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Partial differential equations

Self-adjoint and skew-symmetric extensions of the Laplacian with singular Robin boundary condition

Extensions self-adjointes et anti-symétriques du laplacien, avec condition à la frontière de type Robin singulière

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A B S T R A C T

We study the Laplacian in a bounded domain, with a varying Robin boundary condition singular at one point. The associated quadratic form is not semi-bounded from below, and the corresponding Laplacian is not self-adjoint, it has a residual spectrum covering the whole complex plane. We describe its self-adjoint extensions and exhibit a physically relevant skew-symmetric one. We approximate the boundary condition, giving rise to a family of self-adjoint operators, and we describe its spectrum by the method of matched asymptotic expansions. A part of the spectrum acquires a strange behavior when the small perturbation parameter $\varepsilon > 0$ tends to zero, namely it becomes almost periodic in the logarithmic scale $|\ln \varepsilon|$, and in this way "wanders" along the real axis at a speed $O(\varepsilon^{-1})$.

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R É S U M É

Nous étudions le laplacien dans un domaine borné, avec une condition à la frontière de type Robin, variable et singulière en un point. La forme quadratique associée n'est pas bornée inférieurement, et le laplacien correspondant n'est pas self-adjoint ; son spectre résiduel couvre entièrement le plan complexe. Nous décrivons ses extensions self-adjointes et nous en montrons une anti-symétrique, pertinente en physique. Nous approchons la condition de frontière à l'aide d'une famille d'opérateurs self-adjoints et nous décrivons son spectre par la méthode d'appariement des développements asymptotiques. Une partie du spectre adopte un comportement étrange quand le paramètre $\varepsilon > 0$ de petite perturbation tend vers zéro ; précisément, il devient presque périodique en échelle logarithmique $|\log(\varepsilon)|$, et ainsi « erre » le long de l'axe réel à une vitesse $O(\varepsilon^{-1})$.

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1. Description of the singular problem

In a domain $\Omega \subset \mathbb{R}^2$ enveloped by a smooth simple contour $\partial\Omega$, we consider the Laplacian with a Robin-type boundary condition $a\partial_n u - u = 0$. Here, a is a continuous function defined on $\partial\Omega$, and ∂_n denotes the outward normal derivative to $\partial\Omega$.

If a is positive on $\partial\Omega$, the quadratic form $H^1(\Omega) \ni u \mapsto \|\nabla u; L^2(\Omega)\|^2 - \|a^{-1/2}u, L^2(\partial\Omega)\|^2$ is naturally associated with this problem and, in view of the compact imbedding $H^1(\Omega) \subset L^2(\partial\Omega)$, this form is semi-bounded and closed, and thus defines a self-adjoint operator with compact resolvent. Therefore, the spectrum is an unbounded sequence of real eigenvalues accumulating at $+\infty$. Note that the first eigenvalue is negative, and goes to $-\infty$ if a is a small positive constant, see [5].

Let a become zero at a point $x_0 \in \partial\Omega$. In this note, we will mainly consider the case where a vanishes at order one, i.e. admits the Taylor formula

$$a(s) = a_0 s + O(s^2), \quad s \to 0 \quad \text{with } a_0 > 0, \tag{1}$$

where s is a curvilinear abscissa starting at x_0. For convenience, we denote $b_0 := a_0^{-1}$.

Since we assume a to be continuous, there should exist at least one other point where a vanishes. However, several points of vanishing do not bring any new effect, and we replace the problem by another one: we assume that $\partial\Omega$ is the union of two smooth curves Γ_1 and Γ_2 which meet perpendicularly, with x_0 in the interior of Γ_1, and that a vanishes only at x_0, according to (1). We complete the Robin boundary condition on Γ_1 by a Neumann boundary condition on Γ_2. Therefore, our spectral problem is

$$\begin{cases} -\Delta u = \lambda u \text{ on } \Omega, \\ a\partial_n u - u = 0 \text{ on } \Gamma_1, \text{ and } \partial_n u = 0 \text{ on } \Gamma_2. \end{cases} \tag{2}$$

