Number theory

Odd values of the Rogers–Ramanujan functions

Valeurs impaires des fonctions de Rogers–Ramanujan

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A R T I C L E   I N F O

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A B S T R A C T

Let $g(n)$ and $h(n)$ be the coefficients of the Rogers–Ramanujan identities. We obtain asymptotic formulas for the number of odd values of $g(n)$ for odd $n$, and $h(n)$ for even $n$, which improve Gordon’s results. We also obtain lower bounds for the number of odd values of $g(n)$ for even $n$, and $h(n)$ for odd $n$.

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R É S U M É

Soit $g(n)$ et $h(n)$ les coefficients des identités de Rogers–Ramanujan. Nous obtenons des formules asymptotiques pour le nombre de valeurs impaires de $g(n)$ lorsque $n$ est impair et de $h(n)$ lorsque $n$ est pair. Ces formules améliorent un résultat de Gordon. Nous obtenons également des bornes inférieures pour le nombre de valeurs impaires de $g(n)$ pour $n$ pair et de $h(n)$ pour $n$ impair.

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1. Introduction

The Rogers–Ramanujan identities, first discovered by Rogers in 1894, are the pair of $q$-series identities:

$$G(q) := \prod_{n=0}^{\infty} \frac{1}{(1 - q^{5n+1})(1 - q^{5n+4})} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1 - q)(1 - q^2) \cdots (1 - q^n)},$$

and

$$H(q) := \prod_{n=0}^{\infty} \frac{1}{(1 - q^{5n+2})(1 - q^{5n+3})} = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1 - q)(1 - q^2) \cdots (1 - q^n)}.$$
where \(|q| < 1\). If \(q = e^{2i\pi r}\) and \(\operatorname{Im} r > 0\), then Biagioli [2] showed that \(q^{-\frac{1}{n}} G(q)\) and \(q^{\frac{11}{n}} H(q)\) are modular functions on \(\Gamma_1(5)\) with character. Let \(G(q) = \sum_{n=0}^{\infty} g(n)q^n\) and \(H(q) = \sum_{n=0}^{\infty} h(n)q^n\). We are interested in finding the arithmetic density of the number of odd values of \(g(n)\) and \(h(n)\). Let

\[
\gamma(N) = \#\{1 \leq n \leq N : \ n \equiv 1 \pmod{2} \ \text{and} \ \ g(n) \equiv 1 \pmod{2}\},
\]

and

\[
\delta(N) = \#\{1 \leq n \leq N : \ n \equiv 0 \pmod{2} \ \text{and} \ \ h(n) \equiv 1 \pmod{2}\}.
\]

Gordon [3] proved the order of magnitude of \(\gamma(N)\) and \(\delta(N)\) is \(\frac{N}{\log N}\).

**Theorem 1.1** (Gordon). There exist positive constants \(A\) and \(B\) such that

\[
A \frac{N}{\log N} < \gamma(N), \delta(N) < B \frac{N}{\log N}
\]

for sufficiently large \(N\).

Gordon’s proof is based on the fact that \(g(n)\) with \(n\) odd and \(h(n)\) with \(n\) even are the coefficients of holomorphic modular forms weight 1 mod 2, and hence each such \(n\) can be determined explicitly. We obtain the following asymptotic formulas by refining Gordon’s arguments.

**Theorem 1.2.** For sufficiently large \(N\),

\[
\gamma(N) = \frac{\pi^2}{5} \cdot \frac{N}{\log N} + O \left( \frac{N \log \log N}{\log^2 N} \right),
\]

\[
\delta(N) = \frac{\pi^2}{5} \cdot \frac{N}{\log N} + O \left( \frac{N \log \log N}{\log^2 N} \right).
\]

Let

\[
\gamma'(N) = \#\{1 \leq n \leq N : \ n \equiv 0 \pmod{2} \ \text{and} \ \ g(n) \equiv 1 \pmod{2}\}
\]

and

\[
\delta'(N) = \#\{1 \leq n \leq N : \ n \equiv 1 \pmod{2} \ \text{and} \ \ h(n) \equiv 1 \pmod{2}\}.
\]

A similar question is to bound \(\gamma'(N)\) and \(\delta'(N)\). In this case, \(g(n)\) with \(n\) even and \(h(n)\) with \(n\) odd are the coefficients of holomorphic modular forms weight \(\frac{3}{2}\) mod 2. Applying the ground-breaking work of Bellaïche, Green, and Soundararajan [1] on the non-divisibility of the coefficients of weakly holomorphic modular forms, we obtain the following lower bounds.

