frac{1}{p}}\left\langle g\left(B^{q}\right) \xi, \xi\right\rangle^{\frac{1}{q}},
\end{aligned}
$$

where the first inequality follows from Lemma 3 and the second follows from inequality (10). For the third inequality, we use the log-convexity property of operator monotone decreasing functions [2, Theorem 2.1], and, in the last inequality, we use the fact that, for every positive operators $A, B$ and every $\xi \in \mathcal{H},\left\langle A \not \sharp_{\alpha} B \xi, \xi\right\rangle \leq\langle A \xi, \xi\rangle^{1-\alpha}\langle B \xi, \xi\rangle^{\alpha}$ [4, Lemma 8]. So, we achieve

$$
\left\langle g\left(A^{p_{\Psi_{1}}} B^{q}\right) \xi, \xi\right\rangle \leq \max \left\{K(s)^{R}, K(t)^{R}\right\}\left\langle g\left(A^{p}\right) \xi, \xi\right\rangle^{\frac{1}{p}}\left\langle g\left(B^{q}\right) \xi, \xi\right\rangle^{\frac{1}{q}},
$$

as desired.
Corollary 1. Let $\frac{1}{p}+\frac{1}{q}=1, p, q \geq 1$, and $A, B$ be commuting positive operators with spectra contained in $(0,1)$ such that $0<s A^{p} \leq$ $B^{q} \leq t A^{p}$ for some constants $s, t$. Then, for every unit vector $\xi \in \mathcal{H}$

$$
\begin{equation*}
1-\left\|(A B)^{\frac{1}{2}} \xi\right\|^{2} \leq \max \left\{K(s)^{R}, K(t)^{R}\right\}\left(1-\left\|A^{\frac{p}{2}} \xi\right\|^{2}\right)^{\frac{1}{p}}\left(1-\left\|B^{\frac{q}{2}} \xi\right\|^{2}\right)^{\frac{1}{q}}, \tag{11}
\end{equation*}
$$

and consequently

$$
1-\|A B \xi\|^{2} \leq \max \left\{K\left(s^{2}\right)^{R}, K\left(t^{2}\right)^{R}\right\}\left(1-\left\|A^{p} \xi\right\|^{2}\right)^{\frac{1}{p}}\left(1-\left\|B^{q} \xi\right\|^{2}\right)^{\frac{1}{q}},
$$

where $R=\max \left\{\frac{1}{p}, \frac{1}{q}\right\}$.
Proof. The first inequality is obtained by applying Theorem 1 to the function $g(t)=1-t$ on $(0,1)$ and the fact that $A^{p} \sharp_{\frac{1}{q}} B^{q}=A B$. Also, for every positive operator $A$

$$
\langle A \xi, \xi\rangle=\left\langle A^{\frac{1}{2}} \xi, A^{\frac{1}{2}} \xi\right\rangle=\left\|A^{\frac{1}{2}} \xi\right\|^{2}
$$

For the second inequality, note that since $A B=B A$, from the sandwich condition $0<s A^{p} \leq B^{q} \leq t A^{p}$ we have $0<s^{2} A^{2 p} \leq$ $B^{2 q} \leq t^{2} A^{2 p}$. Now replacing $A^{2}$ and $B^{2}$ with $A$ and $B$ in (11), the assertion is obtained.

Remark 2. Moslehian in [13, Corollary 2.4], showed the operator version of Aczél inequality (1) as follows:

$$
\begin{equation*}
\left(1-\left\|A^{\frac{p}{2}} \xi\right\|^{2}\right)^{\frac{1}{p}}\left(1-\left\|B^{\frac{q}{2}} \xi\right\|^{2}\right)^{\frac{1}{q}} \leq 1-\left\|(A B)^{\frac{1}{2}} \xi\right\|^{2} \tag{12}
\end{equation*}
$$

where $A$ and $B$ are commuting positive operators with spectra contained in $(0,1)$, and $\frac{1}{p}+\frac{1}{q}=1$ for $p, q \geq 1$. As it can be seen, inequality (11) in Corollary 1 provides an upper bound for the operator Aczél inequality (12).

