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Logic

BD entropy and Bernis–Friedman entropy

BD entropie et entropie de Bernis–Friedman

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ABSTRACT

In this note, we propose in the full generality a link between the BD entropy introduced by D. Bresch and B. Desjardins for the viscous shallow-water equations and the Bernis–Friedman (called BF) dissipative entropy introduced to study the lubrication equations. Different dissipative entropies are obtained playing with the drag terms on the viscous shallow-water equations. It helps for instance to prove the global existence of nonnegative weak solutions to the lubrication equations starting from the global existence of nonnegative weak solutions to appropriate viscous shallow-water equations.

RÉSUMÉ

Dans cette note, on propose un lien général entre la BD entropie introduite par D. Bresch et B. Desjardins pour les équations de Saint-Venant visqueuses et l'entropie dissipative de Bernis–Friedman (notée BF) introduite pour étudier les équations de lubrification. Différentes entropies dissipatives sont obtenues suivant le choix des termes de traînée sur Saint-Venant visqueux. Ce lien entre ces deux outils mathématiques aide, par exemple, à prouver l'existence de solutions faibles positives pour les équations de lubrification en partant de l'existence de solutions faibles positives pour des équations de Saint-Venant visqueuses bien choisies.

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Dans cette note, on propose un lien général entre la BD entropie introduite par D. Bresch et B. Desjardins pour les équations de Saint-Venant visqueuses (voir [6] et [7]) et l'entropie dissipative de Bernis–Friedman (que l'on notera BF) introduite (voir [1]) pour étudier les équations de lubrifications. Différentes entropies dissipatives sont obtenues suivant le choix des termes de traînée sur Saint-Venant visqueux, généralisant ainsi quelques travaux importants comme [12], [9]. Ce lien entre les deux outils importants que sont la BD entropie et la BF entropie permet, par exemple, de construire des solutions faibles du modèle de lubrification à partir de solutions faibles du modèle de Saint-Venant. Il permet également d'obtenir certains résultats sur les équations de Saint-Venant en s'inspirant des résultats établis sur les équations de lubrification, qui ont été beaucoup plus étudiées historiquement. Le système de lubrification s'écrit, par exemple,

$$\partial_t h + \partial_x\Big(\frac{1}{\alpha W_e} h^n \partial_x^3 h - \frac{1}{\alpha Fr^2} h^{m-1}\partial_x h\Big) = 0$$

et le modèle de Saint-Venant associé

$$\begin{cases} \partial_t h_\varepsilon + \partial_x(h_\varepsilon \bar{u}_\varepsilon) = 0, \\ \varepsilon\left(\partial_t(h_\varepsilon \bar{u}_\varepsilon) + \partial_x(h_\varepsilon \bar{u}_\varepsilon^2)\right) + \frac{1}{Fr^2} h_\varepsilon^\beta \partial_x(h_\varepsilon) = \varepsilon\left(\frac{4}{R_e}\partial_x(h_\varepsilon \partial_x \bar{u}_\varepsilon)\right) + \frac{1}{W_e} h_\varepsilon \partial_x^3 h_\varepsilon - \alpha \frac{h_\varepsilon^2 \bar{u}_\varepsilon}{h_\varepsilon^n}, \end{cases} \tag{1}$$

où $\beta + n = m$: voir par exemple [2], [3] ou [6]. Nous discuterons les deux outils importants que sont la BF entropie et l'entropie dissipative due à Bernis–Friedman (BF). Nous expliquerons l'intérêt qu'il y a à mettre en exergue une telle relation. On peut, par exemple, étudier ces systèmes et montrer l'existence globale de solutions faibles du modèle de lubrification en partant de solutions faibles positives du modèle de Saint-Venant associé. On peut également considérer des systèmes avec des termes non locaux en s'inspirant de résultats récents, voir par exemple [11], [8].

1. Introduction: lubrication systems and viscous shallow-water equations with drag terms

In this section, we present the formal link between two key tools, respectively, for lubrication systems by Bernis–Friedman and for shallow-water equations by Bresch–Desjardins. We first start by presenting the two quantities and their link on a simple example and then we explain how to get relations in a more general case. Our calculations remain at this stage only formal. We assume that solutions are regular enough.

