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Pluriharmonic Clark measures and analogs of model spaces ☆

Mesures de Clark pluriharmoniques et analogues des espaces modèles

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A B S T R A C T

Let B_d denote the unit ball of $\mathbb{C}^d$, $d \ge 1$. Given an inner function $I : B_d \to B_1$, we study the corresponding family $\sigma_\alpha[I]$, $\alpha \in \partial B_1$, of pluriharmonic Clark measures on the complex sphere. We introduce and investigate related unitary operators U_α mapping analogs of model spaces onto $L^2(\sigma_\alpha)$, $\alpha \in \partial B_1$. In particular, we explicitly characterize the set of $U_\alpha^* f$ such that $f\sigma_\alpha$ is a pluriharmonic measure.

R É S U M É

Soit B_d la boule unité de $\mathbb{C}^d$, $d \ge 1$. Étant donnée une fonction intérieure $I : B_d \to B_1$, nous étudions la famille correspondante $\sigma_\alpha[I]$, $\alpha \in \partial B_1$, de mesures de Clark pluriharmoniques sur la sphère complexe. Nous introduisons et étudions les opérateurs unitaires U_α entre des analogues des espaces modèles et $L^2(\sigma_\alpha)$, $\alpha \in \partial B_1$. En particulier, nous caractérisons explicitement l'ensemble des $U_\alpha^* f$ telles que $f\sigma_\alpha$ soit une mesure pluriharmonique.

1. Introduction

Let B_d denote the open unit ball of $\mathbb{C}^d$, $d \ge 1$. For the unit disk B_1 of $\mathbb{C}$, we also use the notation $\mathbb{D}$. Put $S_d = \partial B_d$ and $\mathbb{T} = \partial\mathbb{D}$. For $z, \zeta \in B_d \cup S_d$ with $\langle z, \zeta\rangle \ne 1$, the equality

$$C(z,\zeta) = (1 - \langle z, \zeta\rangle)^{-d}$$

defines the Cauchy kernel for B_d. The invariant Poisson kernel is given by the formula

$$P(z,\zeta) = \frac{C(z,\zeta)C(\zeta,z)}{C(z,z)} = \left(\frac{1-|z|^2}{|1-\langle z,\zeta\rangle|^2}\right)^d, \quad z \in B_d,\ \zeta \in S_d.$$

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1.1. Pluriharmonic measures

Let $M(S_d)$ denote the space of complex Borel measures on the sphere S_d. A measure $\mu \in M(S_d)$ is called *pluriharmonic* if the invariant Poisson integral

$$P[\mu](z) = \int_{S_d} P(z,\zeta)\, \mathrm{d}\mu(\zeta), \quad z \in B_d,$$

is a pluriharmonic function. Let $PM(S_d)$ denote the set of all pluriharmonic measures. For $\mu \in PM(S_d)$, it is well known that the invariant Poisson integral $P[\mu]$ coincides with the harmonic one. See [7] for this fact, other properties of the invariant Poisson integrals as well as basic results of the function theory in the unit ball B_d.

1.2. Clark measures

Let $\Sigma = \Sigma_d$ denote the normalized Lebesgue measure on the sphere S_d.

Definition 1.1. A holomorphic function $I : B_d \to \mathbb{D}$ is called *inner* if $|I(\zeta)| = 1$ for Σ_d-a.e. $\zeta \in S_d$.

In the above definition, $I(\zeta)$ stands, as usual, for $\lim_{r\to 1-} I(r\zeta)$. Recall that the corresponding limit is known to exist Σ_d-a.e. Also, by the above definition, unimodular constants are not inner functions.

Given an $\alpha \in \mathbb{T}$ and an inner function $I : B_d \to \mathbb{D}$, the quotient

$$\frac{1 - |I(z)|^2}{|\alpha - I(z)|^2} = \mathrm{Re}\left(\frac{\alpha + I(z)}{\alpha - I(z)}\right), \quad z \in B_d,$$

is positive and pluriharmonic. Therefore, there exists a unique positive measure $\sigma_\alpha = \sigma_\alpha[I] \in PM(S_d)$ such that

$$P[\sigma_\alpha](z) = \mathrm{Re}\left(\frac{\alpha + I(z)}{\alpha - I(z)}\right), \quad z \in B_d.$$

Since I is inner, we have

$$P[\sigma_\alpha](\zeta) = \frac{1 - |I(\zeta)|^2}{|\alpha - I(\zeta)|^2} = 0 \quad \Sigma_d\text{-a.e.},$$

thus, σ_α is a singular measure. Here and in what follows, this means that σ_α is singular with respect to Σ_d; in brief, $\sigma_\alpha \perp \Sigma_d$.

