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Deformation of the product of complex Fano manifolds

Déformation du produit de variétés de Fano complexes

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ABSTRACT

Let \mathcal{X} be a connected family of complex Fano manifolds. We show that if some fiber is the product of two manifolds of lower dimensions, then so is every fiber. Combining with previous work of Hwang and Mok, this implies immediately that if a fiber is a (possibly reducible) Hermitian symmetric space of compact type, then all fibers are isomorphic to the same variety.

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RÉSUMÉ

Soit \mathcal{X} une famille connexe des variétés de Fano complexes. On montre que, si une fibre est un produit de deux variétés de dimensions inférieures, alors il en est de même pour chaque fibre. En combinant avec un résultat de Hwang et Mok, ceci implique immédiatement que, si une fibre est un espace Hermitien symétrique de type compact, alors toutes les fibres sont isomorphes à cette variété.

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We consider varieties defined over the field of complex numbers \mathbb{C} . The aim of this short note is to show the following result.

Theorem 1. Let $\pi : \mathcal{X} \to S \ni 0$ be a holomorphic map onto a connected complex manifold S such that all fibers are connected Fano manifolds and $\mathcal{X}_0 \cong \mathcal{X}'_0 \times \mathcal{X}''_0$. Then there exist unique holomorphic maps

 $f': \mathcal{X} \to \mathcal{X}' \ \pi': \mathcal{X}' \to S,$ $f'': \mathcal{X} \to \mathcal{X}'' \ \pi'': \mathcal{X}'' \to S,$

such that \mathcal{X}'_0 and \mathcal{X}''_0 coincide with the fibers of π' and π'' at $0 \in S$, the morphism $\pi = \pi' \circ f' = \pi'' \circ f''$ and $\mathcal{X}_t \cong \mathcal{X}'_t \times \mathcal{X}''_t$ for all $t \in S$.

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It is necessary to assume that each fiber \mathcal{X}_t is Fano in Theorem 1. For example, the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$ can be deformed to a Hirzebruch surface Σ_m with m even. On the other hand, a Hirzebruch surface Σ_m with $m \neq 0$ is neither a Fano manifold nor a product of two curves.

The following vanishing result is standard, which helps to study relative Mori contractions in our setting. The latter is important for our proof of Theorem 1.

Proposition 2. Let $\pi : \mathcal{X} \to \Delta^n$ be a holomorphic map onto the unit disk Δ^n of dimension n such that all fibers are connected Fano manifolds. Suppose that \mathcal{L} is a holomorphic line bundle on \mathcal{X} such that $\mathcal{L}_t := \mathcal{L}|_{\mathcal{X}_t}$ is nef on \mathcal{X}_t for each $t \in \Delta^n$. Then $H^k(\mathcal{X}, \mathcal{L}) = 0$ for all $k \ge 1$.

Proof. Since each \mathcal{X}_t is Fano, $H^k(\mathcal{X}_t, \mathcal{L}_t) = 0$ and $R^k \pi_* \mathcal{L} = 0$ for all $k \ge 1$ by Kodaira vanishing. Then $H^k(\mathcal{X}, \mathcal{L}) = H^k(\Delta^n, \pi_* \mathcal{L}) = 0$ for all $k \ge 1$ since Δ^n is a Stein manifold. \Box

We summarize several facts in Proposition 3 on the relative Mori contraction in our setting. One can consult [3] or [5] for standard notations and results in relative Minimal Model Program. By [7, Theorem 1], the Mori cone of each X_t is invariant in our setting, which we restate as conclusion (ii) in Proposition 3.

Proposition 3. Let $\pi : \mathcal{X} \to \Delta^n$ be as in Proposition 2. Then, for each $t \in \Delta^n$, the following holds.

(i) The natural morphism $Pic(\mathcal{X}) \rightarrow Pic(\mathcal{X}_t)$ is an isomorphism.

