Number theory

# Signature ( $n-2,2$ ) CM types and the unitary Colmez conjecture 

## Type CM signature ( $n-2,2$ ) et la conjecture unitaire de Colmez

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#### Abstract

Colmez conjectured a formula relating the Faltings height of CM abelian varieties to a certain linear combination of logarithmic derivatives of $L$-functions. In this paper, we study the case of unitary CM fields and by studying the class functions that arise, we reduce the conjecture to a special case. Using the Galois action, we prove more cases of the Colmez Conjecture.


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## R É S U M É

Colmez a conjecturé une formule établissant une relation entre la hauteur de Faltings d'une variété abélienne à multiplication complexe et une combinaison linéaire particulière de dérivées logarithmiques de fonctions $L$. Dans cet article, nous restreindrons notre étude aux corps CM unitaires et, par l'étude des fonctions centrales qui se présentent, nous réduirons la conjecture à un cas particulier. En utilisant des actions de Galois, nous démontrerons la conjecture de Colmez pour différents cas.
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## 1. Set up and theorem

The Colmez conjecture gives a formula for the Faltings height of a CM abelian variety in terms of log derivatives of $L$-functions arising from the CM type. This conjecture has proven useful in giving bounds for the Faltings height of CM abelian varieties (see [5] for the case of elliptic curves and [9] where a weaker form of the Colmez conjecture is used in the proof of the André-Oort conjecture for the moduli space of principally polarized abelian varieties).

Definition 1.1. A unitary CM field $E$ is a CM field of the form $E=k F$, where $F$ is a totally real number field and $k \subseteq \mathbb{C}$ is an imaginary quadratic field.

[^0]Given a CM field $E$ of degree $2 n$, a CM type of $E$ consists of $n$ embeddings of $E$ into $\mathbb{C}$ such that no two of the embeddings differ by complex conjugation. For unitary CM fields, we will stratify the CM types by signature.

Definition 1.2. Let $E=k F$ be a unitary CM field. A CM type $\Phi \subseteq \operatorname{Hom}(E, \mathbb{C})$ has signature $(n-r, r)$ if exactly $n-r$ of the embeddings in $\Phi$ restrict to the identity $k \hookrightarrow \mathbb{C}$.

The main theorem of this paper is that, using results of Yang and Yin [10], which rely on deep results of [1], [11], or [3], we can reduce the Colmez conjecture in the unitary case to CM types of signature ( $n-2,2$ ).

Theorem 1.3. Let $E=k F$ be a unitary CM field. Then, the Colmez conjecture holds for $E$ if and only if it holds for all CM types of signature ( $n-2,2$ ).

We spend the remainder of this section providing a brief description of the Colmez conjecture and refer the reader to [10], giving a more thorough background to the conjecture. Section 2 contains the proof of Theorem 1.3 and, in Section 3, we apply Theorem 1.3 to obtain examples of CM fields where the Colmez conjecture holds.

Let $\Phi$ be a CM type of a CM field $E$ and identify $\Phi$ with its characteristic function $\Phi: \operatorname{Hom}(E, \mathbb{C}) \rightarrow\{0,1\}$. If $E^{c}$ denotes the Galois closure of $E$ (which is also a CM field), then the restriction map $\operatorname{Hom}\left(E^{c}, \mathbb{C}\right) \rightarrow \operatorname{Hom}(E, \mathbb{C})$ can be used to extend $\Phi$ to $\Phi^{c}$, a CM type on $E^{c}$.

Choosing an identification of $\operatorname{Hom}\left(E^{c}, \mathbb{C}\right)$ with $\operatorname{Gal}\left(E^{c} / \mathbb{Q}\right)$, we obtain a function $\Phi^{c}: \operatorname{Gal}\left(E^{c} / \mathbb{Q}\right) \rightarrow\{0,1\}$, and we define the reflex CM type $\widetilde{\Phi^{c}}: \operatorname{Gal}\left(E^{c} / \mathbb{Q}\right) \rightarrow\{0,1\}$ by $\widetilde{\Phi}^{c}(g)=\Phi^{c}\left(g^{-1}\right)$.