Corollary 2. Let $g$ be a non-negative operator monotone decreasing function on $(0, \infty)$ and $A, B$ be commuting positive operators such that $0<s A^{p} \leq B^{q} \leq t A^{p}$ for some constants $s, t$. Then

$$
\begin{equation*}
g(A B) \leq \max \left\{K(s)^{R}, K(t)^{R}\right\} g\left(A^{p}\right)^{\frac{1}{p}} g\left(B^{q}\right)^{\frac{1}{q}}, \tag{13}
\end{equation*}
$$

where $R=\max \left\{\frac{1}{p}, \frac{1}{q}\right\}$.
Corollary 3. Let $g$ be a non-negative decreasing function on $(0, \infty)$ and $a_{i}, b_{i}$ be positive numbers such that $0<s \leq \frac{\left(b_{i}\right)^{q}}{\left(a_{i}\right)^{p}} \leq t$ for some constants s, t. Then

$$
\begin{equation*}
\sum_{i=1}^{n} g\left(a_{i} b_{i}\right) \leq \max \left\{K(s)^{R}, K(t)^{R}\right\}\left(\sum_{i=1}^{n} g\left(a_{i}^{p}\right)\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n} g\left(b_{i}^{q}\right)\right)^{\frac{1}{q}} \tag{14}
\end{equation*}
$$

where $R=\max \left\{\frac{1}{p}, \frac{1}{q}\right\}$.
Proof. Let $A\left(x_{1}, x_{2}, \cdots, x_{n}\right)=\left(a_{1} x_{1}, a_{2} x_{2}, \cdots, a_{n} x_{n}\right)$ and $B\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\left(b_{1} x_{1}, b_{2} x_{2}, \cdots, b_{n} x_{n}\right)$ be positive operators acting on Hilbert space $\mathcal{H}=C^{n}$ and $\xi=(1,1, \ldots, 1)$. Now, by applying inequality (9) to operators $A$ and $B$, we get inequality (14).

## 3. Some related results

Dragomir in [7, Theorem 6], gave another reverse inequality for Young's inequality as follows.

Lemma 4. Let $A, B$ be positive operators such that $0<s A \leq B \leq t A$ for some constants $s, t$. Then, for all $\alpha \in[0,1]$

$$
\begin{equation*}
(1-\alpha) A+\alpha B \leq \exp \left(\frac{1}{2} \alpha(1-\alpha)\left(\frac{\max \{1, t\}}{\min \{1, s\}}-1\right)^{2}\right)\left(A \not{ }_{\alpha} B\right) \tag{15}
\end{equation*}
$$

By using this new ratio, we can express some other operator reverse inequalities. The proofs are similar to that of the preceding section.

Proposition 2. Let $g$ be a non-negative operator monotone decreasing function on $(0, \infty)$ and $0<s A \leq B \leq t A$ for some constants $s$, $t$. Then, for all $\alpha \in[0,1]$

$$
g\left(A \sharp_{\alpha} B\right) \leq \exp \left(\frac{1}{2} \alpha(1-\alpha)\left(\frac{\max \{1, t\}}{\min \{1, s\}}-1\right)^{2}\right)\left(g(A) \sharp_{\alpha} g(B)\right)
$$

Proof. The assertion is obtained similar to the proof of Proposition 1, by applying inequality (15) instead of inequality (6). Note that, for every $0 \leq \alpha \leq 1$ and $s, t>0, \exp \left(\frac{1}{2} \alpha(1-\alpha)\left(\frac{\max \{1, t\}}{\min \{1, s\}}-1\right)^{2}\right) \geq 1$.