(ii) There are natural identities as follows:

$$N_1(\mathcal{X}_t) = N_1(\mathcal{X}/\Delta^n) \text{ and } \overline{NE}(\mathcal{X}_t) = \overline{NE}(\mathcal{X}/\Delta^n).$$
⁽¹⁾

(iii) Denote by $\Phi : \mathcal{X} \to \mathcal{Y}$ the relative Mori contraction over Δ^n associated with an extremal face $F \subset \overline{NE}(\mathcal{X}/\Delta^n)$. Then Φ_t is the Mori contraction associated with $F \subset \overline{NE}(\mathcal{X}_t)$ under identification (1). In particular, \mathcal{Y} and \mathcal{Y}_t are normal varieties.

Proof. (i) By Proposition 2, the map $e : \operatorname{Pic}(\mathcal{X}) \to H^2(\mathcal{X}, \mathbb{Z})$, induced by exponential sheaf sequence, is an isomorphism. Since \mathcal{X}_t is a Fano manifold, the map $e_t : \operatorname{Pic}(\mathcal{X}_t) \to H^2(\mathcal{X}_t, \mathbb{Z})$ is an isomorphism. The natural map $r : H^2(\mathcal{X}, \mathbb{Z}) \to H^2(\mathcal{X}_t, \mathbb{Z})$ is also an isomorphism, since Δ^n is a contractible topology space. Then (i) follows from the fact that the natural morphism $\operatorname{Pic}(\mathcal{X}) \to \operatorname{Pic}(\mathcal{X}_t)$ coincides with the composition $e_t^{-1} \circ r \circ e$.

(ii) The linear map $N^1(\mathcal{X}/\Delta^n) \to N^1(\mathcal{X}_t)$ is surjective by (i). Then we have injective linear maps

$$\gamma_t: N_1(\mathcal{X}_t) \to N_1(\mathcal{X}/\Delta^n) \text{ and } \gamma_t^+: \overline{NE(\mathcal{X}_t)} \to \overline{NE}(\mathcal{X}/\Delta^n).$$

By Theorem 1 in [7], the local identification $H_2(\mathcal{X}_0, \mathbb{R}) = H_2(\mathcal{X}_t, \mathbb{R})$ yields the identity of Mori cones $\overline{NE}(\mathcal{X}_0) = \overline{NE}(\mathcal{X}_t)$. Since the local identification $H^2(\mathcal{X}_0, \mathbb{R}) = H^2(\mathcal{X}_t, \mathbb{R})$ gives the identity of $\mathcal{L}|_{\mathcal{X}_0}$ and $\mathcal{L}|_{\mathcal{X}_t}$ for any $\mathcal{L} \in \text{Pic}(\mathcal{X})$, we know by (i) that

$$\gamma_t^+(\overline{NE}(\mathcal{X}_t)) = \gamma_0^+(\overline{NE}(\mathcal{X}_0)), \ \gamma_t(N_1(\mathcal{X}_t)) = \gamma_0(N_1(\mathcal{X}_0)).$$
(2)

The abelian group $Z_1(\mathcal{X}/\Delta^n)$ (resp. semigroup $Z_1^+(\mathcal{X}/\Delta^n)$), generated by reduced irreducible curves that are contracted by π , is isomorphic to $\bigoplus_{s \in \Delta^n} Z_1(\mathcal{X}_s)$ (resp. $\bigoplus_{s \in \Delta^n} Z_1^+(\mathcal{X}_s)$). Then (2) implies that γ_t and γ_t^+ are bijections.

(iii) By the Contraction Theorem, there exist $\mathcal{L} \in \operatorname{Pic}(\mathcal{X})$ and a homomorphism $\Phi : \mathcal{X} \to \mathcal{Y} := \operatorname{Proj}_{\Delta^n} R(\mathcal{L})$ over Δ^n such that $\Phi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{Y}}$ and

$$F = \{\eta \in \overline{NE}(\mathcal{X}/\Delta^n) \mid \deg \mathcal{L}|_{\eta} = 0\},\$$

where $R(\mathcal{L}) := \bigoplus_{m \ge 0} H^0(\mathcal{X}, \mathcal{L}^{\otimes m})$. Then (iii) follows from (ii) and the claim that $H^0(\mathcal{X}, \mathcal{L}^{\otimes m}) \to H^0(\mathcal{X}_t, \mathcal{L}_t^{\otimes m})$ is surjective for all m > 0.

