



## Number theory

## ABC and the Hasse principle for quadratic twists of hyperelliptic curves

*ABC et le principe de Hasse pour les torsions de courbes hyperelliptiques*

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## ABSTRACT

Conditionally on the ABC conjecture, we apply work of Granville to show that a hyperelliptic curve  $C/\mathbb{Q}$  of genus at least three has infinitely many quadratic twists that violate the Hasse Principle iff it has no  $\mathbb{Q}$ -rational hyperelliptic branch points.

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## R É S U M É

En supposant la conjecture ABC, nous utilisons un travail de Granville pour montrer qu'une courbe hyperelliptique  $C/\mathbb{Q}$  de genre au moins trois a une infinité de torsions quadratiques, qui violent le principe de Hasse si et seulement si elle n'a pas de point de branchement hyperelliptique rationnel sur  $\mathbb{Q}$ .

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## 1. Introduction

Let  $C/\mathbb{Q}$  be an algebraic curve. (All our curves will be *nice*: smooth, projective and geometrically integral.) An involution  $\iota$  on  $C$  is an order 2 automorphism of  $C/\mathbb{Q}$ . For any quadratic field  $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ , there is a curve  $\mathcal{T}_d(C, \iota)_{/\mathbb{Q}}$ , the quadratic twist of  $C$  by  $\iota$  and  $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ . After extension to  $\mathbb{Q}(\sqrt{d})$ , the curve  $\mathcal{T}_d(C, \iota)$  is canonically isomorphic to  $C_{/\mathbb{Q}(\sqrt{d})}$ , but the  $\text{Aut}(\mathbb{Q}(\sqrt{d})/\mathbb{Q}) = \langle \sigma_d \rangle$  action on  $C(\mathbb{Q}(\sqrt{d}))$  is “twisted by  $\iota$ ”, meaning that  $\sigma_d : P \in C(\mathbb{Q}(\sqrt{d})) \mapsto \iota(\sigma_d(P))$ . Thus, we have:

$$\mathcal{T}_d(C, \iota)(\mathbb{Q}) = \{P \in C(\mathbb{Q}(\sqrt{d})) \mid \iota(P) = \sigma_d(P)\}.$$

If  $d \in \mathbb{Q}^{\times 2}$ , we put  $\mathcal{T}_d(C, \iota) = C$ , the “trivial quadratic twist.”

Let  $q : C \rightarrow C/\iota$  be the quotient map. Every  $\mathbb{Q}$ -rational point on  $\mathcal{T}_d(C, \iota)$  maps via  $q$  to a  $\mathbb{Q}$ -rational point on  $C/\iota$ . Let  $\overline{P} \in (C/\iota)(\mathbb{Q})$ . If  $\overline{P}$  a branch point of  $\iota$ , the unique point  $P \in C(\mathbb{Q})$  such that  $q(P) = \overline{P}$  is also rational on every quadratic twist. If  $\overline{P}$  is not a branch point of  $\iota$ , there is a unique  $d \in \mathbb{Q}^{\times}/\mathbb{Q}^{\times 2}$  such that the fiber of  $q : \mathcal{T}_d(C, \iota) \rightarrow C/\iota$  consists of two  $\mathbb{Q}$ -rational points.

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Work of Clark and Clark–Stankewicz [2], [3], [4] gives criteria on  $C$  and  $\iota$  for there to be infinitely many  $d \in \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$  such that  $\mathcal{T}_d(C, \iota)_{/\mathbb{Q}}$  violates the Hasse Principle: letting  $\mathbf{A}_{\mathbb{Q}}$  be the adèle ring over  $\mathbb{Q}$ , this means  $\mathcal{T}_d(C, \iota)(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$  but  $\mathcal{T}_d(C, \iota)(\mathbb{Q}) = \emptyset$ . Here is one version.

