On a conjecture of Faulhuber and Steinerberger on the logarithmic derivative of $\vartheta_4$*

De la conjecture de Faulhuber et Steinerberger sur la dérivée logarithmique de $\vartheta_4$

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1. Introduction

The Jacobi $\vartheta$-functions are a classical topic of perennial interest. They appear in many fields of pure and applied mathematics. The analytic properties and the behavior of these functions are crucial for the applications. These properties have been studied, for instance, in [1–9].

In the following, we are interested in the classical Jacobi $\vartheta$-function $\vartheta_4$: we set

$$\vartheta_4(y) = \sum_{k=-\infty}^{\infty} (-1)^k e^{-nk^2y} = \prod_{n=1}^{\infty} \left(1 - e^{-2n\pi y}\right)^2 \left(1 - e^{(2n-1)\pi y}\right)^2.$$
for \( y \in [0, \infty] \). Of course, \( \vartheta_4 \) is usually defined as a modular form in the upper complex half-plane, but as we are only interested in the values on the positive imaginary axis, we employ the common and very convenient abuse of notation of rotating the positive imaginary axis to the positive real axis.

This small note concentrates on proving the following theorem, which was conjectured by Faulhuber and Steinerberger in [6].

**Theorem 1.** The expression \( y^2 \frac{\vartheta_4'(y)}{\vartheta_4(y)} \) is strictly convex and strictly decreasing as a function of \( y \in [0, \infty] \).

The proof will be structured as follows: we prove the convexity in two parts, for small and for large values of \( y \), separately. After this, it is very simple and straightforward to prove that the function is decreasing.

Contrary to what was claimed in an earlier version of this paper, the exponent 2 in the theorem above is in fact the best possible. We are grateful to an anonymous referee for pointing out this. The last section of this paper briefly explains why the exponent 2 cannot be increased by considering the limit \( y \to 0^+ \).

### 2. Results and proofs

We study the function

\[
 f(y) = y^2 \frac{\vartheta_4'(y)}{\vartheta_4(y)},
\]

defined for all \( y \in [0, \infty] \).

**Theorem 2.** The function \( f(y) = y^2 \frac{\vartheta_4'(y)}{\vartheta_4(y)} \) is strictly convex for \( y \in [1, \infty] \).

**Proof.** Notice first that

\[
 f(y) = y^2 \frac{d}{dy} \log \vartheta_4(y) = y^2 \frac{d}{dy} \log \prod_{n=1}^{\infty} \left(1 - e^{-2n\pi y}\right) \left(1 - e^{-(2n-1)\pi y}\right)^2 \\
 = y^2 \frac{d}{dy} \sum_{n=1}^{\infty} \left(\log(1 - e^{-2n\pi y}) + 2\log(1 - e^{-(2n-1)\pi y})\right) \\
 = y^2 \sum_{n=1}^{\infty} \left(\frac{2n\pi e^{-2n\pi y}}{1 - e^{-2n\pi y}} + \frac{(2n-1)\pi e^{-(2n-1)\pi y}}{1 - e^{-(2n-1)\pi y}}\right) \\
 = 2y^2 \sum_{n=1}^{\infty} \left(\frac{n\pi}{e^{2n\pi y} - 1} + \frac{(2n-1)\pi}{e^{(2n-1)\pi y} - 1}\right).
\]

Let us now differentiate:

\[
 f'(y) = 4y \sum_{n=1}^{\infty} \left(\frac{n\pi}{e^{2n\pi y} - 1} + \frac{(2n-1)\pi}{e^{(2n-1)\pi y} - 1}\right) \\
 - 2y^2 \sum_{n=1}^{\infty} \left(2n\pi e^{2n\pi y} (e^{2n\pi y} - 1)^2 + \frac{(2n-1)\pi^2 e^{(2n-1)\pi y}}{(e^{(2n-1)\pi y} - 1)^2}\right).
\]

Let us differentiate again:

\[
 f''(y) = 4 \sum_{n=1}^{\infty} \left(\frac{n\pi}{e^{2n\pi y} - 1} + \frac{(2n-1)\pi}{e^{(2n-1)\pi y} - 1}\right) \\
 - 8y \sum_{n=1}^{\infty} \left(2n^2\pi^2 e^{2n\pi y} (e^{2n\pi y} - 1)^2 + \frac{(2n-1)^2\pi^2 e^{(2n-1)\pi y}}{(e^{(2n-1)\pi y} - 1)^2}\right) \\
 - 2y^2 \sum_{n=1}^{\infty} \left(4n^3\pi^3 e^{2n\pi y} (e^{2n\pi y} - 1)^2 + \frac{(2n-1)^3\pi^3 e^{(2n-1)\pi y}}{(e^{(2n-1)\pi y} - 1)^2}\right) \\
 + 2y^2 \sum_{n=1}^{\infty} \left(2 \cdot 4n^3\pi^3 e^{4n\pi y} (e^{4n\pi y} - 1)^3 + \frac{2(2n-1)^3\pi^3 e^{2(2n-1)\pi y}}{(e^{2(2n-1)\pi y} - 1)^3}\right).
\]
First we can simplify by combining the last two rows by