Let $A_{\Phi}: \operatorname{Gal}\left(E^{c} / \mathbb{Q}\right) \rightarrow \mathbb{C}$ denote the function we obtain by taking a normalized convolution of $\Phi^{c}$ and $\widetilde{\Phi^{c}}$. More concretely,

$$
A_{\Phi}(g)=\frac{1}{\# \operatorname{Gal}\left(E^{c} / \mathbb{Q}\right)} \sum_{\sigma \in \operatorname{Gal}\left(E^{c} / \mathbb{Q}\right)} \Phi^{c}(\sigma) \widetilde{\Phi^{c}}\left(\sigma^{-1} g\right)
$$

To obtain a class function $A_{\Phi}^{0}$, we take the average of $A_{\Phi}$ among conjugates in $\operatorname{Gal}\left(E^{c} / \mathbb{Q}\right)$. That is to say,

$$
A_{\Phi}^{0}(g)=\frac{1}{\# \operatorname{Gal}\left(E^{c} / \mathbb{Q}\right)} \sum_{h \in \operatorname{Gal}\left(E^{c} / \mathbb{Q}\right)} A_{\Phi}\left(h g h^{-1}\right)
$$

As $A_{\Phi}^{0}$ is a class function, we may write

$$
A_{\Phi}^{0}=\sum_{\chi} a_{\chi} \chi
$$

where $a_{\chi} \in \mathbb{C}$ and $\chi$ ranges through the irreducible representations of $\operatorname{Gal}\left(E^{c} / \mathbb{Q}\right)$. Then define the function $Z\left(s, A_{\Phi}^{0}\right)$ by

$$
Z\left(s, A_{\Phi}^{0}\right):=\sum_{\chi} a_{\chi} Z(s, \chi), \quad Z(s, \chi):=\frac{L^{\prime}(s, \chi)}{L(s, \chi)}+\frac{1}{2} \log f_{\chi}
$$

where $L(s, \chi)$ is the Artin $L$-function of $\chi$ and $f_{\chi}$ is the Artin conductor of $\chi$.
Let $\mathbb{Q}^{\mathrm{cm}}$ denote the compositum of all CM number fields. This field is an infinite-degree Galois extension of $\mathbb{Q}$ with a well-defined complex conjugation, which we will denote by $\rho$. Via the quotient map $\operatorname{Gal}\left(\mathbb{Q}^{\mathrm{cm}} / \mathbb{Q}\right) \rightarrow \operatorname{Gal}\left(E^{c} / \mathbb{Q}\right)$, we may consider $A_{\Phi}^{0}$ as a class function on $\operatorname{Gal}\left(\mathbb{Q}^{\mathrm{cm}} / \mathbb{Q}\right)$.

In his 1993 paper [4], Colmez looks at $\mathcal{C} \mathcal{M}^{0}$, the $\mathbb{Q}$ vector space of class functions $f: \operatorname{Gal}\left(\mathbb{Q}^{\mathrm{cm}} / \mathbb{Q}\right) \rightarrow \mathbb{Q}$ such that $f(g)+f(\rho g)$ is independent of $g \in \operatorname{Gal}\left(\mathbb{Q}^{\mathrm{cm}} / \mathbb{Q}\right)$. One can check that, for every CM type $\Phi$, the function $A_{\Phi}^{0}$ is an element of $\mathcal{C} \mathcal{M}^{0}$. Colmez defines a $\mathbb{Q}$-linear height function ht: $\mathcal{C} \mathcal{M}^{0} \rightarrow \mathbb{R}$ such that if $X_{\Phi}$ is an abelian variety with CM by $\left(\mathcal{O}_{E}, \Phi\right)$ with stable Faltings height $\mathrm{h}_{\mathrm{Fal}}\left(X_{\Phi}\right)$, then

$$
\mathrm{h}_{\mathrm{Fal}}\left(X_{\Phi}\right)=-\operatorname{ht}\left(A_{\Phi}^{0}\right)
$$

Here, $E$ is a CM field, $\Phi$ is a CM type of $E$, and $\mathrm{h}_{\text {Fal }}$ denotes the Faltings height of an abelian variety. The $\mathbb{Q}$-linearity of Colmez's ht will be important to us later. Colmez's conjecture is the following alternate formula for ht $\left(A_{\Phi}^{0}\right)$.