The associated quadratic form is defined on $D(q) := \{u \in H^1(\Omega), a^{-\frac{1}{2}} u_{|\Gamma_1} \in L^2(\Gamma_1)\}$ as follows

$$D(q) \ni u \mapsto \int_\Omega |\nabla u|^2 \mathrm{d}x - \int_{\Gamma_1} a^{-1}|u|^2 \mathrm{d}s.$$

It is not semi-bounded anymore. Thus, there is no canonical way for defining a self-adjoint operator associated with problem (2). The natural definition becomes the operator A_0 acting as $-\Delta$ on the domain

$$D(A_0) := \{u \in D(q), \Delta u \in L^2(\Omega), a\partial_n u - u = 0 \text{ on } \Gamma_1, \partial_n u = 0 \text{ on } \Gamma_2\}. \tag{3}$$

Such a problem was studied in [1,6] in a model half-disk, for which the eigenvalue equation had the advantage to decouple in polar coordinates. The authors found that A_0 is non-self-adjoint. In [6], they clarified the "paradox" from [1] stating that, for any $\lambda \in \mathbb{C}$, problem (2) has a nontrivial solution, by showing that the spectrum of A_0^* is residual and coincides with the complex plane.

A three-dimensional version of this spectral problem appears also in the modeling of a spinless particle moving in two thin films with a one-contact point, and has been studied in a model domain in [3].

2. Goal and results

In this note, we explain how to find extensions of A_0 and give a better understanding of their spectrum, arguing with an asymptotic approach. We also exhibit a relevant skew-symmetric extension using a physical argument.

The domain of A_0^* is

$$D(A_0^*) := \{u \in L^2(\Omega), \Delta u \in L^2(\Omega), a\partial_n u - u = 0 \text{ on } \Gamma_1, \ \partial_n u = 0 \text{ on } \Gamma_2\}.$$

To understand how different $D(A_0^*)$ is from $D(A_0)$, we exhibit two possible singular behaviors for functions in $D(A_0^*)$ at the point x_0. Using Kondratiev's theory [4], we investigate a model problem in a half-plane and, as a result, describe $D(A_0^*)$. We deduce, going over the domain Ω, that the deficiency indices of A_0 are (1,1), and we classify its self-adjoint extensions using a parametrization $\theta \mapsto e^{i\theta}$ of the unit circle $\mathbb{S}^1 \subset \mathbb{C}$. The description of A_0^* allows us also to introduce a natural skew-symmetric extension of A_0 corresponding to a Sommerfeld radiation condition at x_0.

Next, we approach our problem by a family of self-adjoint operators by choosing a suitable perturbation of the Robin coefficient a. This is done by means of the non-vanishing discontinuous function

$$a_\varepsilon(s) = a_0 \operatorname{sign}(s)\varepsilon + a(s) \tag{4}$$

satisfying $\inf_{\Gamma_1} |a_\varepsilon| = \varepsilon$, and we study the discrete spectrum of the associated Robin Laplacian as $\varepsilon \to 0$. Using the method of matched asymptotic expansions, we find that its spectrum is related to the eigenvalues of self-adjoint extensions, with a parameter θ_ε oscillating in the logarithmic scale as $\varepsilon \to 0$.

Finally, we describe the differences when the weight function satisfies $a(s) = a_0|s| + O(s^2)$ near the singular point, with $a_0 > 0$. In particular, the number of singularities of functions in $D(A_0^*)$ is now two or four, depending on whether $a_0 > \frac{\pi}{2}$ or not.

A similar result has been obtained in [2], where an operator of the type $\operatorname{div}(\sigma\nabla)$ is considered in a bounded domain, where σ is piecewise constant and changes sign along an interface crossing the boundary.