**Theorem 1.** [4, Thm. 2] *Let  $C/\mathbb{Q}$  be a nice curve, and let  $\iota$  be an involution on  $C$ . Suppose:*

- (T1) *the involution  $\iota$  has no  $\mathbb{Q}$ -rational branch points;*
- (T2) *the involution  $\iota$  has at least one geometric branch point:  $\{P \in C(\overline{\mathbb{Q}}) \mid \iota(P) = P\} \neq \emptyset$ ;*
- (T3) *For some  $d \in \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$  we have  $\mathcal{T}_d(C, \iota)(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$ ;*
- (T4) *The set  $(C/\iota)(\mathbb{Q})$  is finite.*

*Then, as  $X \rightarrow \infty$ , the number of squarefree  $d$  with  $|d| \leq X$  such that  $\mathcal{T}_d(C, \iota)_{/\mathbb{Q}}$  violates the Hasse Principle is  $\gg_c \frac{X}{\log X}$ .*

An involution  $\iota$  on a curve  $C/\mathbb{Q}$  is hyperelliptic if  $C/\iota \cong \mathbb{P}^1$ . A hyperelliptic curve is a pair  $(C, \iota)$  with  $\iota$  a hyperelliptic involution on  $C$ . (A curve of genus at least two admits at most one hyperelliptic involution.) A hyperelliptic curve  $(C, \iota)$  of genus  $g$  has an affine model  $y^2 = f(x)$  with  $f(x) \in \mathbb{Q}[x]$  squarefree of degree  $2g + 2$  and  $\iota : (x, y) \mapsto (x, -y)$ . The twist  $\mathcal{T}_d(C, \iota)$  has affine model  $dy^2 = f(x)$ . The branch points of  $\iota$  are the roots of  $f$  in  $\overline{\mathbb{Q}}$ .<sup>1</sup>

If  $\iota$  is a hyperelliptic involution then  $(C/\iota)(\mathbb{Q}) = \mathbb{P}^1(\mathbb{Q})$  is infinite, so (T4) is *not* satisfied. In this note, we give a conditional complement to Theorem 1 that applies to hyperelliptic curves.

**Theorem 2.** *Assume the ABC conjecture. For a hyperelliptic curve  $(C, \iota)$  of genus  $g \geq 3$ , the following are equivalent:*

- (i) *the hyperelliptic involution  $\iota$  has no  $\mathbb{Q}$ -rational branch points;*
- (ii) *as  $X \rightarrow \infty$ , the number of squarefree integers  $d$  with  $|d| \leq X$  such that  $\mathcal{T}_d(C, \iota)_{/\mathbb{Q}}$  violates the Hasse Principle is  $\gg_c \frac{X}{\log X}$ ;*
- (iii) *some quadratic twist  $\mathcal{T}_d(C, \iota)_{/\mathbb{Q}}$  violates the Hasse Principle.*

Certainly (ii)  $\implies$  (iii). As for (iii)  $\implies$  (i): if  $\iota$  has a  $\mathbb{Q}$ -rational branch point, then this point stays rational on every quadratic twist. So the crux is to show (i)  $\implies$  (ii), which we will do in §2. The global part and the dependence on ABC both come from work of Granville [5]. In §3 we give upper and, in a special case, lower bounds on the number of quadratic twists having adelic points. We use these results to show that when hyperelliptic curves of genus  $g \geq 3$  are ordered by height, for 100% of such curves the number of twists up to  $X$  violating the Hasse Principle is  $o(X)$ , but conditionally on ABC, there are hyperelliptic curves for which the number of twists up to  $X$  violating the Hasse Principle is  $\gg X$ . Some final remarks are given in §4.