\[
\frac{2 \cdot 4n^3 \pi^3 e^{4n\pi y}}{(e^{2n\pi y} - 1)^3} - \frac{4n^3 \pi^3 e^{2n\pi y}}{(e^{2n\pi y} - 1)^2} = \frac{4n^3 \pi^3 e^{2n\pi y}}{(e^{2n\pi y} - 1)^2} \left( e^{2n\pi y} + 1 \right)
\]

and

\[
\frac{2 (2n - 1)^3 \pi^2 e^{(2n-1)\pi y}}{(e^{(2n-1)\pi y} - 1)^3} - \frac{(2n - 1)^3 \pi^2 e^{(2n-1)\pi y}}{(e^{(2n-1)\pi y} - 1)^2} = \frac{(2n - 1)^3 \pi^2 e^{(2n-1)\pi y}}{(e^{(2n-1)\pi y} - 1)^3} \left( e^{(2n-1)\pi y} + 1 \right).
\]

The second derivative can be rewritten as

\[
f''(y) = \sum_{n=1}^{\infty} \left( \frac{4n \pi}{e^{2n\pi y} - 1} - 8y \frac{2n^2 \pi^2 e^{2n\pi y}}{(e^{2n\pi y} - 1)^2} + 2y^2 \frac{4n^3 \pi^3 e^{2n\pi y}}{(e^{2n\pi y} - 1)^3} \left( e^{2n\pi y} + 1 \right) \right)
\]

Let us now look at the terms in the sums, starting with the first sum:

\[
4 \frac{n \pi}{e^{2n\pi y} - 1} - 8y \frac{2n^2 \pi^2 e^{2n\pi y}}{(e^{2n\pi y} - 1)^2} + 2y^2 \frac{4n^3 \pi^3 e^{2n\pi y}}{(e^{2n\pi y} - 1)^3} \left( e^{2n\pi y} + 1 \right)
\]

The first factor is positive, and the second one is certainly positive for all positive \( n \in \mathbb{Z}_+ \) when \( y \geq 2/\pi \). Let us now move to the other sum. Let us treat the case \( n = 1 \) separately:

\[
4 \frac{\pi}{e^{\pi y} - 1} - 8y \frac{\pi^2 e^{\pi y}}{(e^{\pi y} - 1)^2} + 2y^2 \frac{\pi^3 e^{\pi y}}{(e^{\pi y} - 1)^3} \left( e^{\pi y} + 1 \right)
\]

Let us now show that this expression is positive. Define for \( y \in \mathbb{R}_+ \)

\[
g(y) = 2 \left( e^{\pi y} - 1 \right)^2 - 4y \pi e^{\pi y} (e^{\pi y} - 1) + \pi^2 y^2 e^{\pi y} (e^{\pi y} + 1).
\]

We have

\[
g''(y) = 2 e^{\pi y} \pi^2 + 2 e^{2\pi y} \pi^2 + 4 e^{2\pi y} \pi \left( -4\pi + 2\pi^2 y \right) + 2 e^{\pi y} \pi \left( 4\pi + 2\pi^2 y \right)
\]

The last term is clearly positive when \( y > 1/\pi \). Since

\[
4 e^{2\pi y} \pi \left( -4\pi + 2\pi^2 y \right) + 4 e^{2\pi y} \pi^2 \left( 2 - 4\pi y + \pi^2 y^2 \right) = 4 e^{2\pi y} \pi \left( 4\pi^2 \pi^2 y ^2 - 4\pi y - 2\pi \right) > 0,
\]

when \( y > 1/\pi + \sqrt{3}/\pi \), the expression \( g''(y) > 0 \) when \( y > 1/\pi + \sqrt{3}/\pi \). Furthermore, since

\[
g'(1) \approx 3584.5,
\]

the first derivative is also positive. It thus suffices to compute \( g(1) \):

\[
g(1) \approx 55.5 > 0.
\]

Let us now treat the terms with \( n > 1 \):
Proof. The second derivative of \( f(y) \) is \( h(y)/\vartheta_2^3(y) \), where

\[
h(y) = 2 \vartheta_4^4(y) \vartheta_2^2(y) + 4 y \vartheta_4^2(y) \vartheta_2^2(y) + y^2 \vartheta_4^2(y) \vartheta_2^2(y) - 4 y (\vartheta_4^4(y))^2 \vartheta_4(y) - 3 y^2 \vartheta_4^2(y) \vartheta_4(y) \vartheta_4(y) + 2 y^2 (\vartheta_4^4(y))^3.
\]

Since \( \vartheta_4(y) \) is strictly positive, it is enough to prove that \( h(y) > 0 \) for \( y \in [0, 1] \).