3. Description of the adjoint operator

In this section, we investigate the following model problem in the half-plane $\mathbb{R}^2_+$: find $\mathfrak{u} \in L^2_{\text{loc}}(\mathbb{R}^2_+)$ such that

$$\begin{cases} -\Delta\mathfrak{u} = 0 \ \text{ on } \ \mathbb{R}^2_+, \\ \mathfrak{a}\,\partial_n\mathfrak{u} - \mathfrak{u} = 0 \ \text{ on } \ \partial\mathbb{R}^2_+, \end{cases} \tag{5}$$

where $\mathfrak{a}$ is the first-order approximation of a near x_0: $\mathfrak{a}(s) = a_0 s$. Let $(r,\varphi) \in (0,+\infty) \times (-\frac{\pi}{2},\frac{\pi}{2})$ be the associated polar coordinates, the normal derivative reads $\partial_n\mathfrak{u}(s,0) = \mp r^{-1}\partial_\varphi\mathfrak{u}(r,\pm\frac{\pi}{2})$. The boundary condition is decoupled: The problem becomes, in polar coordinates:

$$\begin{cases} -\partial_r^2\mathfrak{u} - r^{-1}\partial_r\mathfrak{u} - r^{-2}\partial_\varphi^2\mathfrak{u} = 0 \ \text{ on } \ (0,+\infty)\times(-\frac{\pi}{2},\frac{\pi}{2}), \\ \forall r > 0: \quad -a_0\partial_\varphi\mathfrak{u} - \mathfrak{u} = 0 \ \text{ at } \varphi = \pm\frac{\pi}{2}. \end{cases}$$

The spectrum of the transverse operator $-\partial_\varphi^2$ is given by solving the eigenvalue problem:

$$-g''(\varphi) = \mu g(\varphi), \quad -a_0 g'(\pm\tfrac{\pi}{2}) - g(\pm\tfrac{\pi}{2}) = 0. \tag{6}$$

Its eigenvalues are $\{\mu_k, k \geq 0\} := \{-b_0^2\} \cup \{k^2, k = 1,2,\ldots\}$. The eigenspace associated with $-b_0^2$ is generated by $g_0(\varphi) = \mathrm{e}^{-b_0\varphi}$, and the one associated with k^2 by

$$g_k(\varphi) = \sin(k(\varphi + \tfrac{\pi}{2})) - ka_0\cos(k(\varphi + \tfrac{\pi}{2})).$$

We introduce two singular solutions to (5):

$$\mathfrak{s}^\pm(r,\varphi) = \mathfrak{c}\, r^{\pm i b_0}\mathrm{e}^{-b_0\varphi} \ \text{ with } \ \mathfrak{c} = (2\sinh(b_0\pi))^{-1/2} \tag{7}$$

where the choice for the normalizing factor $\mathfrak{c}$ will become clear in Proposition 2. Note that $\mathfrak{s}^\pm \notin H^1_{\text{loc}}(\overline{\mathbb{R}^2_+})$.

Let χ be a smooth cut-off function that has a small support and equals one near the point x_0, and let $S^\pm$ be the functions deduced in Ω from $\mathfrak{s}^\pm$: $S^\pm(x) = \chi(x)\mathfrak{s}^\pm(r,\theta)$ through local polar coordinates $\Omega \ni x \mapsto (r,\theta) \in \mathbb{R}^2_+$ near x_0.

As a consequence of the Kondratiev theorem on asymptotics (see [4] and, e.g., [9, Ch. 3]), we get

Proposition 1. *Let $u \in D(A_0^*)$, then there exists $(c_{\text{in}}, c_{\text{out}}) \in \mathbb{C}^2$ such that*

$$u = c_{\text{in}}(u)S^- + c_{\text{out}}(u)S^+ + \widetilde{u} \tag{8}$$

where $\widetilde{u} \in H^2(\Omega) \cap D(q)$. Moreover, there exists $C > 0$ such that, for all $u \in D(A_0^*)$, we have

$$|c_{\text{in}}(u)| + |c_{\text{out}}(u)| + \|\widetilde{u}\|_{L^2(\Omega)} \leq C(\|u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)}). \tag{9}$$

This decomposition of the operator domain is sufficient to deduce the deficiency indices of the operator. On the one hand, since the operator has real coefficient, $\dim(\ker(A_0^* + \mathrm{i})) = \dim(\ker(A_0^* - \mathrm{i}))$. On the other hand, the standard decomposition together with the last proposition implies $\dim(\ker(A_0^* + \mathrm{i})) + \dim(\ker(A_0^* - \mathrm{i})) = \dim(D(A_0^*)/D(A_0)) = 2$. Therefore, $\dim(\ker(A_0^* \pm \mathrm{i})) = 1$, and the deficiency indices are (1,1). As a corollary, the spectrum of A_0 covers the whole complex plane.