## 2. Proof of Theorem 2

### 2.1. Local

**Theorem 3.** *Let  $(C, \iota)_{/\mathbb{Q}}$  be a hyperelliptic curve of genus  $g \geq 1$ . If  $C(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$ , then the set of primes  $p \equiv 1 \pmod{8}$  for which  $\mathcal{T}_p(C, \iota)(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$  has positive density.*

**Proof.** For any place  $\ell \leq \infty$  of  $\mathbb{Q}$ , if  $p \in \mathbb{Q}_\ell^{\times 2}$  then  $\mathcal{T}_p(C, \iota)_{/\mathbb{Q}_\ell} \cong C_{/\mathbb{Q}_\ell}$  and thus  $\mathcal{T}_p(C, \iota)(\mathbb{Q}_\ell) \neq \emptyset$ . In particular, this holds for  $\ell = \infty$ . Henceforth  $\ell$  denotes a prime number.

Let  $M_1 \in \mathbb{Z}^+$  be such that  $C$  extends to a smooth relative curve over  $\mathbb{Z}_\ell$  for all  $\ell > M_1$ . Such an  $M_1$  exists for any nice curve  $C/\mathbb{Q}$  by openness of the smooth locus. Since  $C$  is hyperelliptic, we can take  $M_1$  to be the largest prime dividing its minimal discriminant.

Suppose  $\ell > M := \max(M_1, 4g^2 - 1)$ ,  $\ell \neq p$  and  $p \notin \mathbb{Q}_\ell^{\times 2}$ . Then the minimal regular model  $C_{/\mathbb{Z}_\ell}$  is smooth. We have  $\mathcal{T}_p(C, \iota)_{/\mathbb{Q}_\ell(\sqrt{p})} \cong C_{/\mathbb{Q}_\ell(\sqrt{p})}$ . Since  $\mathbb{Q}_\ell(\sqrt{p})/\mathbb{Q}_\ell$  is unramified and formation of the minimal regular model commutes with étale base change [6, Prop. 10.1.17], it follows that the minimal regular model  $\mathcal{T}_p(C, \iota)_{/\mathbb{Z}_\ell}$  is smooth. By the Riemann hypothesis for curves over a finite field, since  $\ell \geq 4g^2$ , we have  $\mathcal{T}_p(C, \iota)(\mathbb{F}_\ell) \neq \emptyset$ , and then by Hensel’s Lemma we have  $\mathcal{T}_p(C, \iota)(\mathbb{Q}_\ell) \neq \emptyset$ .

Suppose  $\ell \leq M$  and  $\ell \neq p$ . If  $\ell = 2$ , then  $p \in \mathbb{Q}_\ell^{\times 2}$  because  $p \equiv 1 \pmod{8}$ . If  $\ell$  is odd, we require that  $p$  is a quadratic residue modulo  $\ell$ , so again  $p \in \mathbb{Q}_\ell^{\times 2}$ . Either way,  $\mathcal{T}_p(C, \iota)(\mathbb{Q}_\ell) = C(\mathbb{Q}_\ell) \neq \emptyset$ .

Suppose  $\ell = p$ . Let  $P \in C(\overline{\mathbb{Q}})$  be a hyperelliptic branch point. We assume that  $p$  splits completely in  $\mathbb{Q}(P)$ . Then  $P \in C(\mathbb{Q}_p) \cap \mathcal{T}_p(C, \iota)(\mathbb{Q}_p)$ .

All in all, we have finitely many conditions on  $p$ , each of the form that  $p$  splits completely in a certain number field. Taking the compositum of these finitely many number fields and its Galois closure, say  $L$ , we see that if  $p$  splits completely in  $L$  then  $\mathcal{T}_p(C, \iota)(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$ . By (e.g.) the Chebotarev density theorem, this set of primes has positive density.  $\square$

<sup>1</sup> We have chosen a model in which the point at  $\infty$  is not a branch point; this is always possible. There is a model in which the point at  $\infty$  is a branch point iff there is a  $\mathbb{Q}$ -rational branch point.

2.2. Global

**Theorem 4.** (Granville [5, Cor. 1.2]) Assume the ABC conjecture. Let  $(C, \iota)_{/\mathbb{Q}}$  be a hyperelliptic curve of genus  $g \geq 3$ . The number of squarefree integers  $d$  with  $|d| \leq X$  such that  $\mathcal{T}_d(C, \iota)(\mathbb{Q})$  has a point that is not a hyperelliptic branch point is  $\ll_C X^{\frac{1}{g-1}+o(1)} \ll_C X^{2/3}$ .