Conjecture 1.4. For any $C M$ type $\Phi$, $\operatorname{ht}\left(A_{\Phi}^{0}\right)=Z\left(0, A_{\Phi}^{0}\right)$.

## 2. Proof of the theorem

Before we start the proof of Theorem 1.3, let us introduce some notation. If $F$ is a totally real number field of degree $n$, denote by $F^{c}$ the Galois closure of $F$. Let $k$ be an imaginary quadratic field and let $E:=k F$ be a unitary CM field with complex conjugation $\rho$. We will denote the Galois closure of $E$ by $E^{c}$, and thus $E^{c}=k F^{c}$. Furthermore, let $H:=\operatorname{Gal}\left(F^{c} / F\right) \leq$ $\operatorname{Gal}\left(F^{\mathrm{C}} \mathbb{Q}\right)=: G$ and suppose $\# H=h$. Then, we can identify the embeddings of $F$ into $\mathbb{C}$, which we will call $\left\{\sigma_{1}, \ldots, \sigma_{n}\right\}$, with coset representatives for $H \backslash G$.

An embedding $E \hookrightarrow \mathbb{C}$ is uniquely determined by a pair of embeddings $F \hookrightarrow \mathbb{C}$ and $k \hookrightarrow \mathbb{C}$. We denote by $\{1, \rho\}$ the two embeddings of $k$ into $\mathbb{C}$ and for an embedding $\sigma: F \rightarrow \mathbb{C}$, we write $\rho^{i} \sigma$ for the embedding of $E$ into $\mathbb{C}$ given by the pair $\left\{\rho^{i}, \sigma\right\}$. If $i=0$, we simply write $\sigma$ for $1 \sigma$.

A CM type of $E:=k F$ consists of a choice of one of the embeddings $k \hookrightarrow \mathbb{C}$ for each embedding of $F \hookrightarrow \mathbb{C}$. Thus we can parametrize CM types of $E$ via subsets of $\{1,2, \ldots, n\}$. Given $S \subseteq\{1,2, \ldots, n\}$, the corresponding CM type of $E$ is given by

$$
\Phi_{S}=\left\{\rho^{j_{i}} \sigma_{i}: j_{i}=1 \text { if } i \in S, j_{i}=0 \text { if } i \notin S\right\}
$$

Then, $\Phi_{S}$ is a CM type of signature $(n-\epsilon, \epsilon)$, where $\epsilon=\# S$. We will often write CM types as sums,

$$
\begin{aligned}
\Phi_{S} & =\sum_{i \in S} \rho \sigma_{i}+\sum_{i \notin S} \sigma_{i} \\
& =\operatorname{tr}_{E / k}+(\rho-1) \sum_{i \in S} \sigma_{i} .
\end{aligned}
$$

The first step in the proof of the theorem is an explicit calculation of $A_{\Phi_{S}}^{0}$.
Proposition 2.1. Let $S \subseteq\{1,2, \ldots n\}$ be of size $\epsilon$. Then,

$$
A_{\Phi_{S}}^{0}=\frac{1}{2} \operatorname{tr}_{E^{c} / k}-\frac{\epsilon}{n}(1-\rho) \operatorname{tr}_{E^{c} / k}+\frac{\epsilon}{n^{2}}(1-\rho) \chi_{\operatorname{Ind}_{H}^{G}\left(\chi_{0}\right)}+\frac{1}{h n^{2}}(1-\rho) \sum_{g \in G} g\left(\sum_{i \neq j \in S} \sigma_{i} H \sigma_{j}^{-1}\right) g^{-1}
$$