4. Self-adjoint extensions

Once the domain of the adjoint is explicit, it is standard, see [7,10] and others, to find all self-adjoint extensions of A_0 by the use of the symplectic form

$$\psi : (u,v) \mapsto \langle A_0^* u, v\rangle - \langle u, A_0^* v\rangle, \ \text{ defined on } \ D(A_0^*).$$

As a consequence of integration by parts and symplectic algebra, we verify Proposition 2.

Proposition 2. *Let u and v in $D(A_0^*)$, written in the form (8). Then*

$$\psi(u,v) = \mathrm{i}\left(c_{\mathrm{in}}(u)\overline{c_{\mathrm{in}}(v)} - c_{\mathrm{out}}(u)\overline{c_{\mathrm{out}}(v)}\right).$$

As a consequence of Proposition 2, for any $u \in D(A_0^*)$, we obtain

$$\psi(u,u) = \mathrm{i}(|c_{\mathrm{in}}|^2 - |c_{\mathrm{out}}|^2).$$

Therefore, the self-adjoint extensions are the restrictions of A_0^* onto linear spaces of the functions $u \in D(A_0^*)$, for which $|c_{\mathrm{in}}(u)| = |c_{\mathrm{out}}(u)|$. We conclude with Theorem 3.

Theorem 3. *Let $\theta \in \mathbb{R}$, and let $A_0(\theta)$ be the restriction of A_0^* to the domain*

$$D(A_0(\theta)) = \{u \in D(A_0^*), c_{\mathrm{in}}(u) = \mathrm{e}^{\mathrm{i}\theta} c_{\mathrm{out}}(u)\}.$$

Then A^{sa} is a self-adjoint extension of A_0 if and only if there exists $\theta \in \mathbb{R}$ such that $A^{\mathrm{sa}} = A_0(\theta)$.

Each domain of these extensions has compact injection in $L^2(\Omega)$ because a function $u \in D(A_0(\theta))$ differs from $\widetilde{u} \in H^2(\Omega)$ by a linear combination of two functions $S^\pm \in L^2(\Omega)$. Therefore, each of these extensions has compact resolvent. Moreover, it is not semi-bounded from below, and we denote by $(\lambda_k(\theta))_{k\in\mathbb{Z}}$ the increasing sequence of eigenvalues of $A_0(\theta)$.

5. The physical radiation condition and a skew-symmetric extension

In link with the wave equation $-\partial_t^2 W = -\Delta W$, analyzing the propagation of the wave

$$W^\pm(t,x) := \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}\,t} S^\pm(x) = \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}(t \mp \lambda^{-1/2} b_0 \ln r)} g_0(\varphi),$$

in the framework of the Sommerfeld or Mandelstam principles, cf. [9, Ch. 5], we can interpret S^- as propagating from x_0, whereas S^+ would propagate toward x_0. Notice that any other radiation principle leads to the same conclusion.

For a fixed $\lambda \in \mathbb{R}$, the scattering theory (cf. [9, Ch. 5]) provides a solution to (2) in the form

$$\zeta_\lambda = S^+ + \mathrm{e}^{\mathrm{i}\theta_\lambda} S^- + \widetilde{\zeta}_\lambda, \quad \text{with } \widetilde{\zeta}_\lambda \in D(q) \cap H^2(\Omega). \tag{10}$$

This solution is interpreted as the scattering wave initiated with the incident (entering) wave S^+, and $\mathrm{e}^{\mathrm{i}\theta_\lambda}$ is the reflection coefficient, with $|\mathrm{e}^{\mathrm{i}\theta_\lambda}| = 1$, according to the conservation of energy.

Moreover, a natural skew-symmetric extension $\mathfrak{A}_0$ of A_0 can be defined in the domain

$$D(\mathfrak{A}_0) = \{u \in D(A_0^*), c_{\mathrm{in}}(u) = 0\}.$$

This extension corresponds to the natural radiation condition, excluding entering waves.

