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A note on the quasi-ergodic distribution of one-dimensional diffusions

Une note sur la distribution quasi ergodique des diffusions en dimension 1

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ABSTRACT

In this note, we study quasi-ergodicity for one-dimensional diffusions on $(0, \infty)$, where 0 is an exit boundary and $+\infty$ is an entrance boundary. Our main aim is to improve some results obtained by He and Zhang (2016) [3]. In simple terms, the same main results of the above paper are obtained with more relaxed conditions.

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RÉSUMÉ

Nous étudions la quasi-ergodicité des diffusions unidimensionnelles sur $]0, \infty[$, où 0 est une frontière de sortie et ∞ une frontière d'entrée. Notre but est d'améliorer des résultats obtenus par He and Zhang (2016) [3]. Ainsi, nous retrouvons les résultats principaux de ce texte sous des hypothèses moins restrictives.

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1. Introduction

Let $X = (X_t, t \ge 0)$ be a one-dimensional drifted Brownian motion on $(0, \infty)$, i.e.

$$dX_t = dB_t - \alpha(X_t) dt, \qquad X_0 = x > 0,$$
(1.1)

where $(B_t, t \ge 0)$ is a standard one-dimensional Brownian motion and $\alpha \in C^1(0, \infty)$. In this paper, α can explode at the origin. There exists a pathwise unique solution to the stochastic differential equation (1.1) up to the explosion time τ .

Associated with α , we consider the following two functions

$$\Lambda(x) = \int_{1}^{x} e^{Q(y)} dy \quad \text{and} \quad \kappa(x) = \int_{1}^{x} e^{Q(y)} \left(\int_{1}^{y} e^{-Q(z)} dz \right) dy, \tag{1.2}$$

where $Q(y) := 2 \int_{1}^{y} \alpha(x) dx$. Note that Λ is the scale function of the process *X*.

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The other important role will be played by the following measure μ , which is not necessarily finite in this paper, defined on $(0, \infty)$:

$$\mu(\mathrm{d}y) := \mathrm{e}^{-\mathrm{Q}(y)}\mathrm{d}y. \tag{1.3}$$

Note that μ is the speed measure of the process *X*.

Let $T_a := \inf\{0 \le t < \tau : X_t = a\}$ be the hitting time of $a \in (0, \infty)$ for the process *X*. We denote by $T_{\infty} = \lim_{n \to \infty} T_n$ and $T_0 = \lim_{n \to \infty} T_{1/n}$. Since α is regular in $(0, \infty)$, then $\tau = \min\{T_0, T_\infty\}$. Let \mathbb{P}_x be the probability measure under which the process *X* starts at *x*. For any distribution π on $(0, \infty)$, we will use the notation

$$\mathbb{P}_{\pi}(\cdot) := \int_{0}^{\infty} \mathbb{P}_{x}(\cdot) \pi(\mathrm{d}x).$$

We shall denote by \mathbb{E}_x (resp. \mathbb{E}_π) the expectation corresponding to \mathbb{P}_x (resp. \mathbb{P}_π). We denote by $\mathcal{B}(0, \infty)$ the Borel σ -algebra on $(0, \infty)$, $\mathscr{P}(0, \infty)$ the set of all probability measures on $(0, \infty)$ and $\mathbf{1}_A$ the indicator function of A. We define the inner product

$$\langle f, g \rangle_{\mu} = \int_{0}^{\infty} f(u)g(u)\mu(\mathrm{d}u)$$

In this paper, we will use the following hypothesis (H).

Definition 1.1. We say that hypothesis (H) holds if the following explicit conditions on α , all together, are satisfied:

(H1) for all x > 0, $\mathbb{P}_x(\tau = T_0 < T_\infty) = 1$; (H2) for any $\varepsilon > 0$, $\mu(0, \varepsilon) = \infty$; (H3) $S = \int_1^\infty e^{Q(y)} \left(\int_y^\infty e^{-Q(z)} dz \right) dy < \infty$.