Proof. Recall that

$$
\Phi_{S}=\operatorname{tr}_{E / k}+(\rho-1) \sum_{i \in S} \sigma_{i}
$$

Extending $\Phi_{S}$ to $\Phi_{S}^{\mathrm{c}}$, the CM type on $E^{\mathrm{c}}$, amounts to determining which embeddings $E^{\mathrm{c}} \hookrightarrow \mathbb{C}$ when restricted to $E$ are in $\Phi_{S}$. Since $E$ is the fixed field in $E^{c}$ by the subgroup $H, \Phi_{S}^{\mathrm{c}}$ is given by

$$
\Phi_{S}^{\mathrm{c}}=\operatorname{tr}_{E^{\mathrm{c}} / k}+(\rho-1) \sum_{i \in S} \sigma_{i} H
$$

When we write $\Phi_{S}^{\mathrm{c}}$ as a sum in this manner, we are interpreting $\Phi_{S}^{\mathrm{c}}$ as an element of $\mathbb{C}\left[G a l\left(E^{\mathrm{c}} / \mathbb{Q}\right)\right]$ that is isomorphic (as a ring) to the ring (under convolution) of all maps from $\operatorname{Gal}\left(E^{c} / \mathbb{Q}\right)$ to $\mathbb{C}$. Next we find the reflex type $\widetilde{\Phi_{S}^{c}}$ by inverting every element in $\Phi_{S}^{\mathrm{c}}$,

$$
\widetilde{\Phi_{S}^{c}}=\operatorname{tr}_{E^{c} / k}+(\rho-1) \sum_{j \in S} H \sigma_{j}^{-1}
$$

Then take the convolution of $\Phi_{S}^{c}$ and $\widetilde{\Phi_{S}^{c}}$,

$$
\begin{aligned}
A_{\Phi_{S}} & =\frac{1}{\left[E^{\mathrm{c}}: \mathbb{Q}\right]} \Phi^{\mathrm{c}} \widetilde{\Phi}^{\mathrm{c}} \\
& =\frac{1}{2 h n}\left(\operatorname{tr}_{E^{\mathrm{c}} / k}+(\rho-1) \sum_{i \in S} \sigma_{i} H\right)\left(\operatorname{tr}_{E^{\mathrm{c}} / k}+(\rho-1) \sum_{j \in S} H \sigma_{j}^{-1}\right) \\
& =\frac{1}{2} \operatorname{tr}_{E^{\mathrm{c}} / k}-\frac{\epsilon}{n}(1-\rho) \operatorname{tr}_{E^{\mathrm{c}} / k}+\frac{1}{n}(1-\rho) \sum_{i, j \in S} \sigma_{i} H \sigma_{j}^{-1}
\end{aligned}
$$

Finally, we need to project $A_{\Phi_{S}}$ onto the space of class functions to obtain $A_{\Phi_{S}}^{0}$,

$$
\begin{equation*}
A_{\Phi_{S}}^{0}=\frac{1}{2} \operatorname{tr}_{E^{\mathrm{c}} / k}-\frac{\epsilon}{n}(1-\rho) \operatorname{tr}_{E^{\mathrm{c}} / k}+\frac{1}{n}(1-\rho) \frac{1}{h n} \sum_{g \in G} g\left(\sum_{i, j \in S} \sigma_{i} H \sigma_{j}^{-1}\right) g^{-1} \tag{1}
\end{equation*}
$$

The main difficulty in (1) is the final term. We first look at the elements of the sum with $i=j$.