1.1. A lubrication system: energy estimate and Bernis–Friedman (BF) dissipative entropy

In a one-dimensional periodic domain Ω, consider the following thin-film equation, also known as lubrication equation,

$$\partial_t h + \partial_x\Big(\frac{1}{\alpha W_e} F(h)\partial_x^3 h - \frac{1}{\alpha Fr^2} F(h)\partial_x h\Big) = 0. \tag{2}$$

We couple this equation with the initial condition

$$h(x,0) = h_0(x) \quad \text{in } \Omega.$$

System (2) can be rewritten equivalently as a gradient-flow system

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ hu = \frac{1}{\alpha W_e} F(h)\partial_x^3 h - \frac{1}{\alpha Fr^2} F(h)\partial_x h. \end{cases} \tag{3}$$

The corresponding energy is given, for all $t \in (0,T)$, by

$$\int_0^t \int_\Omega \frac{\alpha h^2 u^2}{F(h)}\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_\Omega \frac{h(x,t)^2}{Fr^2} + \frac{(\partial_x h(x,t))^2}{W_e}\,\mathrm{d}x = \frac{1}{2}\int_\Omega \frac{h_0(x)^2}{Fr^2} + \frac{(\partial_x h_0(x))^2}{W_e}\,\mathrm{d}x. \tag{4}$$

In their paper [1], Bernis and Friedman proved the existence of a weak solution to higher-order nonlinear degenerate parabolic equations and suggested a new entropy inequality – referred to as BF entropy – which provides additional estimates serving for increasing the regularity of the weak solution obtained. As for our problem, we adapt the same methodology to obtain the BF entropy of the general lubrication model stated above. Indeed, define the functionals

$$g_\varepsilon(s) = -\int_s^A \frac{1}{F(r)+\varepsilon}\,\mathrm{d}r, \qquad G_\varepsilon(s) = -\int_s^A g_\varepsilon(r)\,\mathrm{d}r,$$

with A being an integer such that $A \geq \max|h(x,t)|$. According to Bernis and Friedman, we multiply (2) by $G_0'(h)$, where $G_0 = \lim_{\varepsilon \to 0} G_\varepsilon$, we get the BF dissipative entropy equality

$$\int_\Omega G_0(h(x,T))\,\mathrm{d}x + \int_0^T \int_\Omega \frac{(\partial_x^2 h)^2}{\alpha W_e} + \frac{(\partial_x h)^2}{\alpha Fr^2}\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega G_0(h_0(x))\,\mathrm{d}x. \tag{5}$$

1.2. A viscous shallow-water system: energy estimate and BD entropy

In a periodic domain Ω, let us consider the viscous shallow-water system with surface tensions and a drag term:

$$\begin{cases} \partial_t h_\varepsilon + \partial_x(h_\varepsilon \bar{u}_\varepsilon) = 0, \\ \varepsilon\left(\partial_t(h_\varepsilon \bar{u}_\varepsilon) + \partial_x(h_\varepsilon \bar{u}_\varepsilon^2)\right) + \frac{h_\varepsilon \partial_x(h_\varepsilon)}{Fr^2} = \varepsilon\left(\frac{4}{R_e}\partial_x(h_\varepsilon \partial_x \bar{u}_\varepsilon)\right) + \frac{1}{W_e} h_\varepsilon \partial_x^3 h_\varepsilon - \alpha \frac{h_\varepsilon^2 \bar{u}_\varepsilon}{F(h_\varepsilon)}. \end{cases} \tag{6}$$

The initial conditions are given by

$$h_\varepsilon|_{t=0} = h_0^\varepsilon, \qquad (h_\varepsilon \bar{u}_\varepsilon)|_{t=0} = m_0^\varepsilon.$$