**Theorem 1.3.** For sufficiently large \(N\), we have

\[
\gamma'(N) \gg \frac{\sqrt{N}}{\log \log N},
\]

\[
\delta'(N) \gg \frac{\sqrt{N}}{\log \log N}.
\]

2. Proof of Theorem 1.2

The parity of \(g(n)\) for odd \(n\) and that of \(h(n)\) for even \(n\) were determined by Gordon [3] explicitly, i.e. \(n\) is odd and \(g(n) \equiv 1 \pmod{2}\) if and only if \(60n - 1 = p^{4\alpha+1}m^2\), where \(\alpha \geq 0\) is an integer and \(p\) is a prime not dividing \(m\). \(n\) is even and \(h(n) \equiv 1 \pmod{2}\) if and only if \(60n + 11 = p^{4\alpha+1}m^2\). Thus \(\gamma(N)\) can be represented as

\[
\gamma(N) = \sum_{\substack{1 \leq n \leq N \ (\text{mod } 2) \ \text{and} \ \ 60n - 1 = p^{4\alpha+1}m^2}} 1.
\]

We denote by \(M = 60N - 1\) for convenience, and by \(\pi(x)\) the number of primes less than \(x\). We split the sum above into two parts according to \(a = 0\) and \(a \geq 1\). The sum over \(a \geq 1\) is bounded by
\[
\]

\[
\ll \sum_{1 \leq a \ll \log N} \sum_{m \leq \sqrt{M}} \pi \left( \left( \frac{M}{m^2} \right)^{\frac{1}{x+1}} \right)
\]

\[
\ll \log N \sum_{m \leq \sqrt{M}} \left( \frac{N}{m^2} \right)^{\frac{1}{2}}
\]

\[
\ll N^{\frac{1}{2}} \log N,
\]

which is negligible. Note that the sum over \(a = 0\) is equivalent to

\[
\sum_{p m^2 \equiv 59 \pmod{120}} 1 = \sum_{m \leq \sqrt{M}} \sum_{p \leq \frac{M}{m^2}} \sum_{m^2 \equiv 59 \pmod{120}} 1.
\]

The contribution for \(\log^2 M < m\) is negligible since

\[
\sum_{\log^2 M < m} \frac{M}{m^2 \log M} \ll \sum_{\log^2 M < m} \frac{M}{m^2} \ll \frac{M}{\log^2 M} \ll \frac{N}{\log^2 N}.
\]

By the prime number theorem for arithmetic progressions, the sum over \(m \leq \log^2 M\) is

\[
\sum_{m \leq \log^2 M} \frac{M}{m^2 \log M} \left(1 + O \left( \frac{\log m}{\log M} \right) \right)
\]

\[
= \frac{1}{32} \sum_{m \leq \log^2 M} \frac{M}{m^2 \log M} \left(1 + O \left( \frac{\log m}{\log M} \right) \right)
\]

\[
= \frac{1}{32} \sum_{m=1}^{\infty} \frac{M}{m^2 \log M} + O \left( \sum_{m > \log^2 M} \frac{M}{m^2 \log M} \right) + O \left( \frac{M \log \log M}{\log^2 M} \right)
\]

\[
= \frac{1}{32} \sum_{m=1}^{\infty} \frac{M}{m^2 \log M} + O \left( \frac{M \log \log M}{\log^2 M} \right).
\]

where \(\phi\) is the Euler’s \(\phi\)-function. Recalling \(M = 60N - 1\), we conclude that

\[
\gamma(N) = \frac{1}{32} \sum_{m=1}^{\infty} \frac{60N - 1}{m^2 \log(60N - 1)} + O \left( \frac{N \log \log N}{\log^2 N} \right)
\]

\[
= \left( \frac{15}{8} \sum_{m=1}^{\infty} \frac{1}{m^2} \right) \frac{N}{\log N} + O \left( \frac{N \log \log N}{\log^2 N} \right)
\]

\[
= \frac{\pi^2}{5} \frac{N}{\log N} + O \left( \frac{N \log \log N}{\log^2 N} \right),
\]

where the constant in the main term is computed by

\[
\frac{15}{8} \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{15}{8} \prod_{p \mid 120} (1 - p^{-2})^{-1} = \frac{6}{5} \zeta(2) = \frac{\pi^2}{5}.
\]