The Cartier divisor $\mathcal{L}^{\otimes m} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{X}_t)$ is nef on \mathcal{X}_s for all $s \in \Delta^n$, which implies that $H^1(\mathcal{X}, \mathcal{L}^{\otimes m} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{X}_t)) = 0$ by Proposition 2. Hence, the homomorphism $H^0(\mathcal{X}, \mathcal{L}^{\otimes m}) \to H^0(\mathcal{X}_t, \mathcal{L}_t^{\otimes m})$, induced by the short exact sequence

$$0 \to \mathcal{L}^{\otimes m} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{X}_t) \to \mathcal{L}^{\otimes m} \to \mathcal{L}_t^{\otimes m} \to 0,$$

is surjective, which proves the claim. \Box

Local verification of Theorem 1 is closely related to Kodaira's stability as follows.

Proposition 4. [4, Theorem 4] Let $p: W \to B$ be a smooth map between complex manifolds such that each fiber F is a compact connected complex manifold with $H^1(F, \mathcal{O}_F) = 0$, and let $\omega : W \to S \ni 0$ be a smooth family of complex manifold with $\omega^{-1}(0) = W$. Then, there exists a commutative diagram



where U is an analytic open neighborhood of $0 \in S$, \mathcal{B} is a complex manifold which is smooth over U, $\mathcal{W}_U := \omega^{-1}(U)$, $\omega_U := \omega|_{\mathcal{W}_U}$, and the fiber $\mathcal{P}_0 : \mathcal{W}_0 \to \mathcal{B}_0$ at $0 \in U$ coincides with $p : W \to B$.

Let us collect two results on split vector bundles from [2] and [1], which will be useful in our proof of Theorem 1.

Proposition 5. [2, page 409] Let \mathcal{V} be a vector bundle over a connected complex manifold X. Suppose that there is a complex subvariety $D \subset X$ and vector bundles \mathcal{V}_1 and \mathcal{V}_2 over $X \setminus D$ such that dim $D \leq \dim X - 2$ and $\mathcal{V}|_{X \setminus D} = \mathcal{V}_1 \oplus \mathcal{V}_2$. Then \mathcal{V}_1 and \mathcal{V}_2 can be extended uniquely as vector bundles \mathcal{V}'_1 and \mathcal{V}'_2 over X, such that $\mathcal{V} = \mathcal{V}'_1 \oplus \mathcal{V}'_2$.

Proof. We repeat briefly the corresponding argument on page 409 of [2], since no explicit statement is given there. The problem is local, and we can assume that $X = \Delta^n$ and \mathcal{V} is the trivial vector bundle $X \times V$. Then there is a holomorphic map $\Phi : X \setminus D \to Y := Gr(r_1, V) \times Gr(r_2, V)$ sending $x \in X \setminus D$ to $((\mathcal{V}_1)_x, (\mathcal{V}_2)_x)$, where r_1 and r_2 is the rank of \mathcal{V}_1 and \mathcal{V}_2 , respectively. The closed subset $E := \{(V_1, V_2) \in Y \mid V_1 \cap V_2 \neq 0\}$ is an ample prime divisor of Y. Since $\Phi(X \setminus D)$ is contained in the affine variety $Y \setminus E$, we can extend Φ to X uniquely by Hartogs' Extension Theorem. \Box

Proposition 6. [1, Theorem 1.4] Let X be a rationally connected manifold such that $T_X = V_1 \oplus V_2$. If V_1 or V_2 is integrable, then X is isomorphic to a product $Y_1 \times Y_2$ such that $V_j = p_{Y_i}^* T_{Y_j}$ for j = 1, 2.

