Assume that \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) is an analytic function germ at the origin with only isolated singularity. Set \( X = f^{-1}(0) \). Let \( S = \mathbb{C}[x_1, \ldots, x_n] \) denote the formal power series ring. Set \( J_f = (\partial f / \partial x_1, \ldots, \partial f / \partial x_n) \) as the Jacobian ideal. Then the Milnor and Tjurina algebras are defined as

\[
M_f = S / J_f, \quad T_f = S / (J_f, f).
\]

Since \( X \) has isolated singularities, \( M_f \) and \( T_f \) are finite dimensional \( \mathbb{C} \)-vector spaces. The corresponding dimension \( \mu \) and \( \tau \) are called the Milnor and Tjurina numbers, respectively. It is clear that \( \mu \geq \tau \).
Consider the following long exact sequence of $\mathbb{C}$-algebras:
\[
0 \to \text{Ker}(f) \to M_f \xrightarrow{f} M_f \to T_f \to 0
\]  
(1)
where the middle map is multiplication by $f$, and Ker$(f)$ is the kernel of this map. Then \( \dim_{\mathbb{C}} \text{Ker}(f) = \tau \).

Recall a well-known result given by J. Briançon and H. Skoda in [1],
\[
f^n \in J_f,
\]
which shows that \( f^n = 0 \) in \( M_f \), i.e. \( (f^{n-1}) \subset \text{Ker}(f) \). Here \( (f^{n-1}) \) is the ideal in \( M_f \) generated by \( f^{n-1} \). The following theorem is a direct application of this result.

**Theorem 1.1.** Assume that \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) is an analytic function germ at the origin with only isolated singularity. Then,
\[
\frac{\mu}{\tau} \leq n.
\]
Moreover, \( \frac{\mu}{\tau} = n \), if and only if, \( \text{Ker}(f) = (f^{n-1}) \).

**Proof.** Since \( f^n = 0 \) in \( M_f \), we have the following finite decreasing filtration:
\[
M_f \supset (f) \supset (f^2) \supset \cdots (f^{n-1}) \supset (f^n) = 0
\]
where \((f^i)\) is the ideal in \( M_f \) generated by \( f^i \).

Consider the following long exact sequence:
\[
0 \to \text{Ker}(f) \cap (f^i) \to (f^i) \xrightarrow{f} (f^i) \to (f^i)/((f^{i+1}) \to 0
\]  
(2)
where the middle map is multiplication by \( f \). Then,
\[
\dim_{\mathbb{C}}((f^i)/((f^{i+1})) = \dim_{\mathbb{C}}(\text{Ker}(f) \cap (f^i)) \leq \dim_{\mathbb{C}} \text{Ker}(f) = \tau.
\]
Therefore,
\[
\mu = \dim_{\mathbb{C}} M_f = \dim_{\mathbb{C}} T_f + \sum_{i=1}^{n-1} \dim_{\mathbb{C}}((f^i)/((f^{i+1})) \leq n \cdot \tau.
\]
\[
\frac{\mu}{\tau} = n \text{ if and only if, for any } 1 \leq i \leq n - 1, \text{Ker}(f) \cap (f^i) = \text{Ker}(f), \text{i.e. Ker}(f) \subset (f^i). \text{On the other hand, } (f^{n-1}) \subset \text{Ker}(f). \text{Hence, Ker}(f) = (f^{n-1}). \quad \square
\]

K. Saito showed ([8]) that \( \frac{\mu}{\tau} = 1 \) holds, if and only if, \( f \) is weighted homogeneous, i.e. analytically equivalent to such a polynomial. It leads to the following natural question.

**Question 1.2.** Is this upper bound of \( \frac{\mu}{\tau} \) optimal? When can the optimal upper bound be obtained?

**Remark 1.3.** Recently, A. Dimca and G.-M. Greuel showed ([3, Theorem 1.1]) that the upper bound \( \frac{\mu}{\tau} \leq 2 \) can never be achieved for the isolated plane curve singularity case unless \( f \) is smooth at the origin. Moreover, they gave ([3, Example 4.1]) a sequence of isolated plane curve singularity with the ratio \( \frac{\mu}{\tau} \) strictly increasing towards \( 4/3 \). In particular, the singularities can be chosen to be all either irreducible, or consisting of smooth branches with distinct tangents. Based on these computations, they asked ([3, Question 4.2]) whether
\[
\frac{\mu}{\tau} < \frac{4}{3}
\]
for any isolated plane curve singularity.

**Example 1.4.** It is clear that \( \frac{\mu}{\tau} > n - 1 \) implies that \( f^{n-1} \notin J_f \).

Consider the function germ:
\[
f = (x_1 \cdots x_n)^2 + x_1^{2n+2} + \cdots + x_n^{2n+2},
\]
which defines an isolated singularity at the origin. B. Malgrange showed ([7]) that the monodromy on the \((n - 1)\)-th cohomology of the Milnor fibre has a Jordan block with size \(n\). Coupled with the theorem by J. Scherk ([9, Theorem]), it gives us that \(f^{n-1} \neq f\). It can be checked with the software SINGULAR that \(\frac{\mu}{\tau} < 1.5\) for \(n \leq 7\), which is far away from our upper bound \(n\).