It is well known (see, e.g., [4], Chapter VI, Theorem 3.2) that (H1) is equivalent to $\Lambda(\infty) = \infty$ and $\kappa(0^+) < \infty$. According to Feller's classification (see [5, Chapter 15]), if (H1) and (H2) are satisfied, then 0 is an exit boundary; if (H1) and (H3) are satisfied, then $+\infty$ is an entrance boundary.

One of the fundamental problems for a killed Markov process conditioned on survival is to study its long-term asymptotic behavior. In order to understand the behavior of the process before extinction, a relevant object to look at is a so-called quasi-ergodic distribution (see [1]). In this paper, we will study the existence and uniqueness of quasi-ergodic distributions for the one-dimensional diffusion process X satisfying hypothesis (H).

Recently, under the conditions that hypothesis (H) holds and the killed semigroup satisfies intrinsic ultracontractivity, He and Zhang [3] proved that there exists a unique quasi-ergodic distribution for the one-dimensional diffusion process X. The main aim of this note is to show that this conclusion still holds only under hypothesis (H) without the intrinsic ultracontractivity. Our main result is Theorem 3.1 (see Section 3).

2. Preliminaries

Before going to our main result, we give some preliminaries. We denote by $L := \frac{1}{2}\partial_{xx} - \alpha \partial_x$ the infinitesimal generator of the one-dimensional diffusion process *X*. From [6], we know that *L* is the generator of a strongly continuous symmetric semigroup of contractions on $\mathbb{L}^2(\mu)$ denoted by $(P_t)_{t\geq 0}$. This semigroup is sub-Markovian, that is, $0 \le P_t f \le 1$ μ -a.e. if $0 \le f \le 1$. Also from [6], we get that when (H1) holds, the semigroup of the process *X* killed at 0 can be given by

$$P_t f(\mathbf{x}) = \mathbb{E}_{\mathbf{x}}[f(X_t), T_0 > t].$$

In this paper, we study quasi-ergodicity for one-dimensional diffusions on $(0, \infty)$, where 0 is an exit boundary and $+\infty$ is an entrance boundary. More formally, the main object of interest of this work would be captured by the following definition.

Definition 2.1. We say that $m \in \mathscr{P}(0, \infty)$ is a quasi-ergodic distribution if there exists a $\pi \in \mathscr{P}(0, \infty)$ such that, for any $A \in \mathcal{B}(0, \infty)$,

$$\lim_{t\to\infty}\mathbb{E}_{\pi}\left(\frac{1}{t}\int_{0}^{t}\mathbf{1}_{A}(X_{s})\,\mathrm{d}s|T_{0}>t\right)=m(A).$$

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We remark that the above limiting distribution is substantially different from the *quasi-limiting distribution* ρ (see, e.g., [1,3]), i.e. there exists $\pi \in \mathscr{P}(0, \infty)$ such that, for all $A \in \mathcal{B}(0, \infty)$,

$$\lim_{t\to\infty} \mathbb{P}_{\pi}(X_t \in A | T_0 > t) = \rho(A).$$

However, it is well known that, for a conservative Markov process $X = (X_t, t \ge 0)$ satisfying some irreducibility conditions, if ρ is the stationary distribution, then starting from any initial distribution π , both $\mathbb{E}_{\pi}\left(\frac{1}{t}\int_{0}^{t} \mathbf{1}_{A}(X_s) ds\right)$ and $\mathbb{P}_{\pi}(X_t \in A)$ converge to $\rho(A)$, when t tends to infinity. This difference is worth further investigation for us. Note that if ρ is a quasilimiting distribution, then it is a *quasi-stationary distribution*, i.e. a probability measure $\rho \in \mathscr{P}(0, \infty)$ such that for all $t \ge 0$ and all $A \in \mathcal{B}(0, \infty)$,

 $\mathbb{P}_{\rho}(X_t \in A | T_0 > t) = \rho(A).$

Under hypothesis (H), the following proposition has been obtained by Littin [6]. This proposition plays an important role in our following arguments.