α is a positive constant, R_e, W_e and Fr are respectively the adimensional Reynold, Weber, and Froude numbers. Note that the terms in the right-hand side of the momentum equation represent, respectively, the viscous term, the capillary term, and the drag term. The energy equation corresponding to (6) is given, for all $t \in (0,T)$ by

$$\left(\int_\Omega \varepsilon \frac{h_\varepsilon(x,t)\bar{u}_\varepsilon^2(x,t)}{2} + \frac{h_\varepsilon(x,t)^2}{2Fr^2} + \frac{(\partial_x h_\varepsilon(x,t))^2}{2W_e}\right)\mathrm{d}x + \int_0^t \int_\Omega \frac{4\varepsilon}{R_e} h_\varepsilon (\partial_x \bar{u}_\varepsilon)^2 + \alpha \frac{h_\varepsilon^2 \bar{u}_\varepsilon^2}{F(h_\varepsilon)}\,\mathrm{d}x\,\mathrm{d}t \tag{7}$$
$$= \frac{1}{2}\int_\Omega \varepsilon \frac{(m_0^\varepsilon)^2}{h_0^\varepsilon} + \frac{(h_0^\varepsilon)^2}{Fr^2} + \frac{(\partial_x h_0^\varepsilon)^2}{W_e}\,\mathrm{d}x.$$

As introduced in [6], the BD entropy equality is obtained by deriving the mass equation in space and multiplying by $4\varepsilon/R_e$, then summing with the momentum equation, and multiplying the sum by the artificial velocity

$$v_\varepsilon = \bar{u}_\varepsilon + \frac{4}{R_e}\partial_x(log(h_\varepsilon)).$$

The BD entropy for the general system is given, for all $t \in (0,T)$, by

$$\frac{\varepsilon}{2}\int_\Omega h_\varepsilon(x,t) v_\varepsilon(x,t)^2\,\mathrm{d}x + \frac{1}{2}\left[\int_\Omega \frac{h_\varepsilon(x,t)^2}{Fr^2} + \frac{(\partial_x h_\varepsilon(x,t))^2}{W_e}\,\mathrm{d}x\right]$$
$$+\alpha \int_0^t \int_\Omega \frac{h_\varepsilon^2 \bar{u}_\varepsilon^2}{F(h_\varepsilon)}\,\mathrm{d}x\,\mathrm{d}t + \frac{4}{R_e}\left[\int_\Omega \int_0^t \frac{(\partial_x h_\varepsilon)^2}{Fr^2} + \frac{(\partial_{xx} h_\varepsilon)^2}{W_e}\,\mathrm{d}x\,\mathrm{d}t + \alpha \underbrace{\int_0^t \int_\Omega \frac{h_\varepsilon \bar{u}_\varepsilon}{F(h_\varepsilon)}\partial_x h_\varepsilon\,\mathrm{d}x\,\mathrm{d}t}_{X}\right] \tag{8}$$
$$= \frac{\varepsilon}{2}\int_\Omega \int_\Omega h_0^\varepsilon(x) v_\varepsilon(x,0)^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \frac{h_0^\varepsilon(x)^2}{Fr^2} + \frac{(\partial_x h_0^\varepsilon(x))^2}{W_e}\,\mathrm{d}x.$$

As for the term X, it can be rewritten as

$$\begin{aligned} X = \int_\Omega h_\varepsilon \bar{u}_\varepsilon \frac{\partial_x h_\varepsilon}{F(h_\varepsilon)}\,\mathrm{d}x &= \int_\Omega h_\varepsilon \bar{u}_\varepsilon \Big(\frac{d}{\mathrm{d}x}\int_A^{h_\varepsilon} \frac{1}{F(y)}\,\mathrm{d}y\Big)\,\mathrm{d}x \\ &= \int_\Omega -\partial_x(h_\varepsilon \bar{u}_\varepsilon)\Big(\int_A^{h_\varepsilon} \frac{1}{F(y)}\,\mathrm{d}y\Big)\,\mathrm{d}x \\ &= \int_\Omega \partial_t h_\varepsilon \Big(\int_A^{h_\varepsilon} \frac{1}{F(y)}\,\mathrm{d}y\Big)\,\mathrm{d}x. \end{aligned}$$

1.3. The link between the BD entropy and the BF dissipative entropy

In view of the terms X and G_0, we noticed the following

$$G_0'(h_\varepsilon) = g_0(h_\varepsilon) = -\int_{h_\varepsilon}^{A} \frac{1}{F(r)}\,\mathrm{d}r = \int_{A}^{h_\varepsilon} \frac{1s}{F(r)}\,\mathrm{d}r.$$