After the famous paper of Clark [1], various properties and applications of the measures σ_α on the unit circle $\mathbb{T}$ have been obtained; see, for example, reviews [5], [6], [8] for further references. To the best of the authors' knowledge, the measures σ_α on the unit sphere S_d, $d \geq 2$, have not been investigated earlier. See [4] for a different extension of the Clark theory motivated by the multivariable operator theory.

1.3. Clark measures and model spaces

For $d \geq 1$, let $Hol(B_d)$ denote the space of holomorphic functions in B_d. The classical Hardy space $H^2 = H^2(B_d)$ consists of those $f \in Hol(B_d)$ for which

$$\|f\|_{H^2}^2 = \sup_{0<r<1} \int_{S_d} |f(r\zeta)|^2\, \mathrm{d}\Sigma_d(\zeta) < \infty.$$

Given an inner function θ on $\mathbb{D}$, the classical model space K_θ is defined as $K_\theta = H^2(\mathbb{D}) \ominus \theta H^2(\mathbb{D})$. Clark [1] introduced and studied a family of unitary operators $U_\alpha : K_\theta \to L^2(\sigma_\alpha)$, $\alpha \in \mathbb{T}$.

For an inner function I in B_d, $d \geq 2$, consider the following natural analogs of K_θ:

$$I^*(H^2) = H^2 \ominus I H^2;$$
$$I_*(H^2) = \{f \in H^2 : I\overline{f} \in H_0^2\},$$

where $H_0^2 = \{f \in H^2 : f(0) = 0\}$. Clearly, we have $I_*(H^2) \subset I^*(H^2)$; if θ is an inner function in $\mathbb{D}$, then $\theta^*(H^2(\mathbb{D})) = \theta_*(H^2(\mathbb{D})) = K_\theta$. In this paper, we define unitary operators

$$U_\alpha : I^*(H^2) \to L^2(\sigma_\alpha), \quad \alpha \in \mathbb{T},$$

and we obtain the following characterization:

Theorem 1.2. *Let I be an inner function in the unit ball B_d, $d \ge 2$, and let $f \in L^2(\sigma_\alpha)$, $\alpha \in \mathbb{T}$. Then the following properties are equivalent:*

(i) $U^*_\alpha f \in I_*(H^2)$;
(ii) $f\sigma_\alpha \in PM(S_d)$.

Auxiliary facts are collected in Section 2. Theorem 1.2 and other results related to $I^*(H^2)$, $I_*(H^2)$ and the unitary operators U_α are discussed in Section 3.

2. Auxiliary results

The following lemma is a particular case of Theorem 1 from [9, Chap. V, §21, Sect. 66].

Lemma 2.1. *Let F be a holomorphic function on $B_d \times B_d$. If $F(z, \overline{z}) = 0$ for all $z \in B_d$, then $F(z, w) = 0$ for all $(z, w) \in B_d \times B_d$.*

Proposition 2.2. *Let $I : B_d \to \mathbb{D}$, $d \ge 2$, be an inner function and let $\sigma_\alpha = \sigma_\alpha[I]$, $\alpha \in \mathbb{T}$. Then*

$$\int_{S_d} C(z, \zeta)C(\zeta, w)\, d\sigma_\alpha(\zeta) = \frac{1 - I(z)\overline{I(w)}}{(1 - \overline{\alpha}I(z))(1 - \alpha\overline{I(w)})} C(z, w)$$

for all $\alpha \in \mathbb{T}$, $z, w \in B_d$.

Proof. The equality

$$\int_{S_d} P(z, \zeta)\, d\sigma_\alpha(\zeta) = \frac{1 - |I(z)|^2}{|\alpha - I(z)|^2}, \quad z \in B_d,$$

and the definition of $P(z, \zeta)$ guarantee that

$$\int_{S_d} C(z, \zeta)C(\zeta, z)\, d\sigma_\alpha(\zeta) = \frac{1 - |I(z)|^2}{|\alpha - I(z)|^2} C(z, z), \quad z \in B_d.$$

It remains to apply Lemma 2.1. □

Corollary 2.3. *Let $I : B_d \to \mathbb{D}$, $d \ge 2$, be an inner function. Then*

$$\int_{S_d} C(z, \zeta)\, d\sigma_\alpha[I](\zeta) = \frac{1}{1 - \overline{\alpha}I(z)} + \frac{\alpha\overline{I(0)}}{1 - \alpha\overline{I(0)}}$$

for all $\alpha \in \mathbb{T}$, $z \in B_d$.