$$
\begin{align*}
\frac{1}{h n} \sum_{g \in G} g\left(\sum_{i \in S} \sigma_{i} H \sigma_{i}^{-1}\right) g^{-1} & =\frac{1}{h n} \sum_{i \in S}\left(\sum_{g \in G} g \sigma_{i} H \sigma_{i}^{-1} g^{-1}\right)  \tag{2}\\
& =\frac{1}{h n} \sum_{i \in S}\left(\sum_{g \in G} g H g^{-1}\right)  \tag{3}\\
& =\frac{\epsilon}{h n} \sum_{g \in G} g H g^{-1} \tag{4}
\end{align*}
$$

The following proposition simplifies (4), and combining the following proposition with equation (1) concludes the proof.
Proposition 2.2. Let $\chi_{0}: H \rightarrow \mathbb{C}$ be the trivial character. As functions $G \rightarrow \mathbb{C}$, we have the relation

$$
\sum_{g \in G} g H g^{-1}=h \chi_{\operatorname{Ind}_{H}^{G}\left(\chi_{0}\right)}
$$

Proof. This is proven on page 18 of [10], but we sketch a proof. Recall that the representation $\operatorname{Ind}_{H}^{G}\left(\chi_{0}\right)$ is given by

$$
\operatorname{Ind}_{H}^{G}\left(\chi_{0}\right)=\{f: G \rightarrow \mathbb{C}: f(x g)=f(g) \quad \forall x \in H, g \in G\}
$$

where $G$ acts by right translation. The space $\operatorname{Ind}_{H}^{G}\left(\chi_{0}\right)$ consists exactly of the functions $f: H \backslash G \rightarrow \mathbb{C}$. Therefore a standard calculation shows that the representation $\operatorname{Ind}_{H}^{G}\left(\chi_{0}\right)$ is isomorphic to the representation arising from the action of $G$ on $H \backslash G$ via $g \cdot H \sigma:=H \sigma g^{-1}$. Recall that we have identified $\left\{\sigma_{1}, \ldots, \sigma_{n}\right\}$ with coset representatives for $H \backslash G$.

It is straightforward to compute the character of a permutation representation, namely it is the number of fixed points. That is to say, for $\sigma \in G$,

$$
\begin{aligned}
\chi_{\operatorname{Ind}_{H}^{G}\left(\chi_{0}\right)}(\sigma) & =\#\left\{i \in\{1, \ldots, n\}: \sigma \cdot H \sigma_{i}=H \sigma_{i}\right\} \\
& =\#\left\{i \in\{1, \ldots, n\}: \sigma \in \sigma_{i} H \sigma_{i}^{-1}\right\} .
\end{aligned}
$$

On the other hand,

$$
\left(\sum_{g \in G} g H g^{-1}\right)(\sigma)=\#\left\{g \in G: \sigma \in g H g^{-1}\right\}
$$

However, for a given $i$ with $\sigma \in \sigma_{i} H \sigma_{i}^{-1}$, then every $g \in \sigma_{i} H$ satisfies $\sigma \in g \mathrm{Hg}^{-1}$ and since $\# H=h$, we obtain $\sum_{g \in G} g H g^{-1}=$ $h \chi_{\operatorname{Ind}_{H}^{G}\left(\chi_{0}\right)}$.

Let us record a few particular cases of Proposition 2.1 which will be of use:

$$
\begin{aligned}
& A_{\Phi_{\varnothing}}^{0}=\frac{1}{2} \operatorname{tr}_{E^{c} / k}, \\
& A_{\Phi_{\{i\}}}^{0}=\frac{1}{2} \operatorname{tr}_{E^{c} / k}-\frac{1}{n}(1-\rho) \operatorname{tr}_{E^{c} / k}+\frac{1}{n^{2}}(1-\rho) \chi_{\operatorname{Ind}_{H}^{G}\left(\chi_{0}\right)}, \\
& A_{\Phi_{\{i, j\}}}^{0}=\frac{1}{2} \operatorname{tr}_{E^{c} / k}-\frac{2}{n}(1-\rho) \operatorname{tr}_{E^{c} / k}+\frac{2}{n^{2}}(1-\rho) \chi_{\operatorname{Ind}_{H}^{G}\left(\chi_{0}\right)}+\frac{1}{h n^{2}}(1-\rho) \sum_{g \in G}\left(g \sigma_{i} H \sigma_{j}^{-1} g^{-1}+g \sigma_{j} H \sigma_{i}^{-1} g^{-1}\right) .
\end{aligned}
$$