6. Wandering of the eigenvalues

Assume that $\lambda_k(\theta)$ is a simple eigenvalue of $A_0(\theta)$, and denote by $C_0(S^+ + \mathrm{e}^{\mathrm{i}\theta} S^-) + \widetilde{u}$ an associated eigenfunction normalized in $L^2(\Omega)$. Then standard computations show that $\partial_\theta \lambda_k(\theta) = -|C_0|^2$. Therefore, an eigenvalue $\lambda_k(\cdot)$ is a non-increasing function of $\theta \in \mathbb{R}$ wherever it is simple; moreover, it is decreasing if $C_0 \neq 0$. If $C_0 = 0$ for some k and θ, then the constant eigenvalue $\lambda_k(\theta) = \lambda$ is associated with a trapped mode, that is a non-trivial solution to problem (2) belonging to $D(q) \cap H^2(\Omega)$, which is in $D(A(\theta))$ for any $\theta \in \mathbb{R}$.

The functions $\lambda_k(\cdot)$ are piecewise analytic; moreover, they cannot be all constant, indeed in that case the range of all the eigenvalues $\lambda_k(\theta)$ would be a discrete set, which contradicts the existence of the physical solution in the form (10) for any $\lambda \in \mathbb{R}$.

Therefore, there exists at least one branch $\lambda_k(\cdot)$ which is decreasing where it is regular. This, combined with the periodicity of the spectrum, shows that

$$\bigcup_{(k,\theta)\in\mathbb{Z}\times\mathbb{R}} \lambda_k(\theta) = \mathbb{R}. \tag{11}$$

7. The method of matched asymptotic expansions

For $\varepsilon > 0$, we recall that a_ε was defined in (4) as an approximation of a. Now the quadratic form

$$u \mapsto \int_\Omega |\nabla u|^2 - \int_{\Gamma_1} a_\varepsilon^{-1}(s)|u|^2$$

is well defined and bounded from below in $H^1(\Omega)$. We denote by A^ε the corresponding self-adjoint operator. The strategy is now to construct quasi-modes for A^ε through eigenfunctions of an extension $A_0(\theta_\varepsilon)$, where θ_ε is to be chosen. The result is given by Theorem 4.

Theorem 4. *For all $k \geq 0$ and for ε small enough, there exists $\theta_\varepsilon \in \mathbb{R}$ such that* dist $(\lambda_k(\theta_\varepsilon), \sigma(A^\varepsilon)) \to 0$ *as $\varepsilon \to 0$. Moreover, the mapping $\varepsilon \mapsto \theta_\varepsilon$ is periodic with respect to* $\ln \varepsilon$, *and*

$$\mathbb{R} = \bigcup_{\varepsilon \downarrow 0} \sigma(A^\varepsilon). \tag{12}$$

The procedure is as follows.

• *Far-field expansion:* outside a fixed neighborhood of x_0, take a function u^{out} as an eigenfunction of $A_0(\theta_\varepsilon)$, where θ_ε is to be chosen. Therefore, it behaves near x_0 as follows:

$$u^{\text{out}} : (r, \varphi) \mapsto C(r^{\mathrm{i}b_0} + \mathrm{e}^{\mathrm{i}\theta_\varepsilon} r^{-\mathrm{i}b_0})\mathrm{e}^{a_0^{-1}\varphi} + \widetilde{u}^{\text{out}},$$

where $\widetilde{u}^{\text{out}}$ is regular and small near 0.

• *Near-field expansion.* In local coordinates near x_0, we perform the scaling $x = \varepsilon\xi$, and considering bounded eigenvalues, we get to solve (5) with $\mathfrak{a}(\xi_1) := a_0(\text{sign}(\xi_1) + \xi_1)$. In order to investigate the behavior of solutions to this problem at infinity, we perform the inversion $\xi \mapsto \eta = |\xi|^{-2}\xi$, which leads to the behavior at the origin $\eta = 0$ of (5), but with the weight function $a_0 : \eta_1 \mapsto \eta_1 + \text{sign}(\eta_1)\eta_1^2$ in the boundary condition. Near the origin $\eta = 0$, we can neglect the part $\text{sign}(\eta_1)\eta_1^2$, and according to Kondratiev's theory [4], there exists a solution to such a problem that behaves as

$$\eta \mapsto \mathfrak{s}^-(\eta) + \mathrm{e}^{\mathrm{i}\theta}\mathfrak{s}^+(\eta) + O(|\eta|), \quad \theta \in \mathbb{R} \text{ fixed}.$$