Differentiating three times the modularity relation

\[
\vartheta_4(y) = y^{-1/2} \vartheta_2(y),
\]

we get first
\begin{align*}
\vartheta'_4(y) &= -\frac{1}{2} y^{-3/2} \vartheta_2 \left( \frac{1}{y} \right) - y^{-5/2} \vartheta'_2 \left( \frac{1}{y} \right), \\
\vartheta''_4(y) &= \frac{3}{4} y^{-5/2} \vartheta_2 \left( \frac{1}{y} \right) + 3 y^{-7/2} \vartheta'_2 \left( \frac{1}{y} \right) + y^{-9/2} \vartheta''_2 \left( \frac{1}{y} \right), \\
\vartheta'''_4(y) &= -\frac{15}{8} y^{-7/2} \vartheta_2 \left( \frac{1}{y} \right) - \frac{45}{4} y^{-9/2} \vartheta'_2 \left( \frac{1}{y} \right) - \frac{15}{2} y^{-11/2} \vartheta''_2 \left( \frac{1}{y} \right) - y^{-13/2} \vartheta'''_2 \left( \frac{1}{y} \right).
\end{align*}

Substituting these back to the definition of \( h(y) \), we are left to prove that the expression
\begin{align*}
\frac{1}{y} = 2 y^{9/2} \left( \vartheta'_2(y) \right)^2 \vartheta_2(y) - 2 y^{9/2} \vartheta''_2(y) \vartheta_2^2(y) - 2 y^{11/2} \left( \vartheta'_2(y) \right)^3 \\
+ 3 y^{11/2} \vartheta''_2(y) \vartheta'_2(y) \vartheta_2(y) - y^{11/2} \vartheta''_2(y) \vartheta'_2^2(y)
\end{align*}
is strictly positive for \( y \in [1, \infty] \).

Using Lemma 3, we can estimate, for \( y \in [1, \infty] \),
\begin{align*}
\frac{1}{y} > 2 y^{9/2} \vartheta_2 \vartheta_2(y) - 2 y^{9/2} \vartheta_2 \vartheta_2(y) + 2 y^{11/2} \vartheta^3 \vartheta_2(y) \\
- 3 y^{11/2} \vartheta_2 \vartheta_2(y) \vartheta_2(y) + y^{11/2} \vartheta_2 \vartheta_2(y) \vartheta_2(y) \\
= y^{9/2} e^{-2\pi y/4} \left( e^{4\pi y} (\alpha y - \beta) + e^{2\pi y} (-\alpha y - \beta - e - \xi) \right),
\end{align*}

with constants \( \alpha \approx 1984.3, \beta \approx 631.7, \gamma \approx 1985.4, \delta \approx 631.8, \varepsilon \approx 1.02 \) and \( \zeta \approx 0.08 \). Thus, we can continue the estimations by
\begin{align*}
\frac{1}{y} > y^{9/2} e^{-2\pi y/4} \left( e^{4\pi y} (1984 y - 632) + e^{2\pi y} (-1986 y - 632) - 2 y - 0.09 \right) \\
\geq y^{9/2} e^{-2\pi y/4} \left( e^{4\pi y} (535 \cdot 1984 y - 535 \cdot 632 - 1986 y - 632) - 2 y - 0.09 \right) \\
> y^{9/2} e^{-2\pi y/4} \left( e^{4\pi y} (533 \cdot 1984 y - 534 \cdot 632) - 2 y - 0.09 \right) > 0. \quad \Box
\end{align*}

**Theorem 5.** The function \( f(y) = y^2 \vartheta'_4(y)/\vartheta_4(y) \) is strictly decreasing for \( y \in \mathbb{R}_+ \).