Hence,

$$X = \int_\Omega \partial_t h_\varepsilon G_0'(h_\varepsilon)\,\mathrm{d}x = \frac{d}{dt}\int_\Omega G_0(h_\varepsilon)\,\mathrm{d}x.$$

Thus, coupled with (7), the BD entropy reads:

$$\begin{aligned}
&\frac{\varepsilon}{2}\int_\Omega h_\varepsilon(x,t)v_\varepsilon(x,t)^2\,\mathrm{d}x - \frac{\varepsilon}{2}\int_\Omega h_\varepsilon(x,t)\bar{u}_\varepsilon(x,t)^2\,\mathrm{d}x - \frac{4\varepsilon}{R_e}\int_0^t\int_\Omega h_\varepsilon(\partial_x\bar{u}_\varepsilon)^2 \\
&+\frac{4}{R_e}\left[\int_\Omega\int_0^t \frac{(\partial_x h_\varepsilon)^2}{Fr^2} + \frac{(\partial_{xx}h_\varepsilon)^2}{W_e}\,\mathrm{d}x\,\mathrm{d}t + \alpha\int_\Omega G_0(h_\varepsilon(x,t))\,\mathrm{d}x\right] \\
&= -\frac{1}{2}\int_\Omega \varepsilon\frac{(m_0^\varepsilon)^2}{h_0^\varepsilon} + \frac{\varepsilon}{2}\int_\Omega h_0^\varepsilon(x)v_\varepsilon(x,0)^2\,\mathrm{d}x + \frac{4\alpha}{R_e}\int_\Omega G_0(h_0^\varepsilon)\,\mathrm{d}x.
\end{aligned} \tag{9}$$

If we assume now that the couple $(h_\varepsilon, \bar{u}_\varepsilon)$, solution to (6), converges in a proper sense to (h, u), then we find that the above BD entropy equality degenerates to the following inequality, which coincides with the dissipative BF entropy of (2)

$$\int_\Omega\int_0^t \frac{(\partial_x h)^2}{\alpha Fr^2} + \frac{(\partial_{xx}h)^2}{\alpha W_e}\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega G_0(h(x,t))\,\mathrm{d}x = \int_\Omega G_0(h_0(x))\,\mathrm{d}x. \tag{10}$$

Of course, these computations are formal and have been written with equalities, but they help to understand that the BF entropy may be obtained from the BD entropy. This provides a way to construct nonnegative solutions to the lubrication equation from nonnegative solutions to the shallow-water equation with appropriate drag terms. Let us present below a general computation with different surface tension and pressure term, showing the relation between the BD entropy and the BF dissipative entropy.

2. A general link between the BD entropy and the BF dissipative entropy

In this part, we will consider the following fourth-order lubrication approximation that has been studied in several papers, see, for instance, [2] and [3]:

$$\partial_t h + \partial_x\left(\frac{1}{\alpha W_e}F(h)\partial_x^3 h - \frac{1}{\alpha Fr^2}D(h)\partial_x h\right) = 0. \tag{11}$$

In [2], for instance, the authors considered the above lubrication model with the following choice of F and D: $F(h) = h^n$ and $D(h) = h^{m-1}$. Indeed, the lubrication equation becomes:

$$\partial_t h + \partial_x\left(\frac{1}{\alpha W_e}h^n\partial_x^3 h - \frac{1}{\alpha Fr^2}h^{m-1}\partial_x h\right) = 0. \tag{12}$$

In fact, they proved the existence of a global in time nonnegative weak solution starting from nonnegative datum for all $n > 0$, and $1 < m < 2$. In particular, the most critical case is the most significantly physical one when $n = 3$ (moving contact line in a thin film). In this case, a distributional solution is proven to exist, where it becomes a strong positive solution in the infinite time limit. In this sequel, we will consider the choices of F and D stated above. Then, the BF entropy corresponding to the latter system is given by

$$\int_\Omega G_0(h(x,T))\,\mathrm{d}x + \int_0^T\int_\Omega \frac{(\partial_x^2 h)^2}{We} + \frac{1}{Fr^2}h^{m-n-1}(\partial_x h)^2\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega G_0(h_0(x))\,\mathrm{d}x. \tag{13}$$