Therefore, we obtain a solution to (5) with weight $\mathfrak{a}$, which produces after rescaling a solution to the Laplace equation in Ω that behaves in a neighborhood of x_0 as

$$u^{\text{in}} : (r, \varphi) \mapsto \widetilde{C}(\varepsilon^{\mathrm{i}b_0} r^{-\mathrm{i}b_0} + \mathrm{e}^{\mathrm{i}\theta}\varepsilon^{-\mathrm{i}b_0} r^{\mathrm{i}b_0})\mathrm{e}^{a_0^{-1}\varphi} + \widetilde{u}^{\text{in}}$$

where $\widetilde{u}^{\text{in}}$ is decaying outside the neighborhood, and $\widetilde{C}$ is a normalization factor.

• *Matching expansions and conclusion.* Matching the two previous expansions, we obtain:

$$\theta_\varepsilon = \theta - 2b_0 \ln \varepsilon \pmod{2\pi}. \tag{13}$$

This formal approach is validated by constructing the quasi-mode from the previous ansätze using cut-off functions: define

$$u^{\text{as}} = \chi^{\text{in}} u^{\text{in}} + \chi^{\text{out}} u^{\text{out}} - \chi^{\text{in}}\chi^{\text{out}} C(\mathfrak{s}^+ + \mathrm{e}^{\mathrm{i}\theta_\varepsilon}\mathfrak{s}^-),$$

where χ^{in} (respectively, χ^{out}) is localized in a bounded neighborhood of x_0 (respectively, outside a neighborhood of x_0 of size $O(\varepsilon)$). Evaluating $(A^\varepsilon - \lambda_k(\theta_\varepsilon))u^{\text{as}}$, we get that $\lambda_k(\theta_\varepsilon)$ is close to the spectrum of A^ε for ε small enough. Note that θ_ε is periodic with respect to $\ln \varepsilon$ and $\mathrm{e}^{\mathrm{i}\theta_\varepsilon}$ runs over $\mathbb{S}^1 \subset \mathbb{C}$ at the rate $O(\varepsilon^{-1})$ as $\varepsilon \to 0$. Then, (12) follows from (11).

8. Further questions

When the weight function satisfies $a(s) = a_0|s| + O(s^2)$, with $a_0 > 0$, the situation depends on the parameter a_0, as described here: The transverse operator in the angular variable in the model half-plane $\mathbb{R}^2_+$ is still $-\partial^2_\varphi$, but the boundary condition at $\varphi = \frac{\pi}{2}$ in (6) now becomes $a_0 g'(\frac{\pi}{2}) - g(\frac{\pi}{2}) = 0$. The negative spectrum of this operator depends on a_0 as follows.

1° If $a_0 > \frac{\pi}{2}$, then there is one negative eigenvalue, and the other ones are positive. It produces two oscillatory solutions.

2° If $a_0 = \frac{\pi}{2}$, there is one negative eigenvalue, and null is also an eigenvalue. There are two additional solutions for the model problem, one has the form $g_0(\varphi)\ln r$, and the other is constant with respect to r.
3° If $a_0 < \frac{\pi}{2}$, then there are two negative eigenvalues. They produce four oscillatory solutions.

Situation 1° can be analyzed exactly in the same way as that we described here. However, situations 2° and 3° are much more different. In particular, the deficiency indices are (2,2), and the self-adjoint extensions are parameterized by two-by-two unitary matrices. The method of the matched asymptotic expansions does not provide an explicit parameter extension as in (13), but a family of unitary matrices depending on ε, cf [8]. This family does not always coincide with the set of all unitary matrices as $\varepsilon \to 0$, but it is sufficient for the construction of approximations.

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