C. R. Acad. Sci. Paris, Ser. I 356 (2018) 657–660

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial differential equations/Probability theory

Convergence in Wasserstein distance for self-stabilizing diffusion evolving in a double-well landscape

Convergence en distance de Wasserstein pour une diffusion auto-stabilisante évoluant dans un paysage à double puits

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ARTICLE INFO

Article history:
Received 30 September 2017
Accepted after revision 19 April 2018
Available online 30 April 2018

Presented by Haïm Brézis

ABSTRACT

It is well-known (see Bolley et al. [3]) that there exists a contraction in Wasserstein distance between the solution to the granular media equation and its unique steady state, provided that the confining potential is strictly convex. Nevertheless, in the nonconvex case, just few is known. In particular, we do not have a unique steady state under easily checked assumptions if the diffusion coefficient is sufficiently small. Consequently, the method of Bolley, Gentil and Guillin can not be applied in this setting. However, here, we present a simple example (for the sake of the simplicity) of a double-well confining potential, and we show the convergence to 0 of the Wasserstein distance between the solution to the granular media equation and a related application (which characterizes the steady states) of this solution.

RÉSUMÉ

Il est bien connu (voir Bolley et al. [3]) qu'il existe une contraction en distance de Wasserstein entre la solution de l'équation des milieux granulaires et son unique état d'équilibre, pour peu que le potentiel de confinement soit strictement convexe. Néanmoins, dans le cas non convexe, on dispose de peu de résultats. En particulier, sous des conditions simples à vérifier, il n'y a pas unicité de l'état d'équilibre. Par conséquent, la méthode de Bolley, Gentil et Guillin ne peut pas être appliquée sous ces conditions. Toutefois, ici, nous présentons un exemple simple (par souci de simplicité) d'un potentiel de confinement à deux puits, et nous montrons la convergence vers 0 de la distance de Wasserstein entre la solution de l'équation des milieux granulaires et une application (qui caractérise les états d'équilibre) de cette solution.

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https://doi.org/10.1016/j.crma.2018.04.020

1. Introduction

In this paper, we are interested in the following so-called granular media equation:

$$\frac{\partial}{\partial t}\mu_t^\sigma(x)=\frac{\partial}{\partial x}\left\{\frac{\sigma^2}{2}\frac{\partial}{\partial x}\mu_t^\sigma(x)+\mu_t^\sigma(x)\Big(V'(x)+F'*\mu_t^\sigma(x)\Big)\right\}\,, \tag{1}$$

where the confining potential V has two wells and the interacting potential F is convex.

This partial differential equation has a natural interpretation in terms of stochastic processes. Indeed, let us consider the following so-called McKean–Vlasov diffusion:

$$\begin{cases} X_t^\sigma=X_0+\sigma B_t-\int_0^t\left(W_s^\sigma\right)'\left(X_s^\sigma\right)\mathrm{d}s \\ W_s^\sigma=V+F*\mathcal{L}\left(X_s^\sigma\right)\end{cases}\,. \tag{2}$$

Here, $*$ denotes the convolution. By μ_t^σ, we denote the law at time t of the process X^σ. It is well known that the family of probability measures $\{\mu_t^\sigma\,;\,t\geq 0\}$ satisfies the granular media equation starting from $\mathcal{L}(X_0)$.

The convergence (as the time goes to infinity) of μ_t^σ towards a steady state has been proved in the convex case (see [1–5]) and in the nonconvex case (see [8,9]).

We now give the assumptions on V and F.

Assumption 1. *The potentials V and F satisfy the following hypotheses:*

- $V(x):=\frac{x^4}{4}-\frac{x^2}{2}$.
- $F(x):=\frac{\alpha}{2}x^2$ *with* $\alpha>0$.

Let us point out that we take a simple example for the sake of the simplicity, but that we can consider a more general double-well confining potential.

We also need an hypothesis of synchronization.

Assumption 2. *We put* $\rho:=\alpha-\sup_{x\in\mathbb{R}}-V''(x)=\alpha-1$. *We assume that ρ is positive.*

Assumption 2 allows us to control μ_t^σ for any $t\geq 0$.

We add a last hypothesis.

Assumption 3. *There are exactly three invariant probabilities.*

This hypothesis is satisfied provided that the diffusion coefficient σ is strictly smaller than a threshold σ_c, see [7,10].

We now introduce the material that will be used in the note.

By $\mathbb{W}_2$, we denote the Wasserstein distance: $\mathbb{W}_2(\mu;\nu)^2=\sqrt{\inf\mathbb{E}\left\{(X-Y)^2\right\}}$, the infimum being taken for all pair of random variables X and Y such that $\mathcal{L}(X)=\mu$ and $\mathcal{L}(Y)=\nu$.