Proposition 2.2. ([6]) Assume that hypothesis (H) holds. Then we have

(i) −L has purely discrete spectrum. The eigenvalues 0 < λ₁ < λ₂ < ··· are simple, lim_{n→∞} λ_n = +∞, and the eigenfunction η_n associated with λ_n has exactly n roots belonging to (0, ∞). The sequence (η_n)_{n≥1} is an orthonormal basis of L²(μ). In particular, η₁ can be chosen to be strictly positive in (0, ∞);

(ii) for any $n \ge 1$, $\eta_n \in \mathbb{L}^1(\mu)$;

(iii) for all x > 0 and all t > 0, there exists some density $r(t, x, \cdot)$ that satisfies

$$\mathbb{E}_{x}[f(X_{t}), T_{0} > t] = \int_{0}^{\infty} r(t, x, y) f(y) \mu(dy)$$
(2.1)

for all bounded Borel function f. Moreover, for all x > 0 and t > 0, the density $r(t, x, \cdot) \in L^2(\mu)$.

(iv) there exists a unique quasi-stationary distribution

$$\nu(\mathrm{d}x) = \frac{\eta_1(x)}{\langle \eta_1, 1 \rangle_{\mu}} \mu(\mathrm{d}x) \tag{2.2}$$

for the process X.

Also, for any x > 0 and any $A \in \mathcal{B}(0, \infty)$,

$$\lim_{t \to \infty} e^{\lambda_1 t} \mathbb{P}_x(T_0 > t) = \eta_1(x) \langle \eta_1, 1 \rangle_\mu, \tag{2.3}$$

$$\lim_{t \to \infty} e^{\lambda_1 t} \mathbb{P}_x(X_t \in A, T_0 > t) = \nu(A)\eta_1(x)\langle \eta_1, 1 \rangle_\mu.$$
(2.4)

This implies that

$$\lim_{t \to \infty} \mathbb{P}_X(X_t \in A | T_0 > t) = \nu(A), \tag{2.5}$$

that is, v is the Yaglom limit distribution. Moreover, any probability measure ρ with compact support in $(0, \infty)$ satisfies

$$\lim_{t \to \infty} e^{\lambda_1 t} \mathbb{P}_{\varrho}(T_0 > t) = \langle \eta_1, 1 \rangle_{\mu} \int_0^\infty \eta_1(x) \varrho(\mathrm{d}x),$$
(2.6)

$$\lim_{t \to \infty} e^{\lambda_1 t} \mathbb{P}_{\varrho}(X_t \in A, T_0 > t) = \nu(A) \langle \eta_1, 1 \rangle_{\mu} \int_0^\infty \eta_1(x) \varrho(\mathrm{d}x),$$
(2.7)

$$\lim_{t \to \infty} \mathbb{P}_{\varrho}(X_t \in A | T_0 > t) = \nu(A).$$
(2.8)

3. Main result

For general Markov processes satisfying positive (Harris) λ -recurrence, Breyer and Roberts established a quasi-ergodic theorem (see [1, Theorem 1]). However, for a general Markov process, checking whether it is positive λ -recurrent is not an easy thing to do. When the reference measure is an infinite measure, it will be more difficult to prove that the λ -invariant measure is a finite measure. Compared to [1], from the proof of our main result, we can see that we not only establish a *fractional Yaglom limit*, i.e. there exists $\pi \in \mathscr{P}(0, \infty)$ such that, for any 0 < q < 1 and all $A \in \mathcal{B}(0, \infty)$, $\lim_{t \to \infty} \mathbb{P}_{\pi}(X_{qt} \in A | T_0 > t) = m(A)$, but also give a relationship between the existence of quasi-limiting distributions and the existence of quasi-ergodic distributions. From the fractional Yaglom limit to the quasi-limiting distribution, we can see that a *phase transition* occurs.

The following theorem is our main result.

Theorem 3.1. Assume that hypothesis (H) holds. Then, for any $\pi \in \mathscr{P}(0, \infty)$ and any bounded measurable function f on $(0, \infty)$, we have

$$\lim_{t\to\infty} \mathbb{E}_{\pi}\left(\frac{1}{t}\int_{0}^{t} f(X_{s}) \,\mathrm{d}s|T_{0}>t\right) = \int_{0}^{\infty} f(y)m(\mathrm{d}y),$$

where m is given by

$$m(\mathrm{d}x) = \eta_1^2(x)\mu(\mathrm{d}x).$$

In particular, m is the unique stationary distribution of the Q -process.