Consider herein the following shallow-water system with a drag term corresponding to a weight $F(h) = h^n$

$$\begin{cases} \partial_t h_\varepsilon + \partial_x(h_\varepsilon \bar{u}_\varepsilon) = 0, \\ \varepsilon\left(\partial_t(h_\varepsilon \bar{u}_\varepsilon) + \partial_x(h_\varepsilon \bar{u}_\varepsilon^2)\right) + \frac{1}{Fr^2} h_\varepsilon^\beta \partial_x(h_\varepsilon) = \varepsilon\left(\frac{4}{R_e}\partial_x(h_\varepsilon \partial_x \bar{u}_\varepsilon)\right) + \frac{1}{W_e} h_\varepsilon \partial_x^3 h_\varepsilon - \alpha \frac{h_\varepsilon^2 \bar{u}_\varepsilon}{h_\varepsilon^n}, \end{cases} \tag{14}$$

where $\beta + n \in (1, 2)$. The energy and BD entropy of system (14) are given, respectively, by

$$\frac{1}{2}\left(\int_\Omega \varepsilon h_\varepsilon(x,T)\bar{u}_\varepsilon^2(x,T) + \frac{1}{Fr^2}\frac{h_\varepsilon(x,T)^{\beta+1}}{\beta(\beta+1)} + \frac{(\partial_x h_\varepsilon(x,T))^2}{W_e}\right) \mathrm{d}x + \int_0^T \int_\Omega \frac{4\varepsilon}{R_e} h_\varepsilon (\partial_x \bar{u}_\varepsilon)^2 + \alpha \frac{h_\varepsilon^2 \overline{u}_\varepsilon^2}{h_\varepsilon^n} \,\mathrm{d}x\,\mathrm{d}t$$
$$= \frac{1}{2}\int_\Omega \varepsilon \frac{(m_0^\varepsilon)^2}{h_0^\varepsilon} + \frac{1}{Fr^2}\frac{(h_0^\varepsilon)^{\beta+1}}{\beta(\beta+1)} + \frac{(\partial_x h_0^\varepsilon)^2}{W_e}\,\mathrm{d}x.$$

and

$$\frac{\varepsilon}{2}\int_\Omega h_\varepsilon(x,T) v_\varepsilon(x,T)^2 - h_\varepsilon(x,T)\bar{u}_\varepsilon(x,t)^2\,\mathrm{d}x + \frac{4}{Re}\Big[\int_0^T \int_\Omega \frac{1}{Fr^2} h_\varepsilon^{\beta-1}(\partial_x h_\varepsilon)^2\,\mathrm{d}x + \frac{1}{W_e}(\partial_{xx} h_\varepsilon)^2\,\mathrm{d}x\,\mathrm{d}t\Big]$$
$$+ \frac{4}{Re}\int_\Omega G_0(h_\varepsilon(x,T))\,\mathrm{d}x - \frac{4\varepsilon}{Re}\int_0^T\int_\Omega h_\varepsilon(\partial_x \bar{u}_\varepsilon)^2\,\mathrm{d}x\,\mathrm{d}t$$
$$= \frac{\varepsilon}{2}\int_\Omega h_0^\varepsilon v_\varepsilon(x,0)^2 - \frac{m_0^\varepsilon}{h_0^\varepsilon}\,\mathrm{d}x + \frac{4}{Re}\int_\Omega G_0(h_0^\varepsilon(x))\,\mathrm{d}x.$$

Under the assumption of the convergence results, and choosing $m = \beta + n \in (1, 2)$, we get that the BD entropy degenerates as well to the BF dissipative entropy of system (12) given by (13). Remark that the link between the BD entropy and the BF entropy may also be done in higher dimensions, see [10] for BF entropy and [7] for BD entropy. This could help to perform the analysis in the bi-dimensional setting.