Corollary 2.3. For any subset $S \subseteq\{1,2, \ldots, n\}$ of size $\epsilon$, we have

$$
A_{\Phi_{S}}^{0}=\sum_{\{i, j\} \subseteq S} A_{\Phi_{\{i, j\}}}^{0}-(\epsilon-2) \sum_{i \in S} A_{\Phi_{\{i\}}}^{0}+\frac{(\epsilon-1)(\epsilon-2)}{2} A_{\Phi_{\varnothing}}^{0}
$$

Theorem 1.3 follows from Corollary 2.3, the linearity of ht and $Z$ and the known equalities $\operatorname{ht}\left(A_{\Phi_{\varnothing}}^{0}\right)=Z\left(0, A_{\Phi_{\varnothing}}^{0}\right)$ (the classical Chowla-Selberg formula) and $\operatorname{ht}\left(A_{\Phi_{(i)}}^{0}\right)=Z\left(0, A_{\Phi_{(i)}}^{0}\right)$ (due to Yang and Yin [10]).

## 3. Galois action on CM types

There is an action of Galois on the set of CM types. Namely, $g \in \operatorname{Gal}\left(E^{c} / \mathbb{Q}\right)$ acts on a CM type $\Phi$ by $g \cdot \Phi:=\{g \sigma: \sigma \in \Phi\}$. It is well known and straightforward to check that if $\Phi_{1}$ and $\Phi_{2}$ are two $C M$ types that are equivalent under this action, then $A_{\Phi_{1}}^{0}=A_{\Phi_{2}}^{0}$.

As in Section 2, let $F$ be a totally real number field with Galois closure $F^{c}$, with $G:=\operatorname{Gal}\left(F^{c} / \mathbb{Q}\right)$ and $H:=\operatorname{Gal}\left(F^{c} / F\right)$. Let $k$ be an imaginary quadratic field and consider the unitary CM field $E:=k F$. We can describe the action of $\operatorname{Gal}\left(E^{\mathrm{c}} / \mathbb{Q}\right) \cong$ $G \times \mathbb{Z} / 2 \mathbb{Z}$ on the set of $C M$ types. The $\mathbb{Z} / 2 \mathbb{Z}$ component acts as a complex conjugation, taking a $C M$ type of signature $(n-\epsilon, \epsilon)$ to a CM type of signature $(\epsilon, n-\epsilon)$. The action of $\operatorname{Gal}\left(E^{\mathrm{C}} / k\right)$ fixes the signature of a CM type and this action on CM types of signature ( $n-\epsilon, \epsilon$ ) is isomorphic to the action of $G$ on the set of subsets of $G / H$ of size $\epsilon$.

One of the main results of [10] is that the Colmez conjecture holds if we average amongst CM types of a given signature. That is to say, if $\Phi(E)_{\epsilon}$ denotes all CM types of $E$ of signature ( $n-\epsilon, \epsilon$ ), then

$$
\sum_{\Phi \in \Phi(E)_{\epsilon}} \operatorname{ht}\left(A_{\Phi}^{0}\right)=\sum_{\Phi \in \Phi(E)_{\epsilon}} Z\left(0, A_{\Phi}^{0}\right)
$$

If there is only one equivalence class of CM types in a given signature, then their result immediately implies that the Colmez conjecture holds for the CM type of that signature. This is the idea behind Yang and Yin's proof that the Colmez conjecture holds for CM types of signature $(n, 0)$ and $(n-1,1)$. In particular, combining these ideas with our Theorem 1.3 gives the following theorem.