2.3. Local–global

We now complete the proof of Theorem 2. Let  $(C, \iota)$  be a hyperelliptic curve of genus  $g \geq 3$  without  $\mathbb{Q}$ -rational hyperelliptic branch points, so  $C$  has an affine model of the form  $y^2 = f(x)$  with  $f(x) \in \mathbb{Z}[x]$  of degree  $2g + 2$ , with distinct roots in  $\overline{\mathbb{Q}}$  and no roots in  $\mathbb{Q}$ . Put  $d_0 := f(1)$ . Then  $(1, 1)$  is a  $\mathbb{Q}$ -point on  $d_0y^2 = f(x)$  and thus on  $\mathcal{T}_{d_0}(C, \iota)$ . The involution  $\iota$  remains  $\mathbb{Q}$ -rational on  $\mathcal{T}_{d_0}(C, \iota)$  (cf. [4, §2.1]). We may thus apply Theorem 3 to the hyperelliptic curve  $(\mathcal{T}_{d_0}(C, \iota), \iota)$ , getting a set of primes  $p \equiv 1 \pmod{8}$  of density  $\delta > 0$  such that

$$\mathcal{T}_{pd_0}(C, \iota)_{/\mathbb{Q}} = \mathcal{T}_p(\mathcal{T}_{d_0}(C, \iota), \iota)_{/\mathbb{Q}}$$

has points everywhere locally. By the Prime Number Theorem in Arithmetic Progressions, for at least  $(\frac{\delta}{d_0} + o(1)) \frac{X}{\log X}$  squarefree  $d$  with  $|d| \leq X$ , we have  $\mathcal{T}_d(C, \iota)(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$ . By Theorem 4, we have  $\mathcal{T}_d(C, \iota)(\mathbb{Q}) \neq \emptyset$  for  $\ll X^{2/3}$  squarefree  $d$  with  $|d| \leq X$ . So the number of squarefree  $d$  with  $|d| \leq X$  such that  $\mathcal{T}_d(C, \iota)_{/\mathbb{Q}}$  violates the Hasse Principle is  $\gg_C \frac{X}{\log X}$ .

3. Counting twists with adelic points

For a hyperelliptic curve  $(C, \iota)_{/\mathbb{Q}}$ , let

$$\mathfrak{L}_C = \{\text{squarefree } d \in \mathbb{Z} \mid \mathcal{T}_d(C, \iota)(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset\}$$

be the set of twists of  $C$  having points everywhere locally. For  $X \geq 1$ , put

$$\mathfrak{L}_C(X) = \#\{\mathfrak{L}_C \cap [-X, X]\}.$$

As we saw above, Theorem 3 gives  $\mathfrak{L}_C(X) \gg \frac{X}{\log X}$ .

Recall that a polynomial  $f \in \mathbb{Z}[x]$  is **intersective** if it has roots modulo  $N$  for all  $N \in \mathbb{Z}^+$ , or equivalently, in  $\mathbb{Z}_p$  for all primes  $p$ . We say a polynomial  $f \in \mathbb{Z}[x]$  is **weakly intersective** if the set of prime numbers  $p$  such that  $f$  has a root modulo  $p$  has density 1.

**Remark 5.** Suppose  $f = a_nx^n + \dots + a_1x + a_0 \in \mathbb{Z}[x]$  has degree  $n \geq 2$ , is weakly intersective and has distinct roots in  $\overline{\mathbb{Q}}$ , with discriminant  $\Delta$ . Let  $G$  be the Galois group of  $f$ .