Proof. (i) We know from part (i) of Proposition 2.2 that $\|\eta_1\|_{L^2(\mu)} = 1$, then *m* is a probability distribution on $(0, \infty)$. Next, we first assume that *f* is positive and bounded. For fixed *u*, we set

 $h_u(x) = \inf\{e^{\lambda_1 r} \mathbb{P}_x(T_0 > r) / \eta_1(x) \langle \eta_1, 1 \rangle_\mu : r \ge u\}.$

From (2.3), we can see that $h_u(x) \uparrow 1$, as $u \to \infty$. Let 0 < q < 1. When $(1 - q)t \ge u$, by the Markov property, we obtain

$$\begin{split} \mathbb{E}_{\pi}(f(X_{qt})|T_{0} > t) &= \frac{\mathbb{E}_{\pi}(f(X_{qt}), T_{0} > t)}{\mathbb{P}_{\pi}(T_{0} > t)} \\ &= \frac{\mathbb{E}_{\pi}[f(X_{qt})\mathbf{1}_{\{T_{0} > qt\}}\mathbb{P}_{X_{qt}}(T_{0} > (1-q)t)]}{\mathbb{P}_{\pi}(T_{0} > t)} \\ &= \frac{e^{\lambda_{1}qt}\mathbb{E}_{\pi}[f(X_{qt})\mathbf{1}_{\{T_{0} > qt\}}e^{\lambda_{1}(1-q)t}\mathbb{P}_{X_{qt}}(T_{0} > (1-q)t)]}{e^{\lambda_{1}t}\mathbb{P}_{\pi}(T_{0} > t)} \\ &\geq \frac{e^{\lambda_{1}qt}\mathbb{E}_{\pi}[f(X_{qt})h_{u}(X_{qt})\eta_{1}(X_{qt})\langle\eta_{1},1\rangle_{\mu}\mathbf{1}_{\{T_{0} > qt\}}]}{e^{\lambda_{1}t}\mathbb{P}_{\pi}(T_{0} > t)}. \end{split}$$

From [3, Proposition 2.3], we know that η_1 is bounded. Moreover, based on (2.3), we have

 $|f(\mathbf{x})h_u(\mathbf{x})\eta_1(\mathbf{x})\langle\eta_1,1\rangle_{\mu}| \leq \langle\eta_1,1\rangle_{\mu} \|f\|_{\infty} \|\eta_1\|_{\infty}.$

Thus, the function $fh_u\eta_1$ is bounded and measurable.

If (H) is satisfied, the same proof as in [2, Theorem 7.3] works, we can deduce that, for any $\pi \in \mathscr{P}(0, \infty)$ and any bounded measurable function g on $(0, \infty)$,

$$\lim_{t \to \infty} \mathbb{E}_{\pi}(g(X_t)|T_0 > t) = \int_0^\infty g(x)\nu(dx), \tag{3.1}$$

where ν is defined in (2.2). This is because the key elements of the proof of [2, Theorem 7.3] only need to know that the process can reach 0 in finite time with probability 1, -L has purely discrete spectrum, $\eta_1 \in \mathbb{L}^1(\mu)$, r(t, x, y) exists and part (iv) of Proposition 2.2 holds. If (H) is satisfied, from Proposition 2.2 we know that all these requirements are fulfilled. Moreover, under hypothesis (H), we also have for any $\pi \in \mathscr{P}(0, \infty)$,

$$\lim_{t \to \infty} e^{\lambda_1 t} \mathbb{P}_{\pi}(T_0 > t) = \langle \eta_1, 1 \rangle_{\mu} \int_0^\infty \eta_1(x) \pi(\mathrm{d}x).$$
(3.2)