Theorem 3.1. Let $k$ be an imaginary quadratic field and let $F$ be a totally real number field with Galois closure $F^{c}$. Let $H:=$ $\operatorname{Gal}\left(F^{\mathrm{c}} / F\right) \leq \operatorname{Gal}\left(F^{\mathrm{c}} / \mathbb{Q}\right)=$ : G. If G acts 2-transitively on $G / H$, then the Colmez conjecture holds for every CM type of the unitary $C M$ field $E:=k F$.

We note that there are examples of such a pair $(G, H)$. The fact that $G$ is the Galois group of the Galois closure of $F$ and $H$ is the subgroup fixing the field $F$ implies that the action of $G$ on $G / H$ induces an embedding $G \hookrightarrow \operatorname{Sym}(G / H)$ of $G$ into the symmetric group on the set of cosets of $G$ by $H$.

Thus, we may apply Theorem 3.1 to any $G$ that is a doubly transitive subgroup of a symmetric group, and we take $H$ to be the stabilizer of any element. As a corollary of the classification of finite simple groups, a classification of doubly transitive subgroups of symmetric groups is known. There are infinite families and sporadic examples. We list this information in the following table, as well as whether or not the group $G$ is known to be the Galois group of a totally real field. We refer the reader to [6] and [2] for further details on these groups and their doubly transitive actions. In the following table, unless otherwise specified $n$ is an integer and $q$ is a prime power.

| G | H | Is $G$ the Galois group of a totally real field? |
| :---: | :---: | :---: |
| $S_{n}$ | $S_{n-1}$ | Yes [7]. |
| $A_{n}$ | $A_{n-1}$ | Yes for $n \leq 16$ [8]. |
| $\mathrm{PSL}_{n}\left(\mathbb{F}_{q}\right)$ | Stabilizer of a point in $\mathbb{P}^{n-1}\left(\mathbb{F}_{q}\right)$ | Yes for $n=2, q \leq 11$ [8]. |
| $\mathrm{PGL}_{n}\left(\mathbb{F}_{q}\right)$ | Stabilizer of a point in $\mathbb{P}^{n-1}\left(\mathbb{F}_{q}\right)$ | Yes for $n=2, q \leq 7$ [8]. |
| $\mathrm{Sp}_{2 m}\left(\mathbb{F}_{2}\right), m \geq 2$ | $\mathrm{GO}_{2 m}^{+}\left(\mathbb{F}_{2}\right)$ |  |
| $\mathrm{Sp}_{2 m}\left(\mathbb{F}_{2}\right), m \geq 2$ | $\mathrm{GO}_{2 m}^{-}\left(\mathbb{F}_{2}\right)$ |  |
| $\mathrm{PSU}_{3}\left(\mathbb{F}_{q}\right)$ | Stabilizer of a isotropic line in $\mathbb{F}_{q^{2}}^{3}$ |  |
| $\mathrm{PGU}_{3}\left(\mathbb{F}_{q}\right)$ | Stabilizer of a isotropic line in $\mathbb{F}_{q^{2}}^{3}$ |  |
| $\mathrm{Sz}(q), q$ an odd power of 2 | Stabilizer in $S_{q^{2}+1}$ |  |
| $\mathrm{R}(q), q$ an odd power of 3 | Stabilizer in $S_{q^{3}+1}$ |  |
| $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$ | $M_{i}$ is a doubly transitive subgroup of $S_{i}$ | Yes for $M_{11}$ [8]. |
| $M_{11}$ | Stabilizer in $S_{12}$ |  |
| $\mathrm{PSL}_{2}\left(\mathbb{F}_{11}\right)$ | Stabilizer in $S_{11}$ | Yes [8]. |
| $A_{7}$ | Stabilizer in $S_{8}$ |  |
| HS | Stabilizer in $S_{176}$ |  |
| $\mathrm{Co}_{3}$ | Stabilizer in $S_{276}$ |  |

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