Proof of Theorem 1. The problem is local, and we can assume $S = \Delta^n \ni 0$. By Proposition 3 we can extend projections $f'_0 : \mathcal{X}_0 \to \mathcal{X}'_0$ and $f''_0 : \mathcal{X}_0 \to \mathcal{X}''_0$ to relative Mori contractions $f' : \mathcal{X} \to \mathcal{X}'$ and $f'' : \mathcal{X} \to \mathcal{X}''$, and $F' \cap F'' = \{0\}$ where F' and F'' are the extremal faces of $\overline{NE}(\mathcal{X}/\Delta)$ associated with f' and f'' respectively. Moreover for any $t \in \Delta^n$, fibers \mathcal{X}'_t and \mathcal{X}''_t are normal, and morphisms f'_t and f''_t are Mori contractions. Denote by $\pi' : \mathcal{X}' \to \Delta^n$ and $\pi'' : \mathcal{X}'' \to \Delta^n$ the corresponding morphisms over Δ^n . The fact $F' \cap F'' = \{0\}$ implies that the morphism

$$f: \mathcal{X} \to \mathcal{X}' \times_{\Delta^n} \mathcal{X}''$$

over Δ^n induced by f' and f'' contracts no curves on \mathcal{X} .

By Proposition 4, there exists an analytic neighborhood U' of $0 \in \Delta^n$ and a holomorphic map $\mathcal{P}' : \mathcal{X}_{U'} \to \mathcal{B}'$ over U' such that the center map \mathcal{P}'_0 coincides with $f'_0 : \mathcal{X}_0 \to \mathcal{X}'_0$, where $\mathcal{X}_{U'} := \pi^{-1}(U')$. The divisor $(f'_0)^*(-K_{\mathcal{X}'_0}) \in \operatorname{Pic}(\mathcal{X}_0)$ is a supporting divisor of f'_0 . Then by Proposition 3, $L' := (\mathcal{P}')^*(-K_{\mathcal{B}'/U'}) \in \operatorname{Pic}(\mathcal{X}_{U'})$ is a supporting divisor of the contraction $f'_{U'} : \mathcal{X}_{U'} \to \mathcal{X}'_{U'}$. Since L' is \mathcal{P}' -trivial, there is a commutative diagram over U' as follows:



We claim that there exists an analytic open neighborhood U of $0 \in U'$ such that the morphism $\phi_U : \mathcal{B}'_U \to \mathcal{X}'_U$ contracts no curves.

Now suppose the claim fails. Then there is a sequence $\{t_k\}_{k=1}^{\infty}$ in U' such that $\lim_{k\to\infty} t_k = 0 \in U'$ and, for each $k \ge 1$, there exists a point $b_k \in \mathcal{B}'_{t_k}$ such that

$$\dim(f')^{-1}(x_k) > \dim(\mathcal{P}')^{-1}(b_k) = \dim \mathcal{X}_0'',$$

where $x_k := \phi(b_k) \in \mathcal{X}'_{t_k}$. By semicontinuity of dimension of the family $\mathcal{X}_{U'}$, one has dim $\mathcal{X}''_{t_k} \le \dim \mathcal{X}''_0$ when k is sufficiently large. By dimension reason, the morphism $f''|_{(f')^{-1}(x_k)} : (f')^{-1}(x_k) \to \mathcal{X}''_{t_k}$ must contain fibers of positive dimension when k is sufficiently large. This contradicts the fact that the morphism $f : \mathcal{X} \to \mathcal{X}' \times_{\Delta^n} \mathcal{X}''$ contracts no curve. Hence, the claim holds.

Denote by $R \subset \mathcal{B}'_U$ the ramification locus of the finite map $\phi_U : \mathcal{B}'_U \to \mathcal{X}'_U$. The composition $\mathcal{B}'_U \to \mathcal{X}'_U \to U$ is a holomorphic map. Thus the scheme-theoretic inverse image of \mathcal{X}'_0 is reduced. Since ϕ_0 is biholomorphic, we know $\mathcal{B}'_0 \notin R$ and $\deg(\phi_U) = 1$. Then the normality of \mathcal{X}'_U implies that ϕ_U is a biholomorphic map.