By Π, we denote the application from the set of probability measures in $\mathbb{R}$ to itself defined by

$$\Pi(\mu)(\mathrm{d}x):=\frac{\exp\left[-\frac{2}{\sigma^2}\left(\frac{x^4}{4}+\frac{\alpha-1}{2}x^2-\alpha m(\mu)x\right)\right]}{\int_{\mathbb{R}}\exp\left[-\frac{2}{\sigma^2}\left(\frac{y^4}{4}+\frac{\alpha-1}{2}y^2-\alpha m(\mu)y\right)\right]}\mathrm{d}x\quad\text{where}\quad m(\mu):=\int_{\mathbb{R}}x\mu(\mathrm{d}x)\,. \tag{3}$$

Thanks to [7], we know that μ is an invariant probability of diffusion (2) if and only if $\mu=\Pi(\mu)$.

In [3], the authors obtained a convergence in Wasserstein distance of μ_t^σ towards the **unique** invariant probability measure by assuming that V is strictly convex, despite it is not uniformly strictly convex. Here, we aim to prove a similar result in the nonconvex case. However, we slightly modify the type of convergence. Indeed, what we show is the following.

Theorem 4. *There exists $K>0$ such that for any $s,t\geq 0$, we have the inequality*

$$\mathbb{W}_2\left(\mu_{t+s}^\sigma\,;\,\Pi\left(\mu_{t+s}^\sigma\right)\right)\leq Ke^{-\rho t}+\frac{\alpha}{\rho}\sup_{r\in[s;s+t]}\left|m\left(\mu_r^\sigma\right)-m\left(\mu_{s+t}^\sigma\right)\right|\,. \tag{4}$$

Thanks to [8,9], we know that μ_t^σ weakly converges to an invariant probability μ^σ so that $m\left(\mu_t^\sigma\right)$ converges to $m\left(\mu^\sigma\right)$ as t goes to infinity. This readily implies the following corollary.

Corollary 5. *We have the limit:* $\lim_{t\to+\infty}\mathbb{W}_2\left(\mu_t^\sigma\,;\,\Pi\left(\mu_t^\sigma\right)\right)=0$.

About the rate of convergence, the first term in (4) decreases exponentially and the second one is related to the rate of convergence of $t \mapsto m\left(\mu_t^\sigma\right)$.

Remark 6. Also, one can wonder what is the limit of μ_t^σ among the three invariant probabilities. However, the question of the basins of attraction is still open.

Let us point out that we are currently dealing with this question in a work in progress. The setting of this work is exactly the same as here: $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$, $F(x) := \frac{\alpha}{2}x^2$ with $\alpha > 1$ and σ is sufficiently small. The main result more or less is the following: if μ_0 is a probability measure with a compact support $\mathcal{K} \subset \mathbb{R}_+^*$, then μ_t^σ weakly converges towards μ_+^σ, the unique invariant probability with a positive mean provided that σ is small enough.

2. Proof of Theorem 4

We now prove Theorem 4. To do so, we consider the following diffusion:

$$Y_u^{s,s+t,\sigma} = X_s^\sigma + \sigma\left(B_{s+u} - B_s\right) - \int_0^u \left[V'\left(Y_r^{s,s+t,\sigma}\right) + F' * \mu_{s+t}^\sigma\left(Y_r^{s,s+t,\sigma}\right)\right] \mathrm{d}r\,. \tag{5}$$

Equivalently to the stochastic differential equation (5), we can look at the partial differential equation

$$\frac{\partial}{\partial u}\mu_u(x) = \frac{\partial}{\partial x}\left\{\frac{\sigma^2}{2}\frac{\partial}{\partial x}\mu_u(x) + \mu_u(x)\Big(V'(x) + F' * \mu_{s+t}^\sigma(x)\Big)\right\}. \tag{6}$$

We put $\Pi_u^{s,s+t} := \mathcal{L}\left(Y_u^{s,s+t,\sigma}\right)$, which is also the solution to Equation 6. The assumptions of the note imply that $V + F * \mu_{s+t}^\sigma$ is uniformly strictly convex: $V'' + \alpha \geq \rho > 0$. However, the unique invariant probability of diffusion (5) is $\Pi\left(\mu_{s+t}^\sigma\right)$. Consequently, we obtain

$$\mathbb{W}_2\left(\Pi_t^{s,s+t}; \Pi\left(\mu_{s+t}^\sigma\right)\right) \leq e^{-\rho t}\mathbb{W}_2\left(\mu_s^\sigma; \Pi\left(\mu_{s+t}^\sigma\right)\right). \tag{7}$$

According to [6, Theorem 2.13], we know that the first moment of μ_t^σ is uniformly bounded. We deduce the existence of a constant $K > 0$ such that, for any $s, t \geq 0$, we have $\mathbb{W}_2\left(\mu_s^\sigma; \Pi\left(\mu_{s+t}^\sigma\right)\right) \leq K$.