For every prime number  $p \nmid a_n\Delta$ , the partition of  $n$  given by the cycle type of a Frobenius element  $\sigma_p$  at  $p$  coincides with the partition of  $n$  given by the degrees of the irreducible factors of the image of  $f$  in  $\mathbb{Z}/p\mathbb{Z}[x]$ . Since  $f$  is weakly intersective, it follows from the Frobenius Density Theorem (see, e.g., [10, §3]) that every  $\sigma \in G$  has a fixed point and thus  $f$  has a root mod  $p$  for all  $p \nmid a_n\Delta$ , and thus by Hensel’s Lemma it has a root in  $\mathbb{Z}_p$  for all but finitely many  $p$ .

Since every  $\sigma \in G$  has a fixed point, it follows from the Cauchy–Frobenius(–“not Burnside”) Lemma that  $f \in \mathbb{Q}[x]$  is not irreducible.

**Theorem 6.** Let  $(C, \iota)_{/\mathbb{Q}}$  be a hyperelliptic curve. Let  $y^2 = f(x)$  be an affine equation for  $C$  with  $f \in \mathbb{Z}[x]$  squarefree of even degree.

a) If  $f$  is weakly intersective then  $\mathfrak{L}_C(X) \gg X$ .

b) If  $f$  is not weakly intersective, let  $\beta$  be the density of the set of prime numbers  $p$  such that  $f$  has no root modulo  $p$ , so  $\beta \in (0, 1)$ .<sup>2</sup> Then  $\mathfrak{L}_C(X) \ll \frac{X}{\log^\beta X}$ .

**Proof.** Let  $\Delta$  be the discriminant of  $f$ .

Step 1: suppose  $f \in \mathbb{Z}[x]$  is weakly intersective. By Remark 5,  $f$  has a root in  $\mathbb{Z}_p$  for all but finitely many  $p$ , and thus the set  $\mathcal{P}$  of prime numbers  $p$  such that  $C(\mathbb{Q}_p) = \emptyset$  is finite. For each  $p \in \mathcal{P}$ , we have  $C_d(\mathbb{Q}_p) \neq \emptyset$  so long as  $d$  lies in the same  $\mathbb{Q}_p$ -adic square class as  $f(1)$ . The set of integers lying in a given  $\mathbb{Q}_p$ -adic square class is a nonempty union of congruence classes modulo  $p^2$  (if  $p > 2$ ) or modulo 16 (if  $p = 2$ ). Applying the Chinese Remainder Theorem, there are  $a, N \in \mathbb{Z}^+$  such that if  $d \equiv a \pmod{N}$  then  $\mathcal{T}_d(C, \iota)(\mathbb{Q}_p) \neq \emptyset$  for all primes  $p$ . Finally, if  $f$  has a real root then  $\mathcal{T}_d(C, \iota)(\mathbb{R}) \neq \emptyset$  for all  $d$ ; otherwise  $\mathcal{T}_d(C, \iota)(\mathbb{R}) \neq \emptyset$  iff  $df(1) > 0$ . Thus  $\mathfrak{L}_C(X) \gg X$ . (The implied constant can be made explicit in terms of  $\Delta$ .)

Step 2: suppose  $f$  is not weakly intersective. Let  $E'$  be the set of all squarefree integers  $d$  such that for all primes  $p \mid d$ , either  $p \mid 2\Delta$  or  $f$  has a root modulo  $p$ . Let  $E$  be the set of all squarefree integers that do not lie in  $E'$ . Thus for all  $d \in E$ ,

<sup>2</sup> The polynomial  $f$  has a root modulo every prime  $p$  that splits completely in the splitting field of  $f$ , so  $\beta > 0$ .

there is an odd prime  $p \mid d$  such that the image of  $f$  in  $\mathbb{Z}/p\mathbb{Z}$  is squarefree and has no root modulo  $p$ . By a result of Sadek [8, Cor. 4.2], this implies that  $\mathcal{T}_d(C)(\mathbb{Q}_p) = \emptyset$ . It follows that

$$\mathfrak{U}_C \subset E'.$$

Let  $E'(X)$  be the number of  $d \in E'$  with  $|d| \leq X$ . Then [9, Thm. 2.4] implies that if  $0 < \beta < 1$  then there is  $c > 0$  such that  $E'(X) \sim \frac{cX}{\log^\beta X}$ .  $\square$

We call a hyperelliptic curve  $(C, \iota)_{/\mathbb{Q}}$  **weakly intersective** if it has a weakly intersective squarefree, integral, even degree defining polynomial.<sup>3</sup> Since no weakly intersective polynomial is irreducible, when genus  $g$  hyperelliptic curves are ordered by height, 0% of them are weakly intersective.