3. Mathematical results obtained using the link between BD and BF entropies

In this part, we aim at proving the existence of a global in time weak solution for the lubrication model by passing to the limit in the viscous shallow-water model with two different choices of the drag term, corresponding to two weights: $F(h) = h^2$, which results in a linear drag term, and $F(h) = h^2 + h^3$, which yields a nonlinear drag term. The latter weight has been used by A.L. Bertozzi in the physical and mathematical justification of the lubrication model [2]. The main theorem states the following.

Theorem 3.1. *Given a sequence $(h_\varepsilon, \bar{u}_\varepsilon)_\varepsilon$ a global weak solution to (6), where $h_0^\varepsilon \geq 0$, then there exists a subsequence of $(h_\varepsilon, \bar{u}_\varepsilon)$ such that $(h_\varepsilon, \bar{u}_\varepsilon)$ converges to (h,u), a global weak solution to the lubrication system (2) satisfying $h \geq 0$, and the initial condition $h|_{t=0} = h_0$, where h_0 is the weak limit of h_0^ε in $H^1(\Omega)$.*

The proof of the limit of the viscous shallow-water model into a lubrication model is summarized in the following steps:

(i) assuming that (6) possesses a weak solution $(h_\varepsilon, \bar{u}_\varepsilon)$, bring first the physical energy and BD entropy estimates to get uniform bounds of the system's unknowns and thus get weak convergence up to a subsequence of these terms; such solution has been constructed by D. Bresch and B. Desjardins in [6]. See also [13] for more general interesting studies related to $1D$ compressible Navier–Stokes;
(ii) use compactness theory to obtain strong convergence (mainly for h_ε);
(iii) pass the limit in the weak formulation of (6) to obtain that the solution is a weak solution to the lubrication theory.

It is important to remark that some studies have already analyzed the limit process from shallow-water to lubrication systems but with special pressure terms, see the nice papers [14], [12] and recently [9]. See also the recent paper [4] where dissipative systems may be obtained from a shallow-water-type system through a quadratic change of time and no need for *a priori* drag terms. Note that in [12] and [9], the BD and BF entropies are interconnected for $F(h) = h$. In [11], we can find a result concerning the global existence of non-negative solutions for electrified thin films. Such systems contain nonlocal terms. As an example consider the following system

$$\partial_t h + \partial_x(h^3 \partial_x(\partial_x^2 h - I(h))) = 0 \text{ in } \Omega = (0, 1)$$

where $I(h)$ is a non-local elliptic operator of order 1 given by

$$I(h) = \int_{\Omega} (h(y) - h(x))\nu(x, y)\,\mathrm{d}y$$

where for all $x, y \in \Omega$

$$\nu(x, y) = \frac{\pi}{2}\Big(\frac{1}{1-\cos(\pi(x-y))} + \frac{1}{1-\cos(\pi(x+y))}\Big).$$

The system is supplemented by the following boundary and initial conditions

$$\partial_x h = h^3\partial_x(\partial_x^2 h - I(h)) = 0 \text{ on } \partial\Omega, \qquad h|_{t=0} = h_0 \text{ for } x \in \Omega.$$

It is interesting to see that, in order to construct a solution to this lubrication equation, one can consider the following shallow-water model

$$\begin{cases} \partial_t h_\varepsilon + \partial_x(h_\varepsilon \bar{u}_\varepsilon) = 0, \\ \varepsilon\left(\partial_t(h_\varepsilon\bar{u}_\varepsilon) + \partial_x(h_\varepsilon\bar{u}_\varepsilon^2)\right) + h_\varepsilon\partial_x I(h_\varepsilon) = \varepsilon\partial_x(h_\varepsilon\partial_x\bar{u}_\varepsilon) + h_\varepsilon\partial_x^3 h_\varepsilon - \alpha\frac{\bar{u}_\varepsilon}{h_\varepsilon}, \end{cases} \tag{15}$$

with appropriate boundary conditions. Then using the energy estimate and the BD entropy, we can pass to the limit and get the global existence of non-negative solutions to the system studied in [11]. Note that the compressible Navier–Stokes system with constant viscosities and non-local term has been studied recently in [8] together with the long-time behavior of its solutions. The details will be given in the forthcoming paper [5].

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