The proof of the result for \(\delta(N)\) is similar, so is omitted.
3. Proof of Theorem 1.3

Let \( f = \sum_{n=0}^{\infty} a(n) q^n \) be a weakly holomorphic modular form of half-integral weight \( k \) and level \( \Gamma_1(N) \) with integer coefficients. Bellaïche, Green, and Soundararajan \[1\] showed that if \( f \neq 0 \pmod{2} \), then

\[
\varepsilon(n \leq N : a(n) \equiv 1 \pmod{2}) \gg \frac{\sqrt{N}}{\log \log N}.
\]

Thus, to prove Theorem 1.3, it suffices to show the generating functions for \( g(2n) \) and \( h(2n+1) \pmod{2} \) are weakly holomorphic modular forms of half-integral weight for some levels. The generating functions for \( g(2n) \) and \( h(2n+1) \) were computed by Hirschhorn in \[4\]. In view of Theorem 1 of \[4\], we have

\[
\sum_{n=0}^{\infty} g(2n) q^n = \prod_{n=1}^{\infty} \frac{(1 - q^{4n})}{(1 - q^n)(1 - q^{40n-8})(1 - q^{40n-32})},
\]

\[
\sum_{n=0}^{\infty} h(2n+1) q^n = \prod_{n=1}^{\infty} \frac{(1 - q^{4n})}{(1 - q^n)(1 - q^{40n-16})(1 - q^{40n-24})}.
\]

Let \( \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \) be the Dedekind’s eta function and denote by \( f_1(\tau) = q^{-1} G(q) \) and \( f_2(\tau) = q^{11} H(q) \), where \( q = e^{2\pi i \tau} \). We see that

\[
\sum_{n=0}^{\infty} g(2n) q^n = \prod_{n=1}^{\infty} \frac{(1 - q^n)^3}{(1 - q^{5n-1})^8(1 - q^{5n-4})^8} = q^{126} f_1^8(\tau) \eta^3(\tau) \pmod{2},
\]

\[
\sum_{n=0}^{\infty} h(2n+1) q^n = \prod_{n=1}^{\infty} \frac{(1 - q^n)^3}{(1 - q^{5n-2})^8(1 - q^{5n-3})^8} = q^{-71} f_2^8(\tau) \eta^3(\tau) \pmod{2}.
\]

If \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(5) \), by Proposition 2.5 of \[2\], the transformations of \( f_1(\tau) \) and \( f_2(\tau) \) are given by

\[
f_1(A\tau) = e^{4\pi i \text{ab}} v_1(A) f_1(\tau),
\]

\[
f_2(A\tau) = e^{-4\pi i \text{ab}} v_1(A) f_2(\tau),
\]

where \( v_1(A) \) denotes the multiplier system of \( \eta^{14}(\tau) \), i.e.

\[
v_1\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{cases} \exp\left( \frac{2\pi i}{6} (-3c - b d (c^2 - 1) + c (a + d)) \right), & c \text{ odd,} \\ \exp\left( \frac{2\pi i}{6} (3d - 3 - ac (d^2 - 1) + d (b - c)) \right), & d \text{ odd.} \end{cases}
\]

Using these formulas, we can easily verify that \( f_1(120\tau) \) and \( f_2(120\tau) \) are invariant on \( \Gamma_1(2880) \). Since \( \eta(24\tau) \) is a modular form weight \( \frac{1}{2} \) on \( \Gamma_0(576) \) with character \( \chi(2) \) (see, for example, \[5\], Corollary 1.62), it follows that \( \sum_{n=0}^{\infty} h(2n+1) q^{120n+71} \pmod{2} \) and \( \sum_{n=0}^{\infty} h(2n+1) q^{120n+71} \pmod{2} \) are weakly holomorphic form weight \( \frac{1}{2} \) and level \( \Gamma_1(2880) \), as desired.

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