Theorems 2 and 6 immediately imply the following:

**Corollary 7.** *Let  $(C, \iota)_{/\mathbb{Q}}$  be a hyperelliptic curve of genus  $g$  without  $\mathbb{Q}$ -rational branch points.*

a) *If  $C$  is weakly intersective and  $g \geq 3$ , then conditionally on ABC, as  $X \rightarrow \infty$  the number of quadratic twists of  $(C, \iota)$  that violate the Hasse Principle is  $\gg X$ .*

b) *If  $C$  is not weakly intersective, then as  $X \rightarrow \infty$ , the number of quadratic twists of  $(C, \iota)$  that violate the Hasse Principle is  $o(X)$ .*

**Example 8.**

a) For any coprime, nonsquare integers  $a, b > 1$ , the polynomial  $(x^2 - a)(x^2 - b)(x^2 - ab)$  is weakly intersective and without rational roots. The polynomial  $(x^2 - 2)(x^2 - 3)(x^2 - 6)$  is not intersective – it has no root in  $\mathbb{Q}_2$ . The polynomial  $(x^2 - 2)(x^2 - 17)(x^2 - 34)$  is intersective.

b) For  $g \geq 3$ , let  $h(x) \in \mathbb{Z}[x]$  be monic of degree  $2g - 4$ , with nonzero discriminant, without rational roots and such that  $h(\pm\sqrt{2}), h(\pm\sqrt{3}), h(\pm\sqrt{6}) \neq 0$ . Then

$$C_{/\mathbb{Q}} : y^2 = 2(x^2 - 2)(x^2 - 3)(x^2 - 6)h(x)$$

is a weakly intersective hyperelliptic curve of genus  $g \geq 3$  without  $\mathbb{Q}$ -rational branch points. So conditionally on ABC, a positive proportion of the quadratic twists of  $C$  violate the Hasse principle.

c) For every even  $n \geq 2$ , there is a cyclic Galois extension  $F/\mathbb{Q}$  of degree  $n$ , and there is a monic polynomial  $f \in \mathbb{Z}[x]$  such that  $\mathbb{Q}[x]/(f) \cong F$ . The hyperelliptic curve  $C_{/\mathbb{Q}} : y^2 = 2f(x)$  has genus  $\frac{n}{2} - 1$  and  $\mathfrak{U}_C(X) \ll \frac{X}{\log^{1-\frac{1}{n}} X}$ .

**4. Some remarks**

In [5, Conj. 1.3], Granville conjectures that for all  $g \geq 2$ , if  $f \in \mathbb{Z}[x]$  has degree  $2g + 1$  or  $2g + 2$  and distinct roots in  $\overline{\mathbb{Q}}$ , then there is a constant  $\kappa'_f > 0$  such that the number of squarefree  $d$  with  $|d| \leq X$  such that  $dy^2 = f(x)$  has a  $\mathbb{Q}$ -point that is not a hyperelliptic branch point is  $\sim \kappa'_f X^{\frac{1}{g+1}}$ . The above arguments apply verbatim to show that conditionally on Granville’s conjecture, for all  $g \geq 2$ , a hyperelliptic curve  $C_{/\mathbb{Q}}$  has  $\gg_c \frac{X}{\log X}$  twists that violate the Hasse principle iff  $C$  has no  $\mathbb{Q}$ -rational branch points. On the other hand, Vatsal has exhibited a genus-one hyperelliptic curve  $(C, \iota)_{/\mathbb{Q}}$ , for which a positive proportion of the quadratic twists have infinitely many rational points [11]. Still, it may be true that every hyperelliptic curve of genus 1 without  $\mathbb{Q}$ -rational branch points has infinitely many twists that violate the Hasse Principle.