Theorem 2. Let $g$ be a non-negative operator monotone decreasing function on ( $0, \infty$ ), $\frac{1}{p}+\frac{1}{q}=1, p, q>1$, and $0<s A^{p} \leq B^{q} \leq t A^{p}$ for some constants $s, t$. Then, for all $\xi \in \mathcal{H}$

$$
\begin{aligned}
& g\left(A^{p} \sharp_{\frac{1}{q}} B^{q}\right) \leq \exp \left(\frac{1}{2 p q}\left(\frac{\max \{1, t\}}{\min \{1, s\}}-1\right)^{2}\right)\left(g\left(A^{p}\right) \sharp_{\frac{1}{q}} g\left(B^{q}\right)\right), \\
& \left\langle g\left(A^{p} \sharp_{\frac{1}{q}} B^{q}\right) \xi, \xi\right\rangle \leq \exp \left(\frac{1}{2 p q}\left(\frac{\max \{1, t\}}{\min \{1, s\}}-1\right)^{2}\right)\left\langle g\left(A^{p}\right) \xi, \xi\right\rangle^{\frac{1}{p}}\left\langle g\left(B^{q}\right) \xi, \xi\right\rangle^{\frac{1}{q}} .
\end{aligned}
$$

In [9, Theorem B], another reverse Young inequality is presented as follows.

Lemma 5. Let $A$ and $B$ be positive operators such that $0<s A \leq B \leq A$ for a constant $s$ and $\alpha \in[0,1]$. Then

$$
(1-\alpha) A+\alpha B \leq M_{\alpha}(s)\left(A \not \sharp_{\alpha} B\right),
$$

where $M_{\alpha}(s)=1+\frac{\alpha(1-\alpha)(s-1)^{2}}{2 s^{\alpha+1}}$.
Now, by using this new constant, the similar reverse Aczél inequalities are obtained. Note that $M_{\alpha}(s) \geq 1$ for every $\alpha \in[0,1]$. See [9] for more properties of $M_{\alpha}(s)$.

Proposition 3. Let $g$ be a non-negative operator monotone decreasing function on ( $0, \infty$ ), $0<s A \leq B \leq A$ for a constant $s$ and $\alpha \in[0,1]$. Then

$$
g\left(A \not \sharp_{\alpha} B\right) \leq M_{\alpha}(S)\left(g(A) \not \sharp_{\alpha} g(B)\right) .
$$

Theorem 3. Let $g$ be a non-negative operator monotone decreasing function on $(0, \infty), \frac{1}{p}+\frac{1}{q}=1, p, q>1$, and $0<s A^{p} \leq B^{q} \leq A^{p}$ for a constant s. Then, for all $\xi \in \mathcal{H}$,

$$
\begin{aligned}
& g\left(A^{p_{\sharp_{1}}} B^{q}\right) \leq M_{\frac{1}{q}}(s)\left(g\left(A^{p}\right) \sharp_{\frac{1}{q}} g\left(B^{q}\right)\right), \\
& \left\langle g\left(A^{p_{\sharp_{1}}} B^{q}\right) \xi, \xi\right\rangle \leq M_{\frac{1}{q}}(s)\left\langle g\left(A^{p}\right) \xi, \xi\right\rangle^{\frac{1}{p}}\left\langle g\left(B^{q}\right) \xi, \xi\right\rangle^{\frac{1}{q}} .
\end{aligned}
$$

Remark 3. We clearly see that the condition $0<s A \leq B \leq t A$ for some $s \leq t$ in Lemma 4 is more general than the condition $0<s A \leq B \leq A$ for $s \leq 1$ in Lemma 5. But under the same condition $0<s A \leq B \leq A$, the constant appearing in Lemma 5 gives a better estimate than the ones in Lemma 4. In fact, we have

$$
M_{\alpha}(s) \leq \exp \left(\frac{1}{2} \alpha(1-\alpha)\left(\frac{1}{s}-1\right)^{2}\right)
$$