**Proof.** We have proved that the function is strictly convex, namely, that the second derivative is positive. Hence, it suffices to prove that the first derivative is negative for large values of \( y \). The first derivative is
\begin{align*}
f'(y) = 4 y \sum_{n=1}^{\infty} \left( \frac{n \pi}{e^{2n\pi y} - 1} + \frac{(2n - 1) \pi}{e^{(2n-1) \pi y} - 1} \right) \\
- 2 y^2 \sum_{n=1}^{\infty} \left( \frac{2n^2 \pi^2 e^{2n\pi y}}{(e^{2n\pi y} - 1)^2} + \frac{(2n - 1)^2 \pi^2 e^{(2n-1) \pi y}}{(e^{(2n-1) \pi y} - 1)^2} \right).
\end{align*}

Let us first look at the terms
\[ \frac{4n\pi y}{e^{2n\pi y} - 1} - \frac{4n^2 \pi^2 y^2 e^{2n\pi y}}{(e^{2n\pi y} - 1)^2} = \frac{4n\pi y}{e^{2n\pi y} - 1} \left( e^{2n\pi y} - 1 - yn\pi e^{2n\pi y} \right). \]
The first factor is clearly positive, while the second factor is clearly negative when \( y \) is sufficiently large, say \( y \geq 1 \).

Let us now move to the other terms:
\[ \frac{4y \pi (2n - 1)}{e^{(2n-1) \pi y} - 1} - \frac{2(2n - 1)^2 \pi^2 y^2 e^{(2n-1) \pi y}}{(e^{(2n-1) \pi y} - 1)^2} \]
\[ = \frac{2(2n - 1) \pi y}{e^{(2n-1) \pi y} - 1} \left( e^{(2n-1) \pi y} - 1 - (2n - 1) \pi y e^{(2n-1) \pi y} \right). \]
The first factor is clearly positive, while the second one is negative for large \( y \), say \( y \geq 1 \), so the product is negative. The function is thus decreasing. \( \Box \)
3. Why cannot the exponent 2 in Theorem 1 be increased?

By using the modularity relation connecting $\vartheta_4$ and $\vartheta_2$, we see that
\[
f(y) = \frac{y^2 \vartheta_4'(y)}{\vartheta_4(y)} = y^2 - \frac{1}{2} y^{-3/2} \vartheta_2(1/y) - y^{-5/2} \vartheta_2'(1/y) = -\frac{y}{2} - \frac{\vartheta_2'(1/y)}{\vartheta_2(1/y)}
\]
for $y \in \mathbb{R}_+$. Let us make the change of variables $q = e^{-2\pi y}$, and obtain $q$-series
\[
q^{-1/8} \vartheta_2(-(\log q)/(2\pi)) = 2 \sum_{n=0}^{\infty} q^{(n^2+n)/2}
\]
and
\[
q^{-1/8} \vartheta_2'(-(\log q)/(2\pi)) = -\frac{\pi}{2} \sum_{n=0}^{\infty} (2n+1)^2 q^{(n^2+n)/2}.
\]
These can be interpreted as functions of $q$ defined first for $q \in [0, 1]$. Furthermore, they extend analytically and are nonzero for small $q \in \mathbb{C}$, and so their quotient is given by power series for small values of $q$. This leads to a representation of $f(y)$, when $y \in \mathbb{R}_+$ is small, in the form
\[
f(y) = -\frac{y}{2} + \pi \frac{4}{y} + \sum_{n=1}^{\infty} c(n) e^{-2\pi n/y},
\]
with real coefficients $c(1), c(2), \ldots$. Furthermore, we may differentiate this representation termwise to obtain the asymptotics
\[
f(y) = \frac{\pi}{4} + O(y), \quad f'(y) = O(1), \quad \text{and} \quad f''(y) = O(y^{-4} e^{-2\pi/y}),
\]
which hold when $y \to 0+$.

Take now any fixed $\alpha \in [0, 1]$ and consider the function
\[
g(y) = \frac{y^{2+\alpha} \vartheta_4'(y)}{\vartheta_4(y)} = y^{\alpha} f(y),
\]
defined for all $y \in \mathbb{R}_+$. Its first derivative is
\[
g'(y) = \alpha y^{\alpha-1} f(y) + y^{\alpha} f'(y) = \alpha y^{\alpha-1} \frac{\pi}{4} + O(y^{\alpha}) \sim \alpha \frac{\pi}{4} y^{\alpha-1},
\]
as $y \to 0+$, and its second derivative is
\[
g''(y) = \alpha (\alpha - 1) y^{\alpha-2} f(y) + 2\alpha y^{\alpha-1} f'(y) + y^{\alpha} f''(y)
\]
\[
= \alpha (\alpha - 1) y^{\alpha-2} \frac{\pi}{4} + O(y^{\alpha-1}) \sim \alpha (\alpha - 1) \frac{\pi}{4} y^{\alpha-2},
\]
as $y \to 0+$. The former asymptotics say that $g(y)$ is increasing for small $y$, and since the coefficient $\alpha (\alpha - 1)$ is negative, the latter asymptotics say that $g'(y)$ is negative for small values of $y$. Thus the exponent 2 in Theorem 1 cannot be increased.

References