We can analyze f'' in a way similar to the discussion above. By shrinking U, we can conclude that there exists an analytic open neighborhood U of $0 \in \Delta^n$ such that \mathcal{X}'_U and \mathcal{X}''_U are complex manifolds, and

$$\begin{split} f'_{U} &: \mathcal{X}_{U} \to \mathcal{X}'_{U} \ \pi'_{U} : \mathcal{X}'_{U} \to U, \\ f''_{U} &: \mathcal{X}_{U} \to \mathcal{X}''_{U} \ \pi''_{U} : \mathcal{X}''_{U} \to U, \end{split}$$

are holomorphic maps. Then by the generic smoothness property of f' and f'', there is a closed subvariety $B \subset \Delta^n \setminus U$ such that \mathcal{X}'_{Ω} and \mathcal{X}''_{Ω} are complex manifolds, and

$$\begin{split} f'_{\Omega} &: \mathcal{X}_{\Omega} \to \mathcal{X}'_{\Omega} \, \pi'_{\Omega} : \mathcal{X}'_{\Omega} \to \Omega, \\ f''_{\Omega} &: \mathcal{X}_{\Omega} \to \mathcal{X}''_{\Omega} \, \pi''_{\Omega} : \mathcal{X}''_{\Omega} \to \Omega, \end{split}$$

are holomorphic maps, where $\Omega := \Delta^n \setminus B$. In particular, dim $B \le n - 1$.

As a projective morphism with finite fibers, $f : \mathcal{X} \to \mathcal{X}' \times_{\Delta^n} \mathcal{X}''$ is a finite morphism. It follows that there exists a positively codimensional closed subset D of $\mathcal{X}_B := \pi^{-1}(B)$ such that the relative tangent bundles $T^{f'}$ and $T^{f''}$ are holomorphic on $\mathcal{X} \setminus D$ and

$$T^{\pi}|_{\mathcal{X}\setminus D}=V'\oplus V'',$$

where $V' := (T^{f'})|_{\mathcal{X}\setminus D}$ and $V'' := (T^{f''})|_{\mathcal{X}\setminus D}$. By Proposition 5, we can extend V' and V'' uniquely as vector bundles \widetilde{V}' and \widetilde{V}'' over \mathcal{X} such that $T^{\pi} = \widetilde{V}' \oplus \widetilde{V}''$. The distributions \widetilde{V}' and \widetilde{V}'' are both integrable, since

$$\widetilde{V}'|_{\mathcal{X}\setminus D} = V' = (T^{f'})|_{\mathcal{X}\setminus D}, \ \widetilde{V}''|_{\mathcal{X}\setminus D} = V'' = (T^{f''})|_{\mathcal{X}\setminus D}.$$
(4)

We can apply Proposition 6 to \mathcal{X}_t with $t \in \Delta^n$, since Fano manifolds are rationally connected. It follows that $\mathcal{X}_t \cong \mathcal{X}'_t \times \mathcal{X}''_t$ for each $t \in \Delta^n$.

The leaves of \widetilde{V}' (resp. \widetilde{V}'') are all Fano manifolds and hence simply connected. There is a submersion $\phi' : \mathcal{X} \to \mathcal{Y}'$ (resp. $\phi'' : \mathcal{X} \to \mathcal{Y}''$) to a smooth complex variety \mathcal{Y}' (resp. \mathcal{Y}'') over Δ^n such that $\widetilde{V}' = T^{\phi'}$ (resp. $\widetilde{V}'' = T^{\phi''}$), where the smoothness of \mathcal{Y}' (resp. \mathcal{Y}'') at an arbitrary point follows from the simple-connectedness of the corresponding \widetilde{V}' -leaf (resp. \widetilde{V}'' -leaf), see [6, Proposition 3.7]. Then (4) implies that $\mathcal{Y}', \mathcal{Y}'', \phi'$ and ϕ'' coincide with $\mathcal{X}', \mathcal{X}'', f'$ and f'' respectively. The conclusion follows. \Box

It is proved by Hwang and Mok [2] that irreducible Hermitian symmetric spaces of compact type are rigid under Kähler deformation. Combining with Theorem 1, we get the following corollary immediately.

Corollary 7. Let $\pi : \mathcal{X} \to S \ni 0$ be a smooth family of connected Fano manifolds over a connected complex manifold S. Suppose that \mathcal{X}_0 is a (possibly reducible) Hermitian symmetric space of compact type. Then we have $\mathcal{X}_t \cong \mathcal{X}_0$ for all $t \in S$.

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