for every $\alpha \in[0,1]$ and $0<s \leq 1$ [8, Proposition 2.10].
In [11], it is shown that if $f:[0, \infty) \longrightarrow[0, \infty)$ is an operator monotone function and $0<s A \leq B \leq t A$ for some constants $s, t$, then for all $\alpha \in[0,1]$

$$
f(A) \not \sharp_{\alpha} f(B) \leq \max \{S(s), S(t)\} f\left(A \not \sharp_{\alpha} B\right),
$$

where $S(t)=\frac{t^{\frac{1}{t-1}}}{e \log t^{\frac{1}{t-1}}}$ for $t>0$ is the so-called Specht's ratio [10,15]. As a result, we can show, for a non-negative operator monotone decreasing function $g$ on $(0, \infty), 0<s A \leq B \leq t A$, and $\alpha \in[0,1]$

$$
\begin{equation*}
g\left(A \sharp_{\alpha} B\right) \leq \max \{S(s), S(t)\}\left(g(A) \sharp_{\alpha} g(B)\right) . \tag{16}
\end{equation*}
$$

Hence, one can deduce another reverse of operator Aczél inequality with the constant $\max \{S(p), S(q)\}$, which is independent of $\alpha$.

Remark 4. In this paper, three evaluation expressions are derived. In the following, we show that there is no ordering between the appeared estimates.
(1) Comparison of the constants in Lemma 2 and in Lemma 4:

Let $0<s A \leq B \leq t A$ for some constants $s, t$ and $\alpha \in[0,1]$. Also, with no loss of generality let $s<t<1$. Since $K$ is decreasing function on $(0,1)$, by Lemma 2 , we have

$$
(1-\alpha) A+\alpha B \leq K(s)^{R}\left(A \sharp_{\alpha} B\right)
$$

where $R=\max \{\alpha, 1-\alpha\}$. Also, be Lemma 4,

$$
(1-\alpha) A+\alpha B \leq \exp \left(\frac{1}{2} \alpha(1-\alpha)\left(\frac{1}{s}-1\right)^{2}\right) A \not \sharp_{\alpha} B .
$$

Now, the following numerical examples show that there is no ordering between them:
(i) take $\alpha=0.9$ and $s=0.3$, then we have

$$
\max \left\{K(s)^{\alpha}, K(s)^{1-\alpha}\right\}-\exp \left(\frac{1}{2} \alpha(1-\alpha)\left(\frac{1}{s}-1\right)^{2}\right) \simeq 0.0833059
$$

(ii) take $\alpha=0.3$ and $s=0.3$, then we have

$$
\max \left\{K(s)^{\alpha}, K(s)^{1-\alpha}\right\}-\exp \left(\frac{1}{2} \alpha(1-\alpha)\left(\frac{1}{s}-1\right)^{2}\right) \simeq-0.500368
$$

(2) Comparison of the constants in inequality (16) and in inequality (7) of Proposition 1.

Let $0<s A \leq B \leq t A$ for some constants $s, t$ and $\alpha \in[0,1]$. Also, with no loss of generality let $1 \leq s \leq t$. Then, from inequality (16), we have

$$
g\left(A \nVdash_{\alpha} B\right) \leq S(t)\left(g(A) \sharp_{\alpha} g(B)\right),
$$

and from inequality (7)

$$
g\left(A \sharp_{\alpha} B\right) \leq K(t)^{R}\left(g(A) \not \sharp_{\alpha} g(B)\right),
$$

where $R=\max \{\alpha, 1-\alpha\}$. We compare the coefficients of these inequalities as follows:
(i) take $\alpha=0.8$ and $t=9$, then

$$
\max \left\{K(t)^{\alpha}, K(t)^{1-\alpha}\right\}-S(t) \simeq 0.501632
$$

(ii) take $\alpha=0.1$ and $t=9$, then

$$
\max \left\{K(t)^{\alpha}, K(t)^{1-\alpha}\right\}-S(t) \simeq-0.655227
$$

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