The proof of this property is similar to that of [2, Theorem 7.3], since it only uses the same properties above. So, by (3.1) and (3.2), we get

$$\begin{split} \liminf_{t \to \infty} \mathbb{E}_{\pi}(f(X_{qt})|T_0 > t) &\geq \lim_{t \to \infty} \frac{e^{\lambda_1 qt} \mathbb{E}_{\pi}[f(X_{qt})h_u(X_{qt})\eta_1(X_{qt})\langle \eta_1, 1 \rangle_{\mu} \mathbf{1}_{\{T_0 > qt\}}]}{e^{\lambda_1 t} \mathbb{P}_{\pi}(T_0 > t)} \\ &= \int_0^\infty f(x)h_u(x)\eta_1(x)\langle \eta_1, 1 \rangle_{\mu} \nu(dx) \\ &= \int_0^\infty f(x)h_u(x)m(dx). \end{split}$$

Based on the monotone convergence theorem, by letting $u \to \infty$ in the above formula, we have

$$\liminf_{t \to \infty} \mathbb{E}_{\pi}(f(X_{qt})|T_0 > t) \ge \int_0^\infty f(x)m(\mathrm{d}x).$$
(3.3)

On the other hand, since f is bounded, we can repeat the argument, replacing f by $||f||_{\infty} - f$, which gives

$$\limsup_{t \to \infty} \mathbb{E}_{\pi}(f(X_{qt})|T_0 > t) \le \int_0^\infty f(x)m(\mathrm{d}x).$$
(3.4)

Combining (3.3) and (3.4), for positive and bounded function f, we have

$$\lim_{t \to \infty} \mathbb{E}_{\pi} \left(f(X_{qt}) | T_0 > t \right) = \int_0^\infty f(x) m(\mathrm{d}x).$$
(3.5)

For (3.5), we can extend it to arbitrary bounded f by subtraction.

So, by change of variable in the Lebesgue integral and the dominated convergence theorem, we obtain

$$\lim_{t \to \infty} \mathbb{E}_{\pi} \left(\frac{1}{t} \int_{0}^{t} f(X_{s}) \, ds | T_{0} > t \right) = \lim_{t \to \infty} \mathbb{E}_{\pi} \left(\int_{0}^{1} f(X_{qt}) \, dq | T_{0} > t \right)$$
$$= \lim_{t \to \infty} \int_{0}^{1} \mathbb{E}_{\pi}(f(X_{qt}) | T_{0} > t) \, dq$$
$$= \int_{0}^{\infty} f(x) m(dx).$$

(ii) If (H) is satisfied, the same proof as in [2, Corollary 6.1] works, we can deduce that the Q-process exists (the reason is as described above). More precisely, if (H) is satisfied, then the family $(\mathbb{Q}_x)_{x>0}$ of probability measures on Ω defined by

$$\mathbb{Q}_{x}(A) = \lim_{t \to \infty} \mathbb{P}_{x}(A|T_{0} > t), \quad \forall A \in \mathcal{F}_{s}, \ \forall s \ge 0,$$

is well defined, and the process $(\Omega, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{Q}_x)_{x>0})$ is a diffusion process on $(0, \infty)$ with transition probability densities (w.r.t. the Lebesgue measure) given by

$$q(s, x, y) = e^{\lambda_1 s} \frac{\eta_1(y)}{\eta_1(x)} r(s, x, y) e^{-Q(y)}.$$

So, from [2, Corollary 6.2], we know that the Q-process admits a unique invariant probability measure

$$\vartheta(\mathrm{d} x) = \eta_1^2(x)\mu(\mathrm{d} x).$$

Thus, *m* coincides with the unique stationary distribution ϑ of the Q-process. This shows the result. \Box

Finally, let us remark that the proof developed in [1, Theorem 1] could be used here in order to prove the same result with $\pi = \delta_x$ for any $x \in (0, \infty)$, where δ_x denotes the Dirac measure at x. Indeed, it only uses the fact that

$$\lim_{t\to\infty} \mathbb{Q}_{\mathbf{x}}(f(X_t)) = \int_0^\infty f(y)m(\mathrm{d} y),$$

for all bounded measurable function f.

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