2. Applications

Theorem 1.1 implies a well-known result in complex singularity theory, which states that the Milnor number of an analytic function germ is finite (or non-zero) if and only if the Tjurina number is so (see [4, Lemma 2.3, Lemma 2.44]).

2.1. A lower bound for the Tjurina number

First we recall a well-known lower bound for \(\mu\) in terms of \(n\) and the multiplicity \(m\) of \(f\) at the origin. The following description can be found in [5].

The sectional Milnor numbers associated with the germ \(X\) are introduced by Teissier [10]. The \(i\)-th sectional Milnor number of the germ \(X\), denoted \(\mu^i\), is the Milnor number of the intersection of \(X\) with a general \(i\)-dimensional plane passing through the origin (it does not depend on the choice of the generic planes). Then \(\mu = \mu^n\). The Minkowski inequality for mixed multiplicities says that the sectional Milnor numbers always form a log-convex sequence [11]. In other words, we have
\[
\frac{\mu^n}{\mu^{n-1}} \geq \frac{\mu^{n-1}}{\mu^{n-2}} \geq \cdots \geq \frac{\mu}{\mu^0},
\]
where \(\mu^0 = 1\) and \(\mu^1 = m - 1\). Then
\[
\mu \geq (m - 1)^n. \tag{3}
\]
Moreover, the equality holds if and only if \(f\) is a semi-homogeneous function (i.e. \(f = f_m + g\), where \(f_m\) is a homogeneous polynomial of degree \(m\) defining an isolated singularity at the origin and \(g\) consists of terms of degree at least \(m + 1\)) after a biholomorphic change of coordinates. For a detailed proof, see [13, Proposition 3.1].

The next corollary is a direct consequence of Theorem 1.1 and (3).

Corollary 2.1. Assume that \(f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)\) is an analytic function germ at the origin with only isolated singularity. Then,
\[
\tau \geq \frac{(m - 1)^n}{n}.
\]
It is clear that, even for the homogeneous polynomial case, this lower bound can never be obtained when \(n > 1\). In fact, in this case, \(\tau = \mu = (m - 1)^n > \frac{(m - 1)^n}{n}\).

2.2. Another lower bound for the Tjurina number

Another lower bound for \(\mu\) is given by A. G. Kushnirenko using the Newton number ([6]). Let \(\Gamma\) be the boundary of the Newton polyhedron of \(f\), i.e., \(\Gamma\) is a polyhedron of dimension \(n - 1\) in \(\mathbb{N}^n\) (where \(\mathbb{N} = \{0, 1, 2, \cdots\}\)) determined in the usual way by the non-zero coefficients in \(f\). Then \(f\) is said to be convenient if \(\Gamma\) meets each of the coordinate axes of \(\mathbb{R}^n\). Let \(S\) be the union of all line segments in \(\mathbb{R}^n\) joining the origin to points of \(\Gamma\). For a convenient \(f\), the Newton number \(\nu(f)\) is defined as:
\[
\nu = n!V_n - (n - 1)!V_{n-1} + \cdots + (-1)^{n-1}1!V_1 + (-1)^n,
\]
where \(V_n\) is the \(n\)-dimensional volume of \(S\) and for \(1 \leq q \leq n - 1\), \(V_q\) is the sum of the \(q\)-dimensional volumes of the intersection of \(S\) with the coordinate planes of dimension \(q\). A. G. Kushnirenko showed that, if \(f\) is convenient, then,
\[
\mu \geq \nu.
\]
Moreover, \(\mu = \nu\) holds, if \(f\) is non-degenerate. (For the definition of non-degenerate, see [6, Definition 1.19].) Again, this gives us a corresponding lower bound for the Tjurina number.

Corollary 2.2. Assume that \(f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)\) is an analytic function germ at the origin with only isolated singularity, which is convenient. Then,
\[
\tau \geq \frac{\nu}{n},
\]
where \(\nu\) is the Newton number.

**Question 2.3.** Are the lower bounds of \(\tau\) in Corollary 2.1 and 2.2 optimal? When can the optimal lower bounds be obtained?

For some special class of polynomials, the bound for the ratio \(\frac{\mu}{\tau}\) can be improved. For example, A. Dimca showed that \(f^2 \in f_f\) for semi-weighted homogeneous polynomials ([2, Example 3.5]), hence \(\frac{\mu}{\tau} \leq 2\) and \(\tau \geq \frac{(m-1)^n}{2}\) in this case.

**Example 2.4.** Choose \(f = x^n + y^n + z^n + g\), where \(g\) has degree at least \(m + 1\). Then \(\mu = (m-1)^3\). It is shown in [12, Example 4.7] that \(\tau_{\text{min}} = (2m-3)(m+1)(m-1)/3\), when \(g\) varies.

**Acknowledgements**

The author is grateful to Alexandru Dimca for useful discussions. The author is partially supported by Nero Budur’s research project G0B2115N from the Research Foundation of Flanders.

**References**