The present work should be compared to two other works that apply Theorem 1 (or its predecessor [2, Thm. 2]) and Faltings’ Theorem to get Hasse Principle violations. Namely, Ozman [7] works with the Atkin–Lehner involution  $w_N$  on a modular curve  $X_0(N)$  for a prime  $N \equiv 1 \pmod{4}$  and Clark–Stankewicz [4] work with the Atkin–Lehner involution  $w_D$  on a Shimura curve  $X^D$  for a squarefree  $D > 1$ . Taking  $N > 131$  (resp.  $D > 546$ ) ensures that  $X_0(N)/\langle w_N \rangle$  (resp.  $X^D/\langle w_D \rangle$ ) has genus at least 2 and thus finitely many  $\mathbb{Q}$ -points. In each work, there is an analysis of  $\mathfrak{U}_C(X)$ , the number of twists up to  $X$  with adelic points. For modular curves  $X_0(N)$ , Ozman shows that  $\mathfrak{U}_C(X) \sim \frac{cX}{\log^\gamma X}$  for a positive constant  $C$  and a  $\gamma \in [0, 1]$  determined in terms of the class group of  $\mathbb{Q}(\sqrt{-N})$  [7, Thm. 5.4] (and cf. [4, p. 2841, footnote 5]). In the case of Shimura curves  $X^D$ , Clark–Stankewicz show [4, Thm. 8] that

$$\frac{X}{\log^{\alpha_D} X} \ll \mathfrak{U}_{X^D}(X) \ll \frac{X}{\log^{\beta_D} X}$$

for constants  $0 < \beta_D < \alpha_D < 1$  determined in terms of  $D$ , such that  $\lim_{D \rightarrow \infty} \alpha_D - \beta_D = 0$ .

<sup>3</sup> By Theorem 6, if one defining polynomial is weakly intersective, then all are.

There is some overlap: for a finite nonempty set of  $N$  (resp. of  $D$ ), the pair  $(X_0(N), w_N)$  (resp.  $(X^D, w_D)$ ) is hyperelliptic. E.g., the pair  $(X_0(41), w_{41})$  is hyperelliptic of genus 3 and [7, *loc. cit.*] gives  $\mathfrak{L}_{X_0(41)}(X) \sim \frac{CX}{\log^{11} X}$ . Similarly, the pair  $(X^{35}, w_{35})$  is hyperelliptic of genus 3 and [4, *loc. cit.*] gives  $\frac{X}{\log^{15} X} \ll \mathfrak{L}_{X^{35}}(X) \ll \frac{X}{\log^{11} X}$ .

It can be shown that for all hyperelliptic curves  $(C, \iota)_{/\mathbb{Q}}$ , there is  $\alpha = \alpha(C) < 1$  such that  $\mathfrak{L}_C(X) \gg \frac{X}{\log^\alpha X}$ . In fact, the same conclusion should hold for any  $(C, \iota)_{/\mathbb{Q}}$  satisfying (T1), (T2), and (T3) in Theorem 1, which amounts to a quantitative strengthening of the local part of this result. We hope to return to this in a future work.

Recent work of Bhargava–Gross–Wang [1] shows that, for each fixed  $g \geq 1$ , when genus- $g$  hyperelliptic curves  $(C, \iota)_{/\mathbb{Q}}$  are ordered by height, a positive proportion violate the Hasse Principle. This work is unconditional; moreover, the positive proportion result should be contrasted with Corollary 7b). On the other hand, since all quadratic twists of a hyperelliptic curve induce the same point of the moduli space  $\mathcal{H}_g$  of hyperelliptic curves of genus  $g$ , our result gives, conditionally on ABC, Hasse Principle violations on the largest possible subset of  $\mathcal{H}_g$ .

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