We now provide an upper-bound for the quantity $\mathbb{W}_2\left(\mu_{s+t}^\sigma; \Pi_t^{s,s+t}\right)$. We remark that

$$d\left(X_{s+u}^\sigma - Y_u^{s,s+t,\sigma}\right) = \left[-V'\left(X_{s+u}^\sigma\right) - F' * \mu_{s+u}^\sigma\left(X_{s+u}^\sigma\right) + V'\left(Y_u^{s,s+t,\sigma}\right) + F' * \mu_{s+t}^\sigma\left(Y_u^{s,s+t,\sigma}\right)\right] \mathrm{d}u\,.$$

As $F'(x) = \alpha x$, we can write

$$\begin{aligned} d\left(X_{s+u}^\sigma - Y_u^{s,s+t,\sigma}\right) &= -\Big[V'\left(X_{s+u}^\sigma\right) + \alpha\left(X_{s+u}^\sigma - \mathbb{E}\left[X_{s+u}^\sigma\right]\right) \\ &\qquad - V'\left(Y_u^{s,s+t,\sigma}\right) - \alpha\left(Y_u^{s,s+t,\sigma} - \mathbb{E}\left[X_{s+t}^\sigma\right]\right)\Big] \mathrm{d}u \\ &= -\Big[\left(W_{s+u}^\sigma\right)'\left(X_{s+u}^\sigma\right) - \left(W_{s+u}^\sigma\right)'\left(Y_u^{s,s+t,\sigma}\right) - \alpha\left(\mathbb{E}\left[X_{s+u}^\sigma\right] - \mathbb{E}\left[X_{s+t}^\sigma\right]\right)\Big] \mathrm{d}u\,, \end{aligned}$$

where $W_{s+u}^\sigma := V + F * \mu_{s+u}^\sigma$. Consequently, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u}\left|X_{s+u}^\sigma - Y_u^{s,s+t,\sigma}\right|^2 &= -2\left(X_{s+u}^\sigma - Y_u^{s,s+t,\sigma}\right)\left\{\left(W_{s+u}^\sigma\right)'\left(X_{s+u}^\sigma\right) - \left(W_{s+u}^\sigma\right)'\left((Y_u^{s,s+t,\sigma}\right)\right\} \\ &\quad + 2\alpha\left(X_{s+u}^\sigma - Y_u^{s,s+t,\sigma}\right)\left[m\left(\mu_{s+u}^\sigma\right) - m\left(\mu_{s+t}^\sigma\right)\right]. \end{aligned}$$

Since W_{s+u}^σ is uniformly strictly convex ($\left(W_{s+u}^\sigma\right)'' \geq \rho > 0$), we deduce the following

$$\frac{\mathrm{d}}{\mathrm{d}u}\left|X_{s+u}^\sigma - Y_u^{s,s+t,\sigma}\right|^2 \leq -2\rho\left|X_{s+u}^\sigma - Y_u^{s,s+t,\sigma}\right|^2 + 2\alpha\left|X_{s+u}^\sigma - Y_u^{s,s+t,\sigma}\right| \sup_{u\in[0;t]}\left|m\left(\mu_{s+u}^\sigma\right) - m\left(\mu_{s+t}^\sigma\right)\right|.$$

Consequently, the quantity $\frac{\mathrm{d}}{\mathrm{d}u}\left|X_{s+u}^\sigma - Y_u^{s,s+t,\sigma}\right|^2$ is nonpositive provided that $\left|X_{s+u}^\sigma - Y_u^{s,s+t,\sigma}\right| \geq \frac{\alpha}{\rho}\sup_{u\in[0;t]}\left|m\left(\mu_{s+u}^\sigma\right) - m(\mu_{s+t}^\sigma)\right|$. Since $X_s^\sigma = Y_0^{s,s+t,\sigma}$, we immediately obtain that:

$$\left|X_{s+u}^\sigma - Y_u^{s,s+t,\sigma}\right| \leq \frac{\alpha}{\rho}\sup_{u\in[0;t]}\left|m\left(\mu_{s+u}^\sigma\right) - m\left(\mu_{s+t}^\sigma\right)\right|.$$

Consequently, $\mathbb{W}_2\left(\mu_{s+t}; \Pi_t^{s,s+t}\right)^2 \leq \mathbb{E}\left\{\left|X_{s+u}^\sigma - Y_u^{s,s+t,\sigma}\right|^2\right\} \leq \frac{\alpha^2}{\rho^2}\left(\sup_{u\in[0;t]}\left|m\left(\mu_{s+u}^\sigma\right) - m\left(\mu_{s+t}^\sigma\right)\right|\right)^2$, which ends the proof.

Acknowledgements

We are grateful to the anonymous referee, his or her interest in the present work and the remarks that he or she made to help us to improve the note.

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