By definition, the ball algebra $A(B_d)$ consists of those $f \in C(\overline{B_d})$ that are holomorphic in B_d. For $z \in B_d$, let $M_z(S_d)$ denote the set of those probability measures $\rho \in M(S_d)$ that represent the point z for $A(B_d)$, that is,

$$\int_{S_d} f\, d\rho = f(z) \quad \text{for all } f \in A(B_d).$$

Elements of $M_z(S_d)$ are called representing measures.

Definition 2.4. A measure $\mu \in M(S_d)$ is said to be *totally singular* if $\mu \perp \rho$ for all $\rho \in M_0(S_d)$.

It is easy to check that the notion introduced in Definition 2.4 does not change if $M_0(S_d)$ is replaced by $M_z(S_d)$ for any $z \in B_d$; see, for example, [7, Sect. 9.1.3].

Theorem 2.5 *([3, Theorem 10]). Let $\mu \in PM(S_d)$. Then the singular part of μ is totally singular.*

Corollary 2.6. *Let I be an inner function in B_d, $d \ge 2$. Then $\sigma_\alpha = \sigma_\alpha[I]$ is totally singular for any $\alpha \in \mathbb{T}$.*

Definition 2.7 *(see [7, Sect. 9.1.5]).* We say that $\mu \in M(S_d)$ is a *Henkin measure* if

$$\lim_{j\to\infty} \int_{S_d} f_j \,\mathrm{d}\mu = 0$$

for any bounded sequence $\{f_j\}_{j=1}^{\infty} \subset A(B_d)$ with the following property:

$$\lim_{j\to\infty} f_j(z) = 0 \quad \text{for any } z \in B_d.$$

Lemma 2.8. *Let I be an inner function in B_d and let $\sigma_\alpha = \sigma_\alpha[I]$, $\alpha \in \mathbb{T}$. Then the ball algebra $A(B_d)$ is dense in $L^2(\sigma_\alpha)$.*

Proof. Assume that $A(B_d)$ is not dense in $L^2(\sigma_\alpha)$. Then there exists a non-trivial function $h \in L^2(\sigma_\alpha)$ such that $h\sigma_\alpha \in A(B_d)^\perp$, that is,

$$\int_{S_d} fh \,\mathrm{d}\sigma_\alpha = 0 \quad \text{for all } f \in A(B_d).$$

So, $h\sigma_\alpha$ is clearly a Henkin measure. Hence, by the Cole–Range theorem (see [2] or [7, Theorem 9.6.1]), $h\sigma_\alpha \ll \rho$ for some representing measure $\rho \in M_0(S_d)$. However, $h\sigma_\alpha \perp \rho$ by Corollary 2.6. This contradiction finishes the proof of the lemma. □

3. Two analogs of model spaces

For an inner function θ on $\mathbb{D}$, the classical model space $K_\theta = K_\theta(\mathbb{D})$ is defined as $K_\theta = H^2(\mathbb{D}) \ominus \theta H^2(\mathbb{D})$. Given an inner function I in B_d, $d \ge 2$, recall that we consider the following analogs of the model space: $I_*(H^2) = \{f \in H^2 : I\overline{f} \in H^2_0\}$ and $I^*(H^2) = H^2 \ominus IH^2$, where $H^2 = H^2(B_d)$. Clearly, $I_*(H^2) \subset I^*(H^2)$.

Let $\alpha \in \mathbb{T}$. In the present section, we construct a unitary operator U_α from $I^*(H^2)$ onto $L^2(\sigma_\alpha)$; see Theorem 3.1 below. Next, in Section 3.2, we prove that (ii)⇒(i) in Theorem 1.2; also, we outline the proof of the reverse implication.

3.1. A unitary operator from $I^*(H^2)$ onto $L^2(\sigma_\alpha)$

Observe that

$$K(z,w) \stackrel{\text{def}}{=} \frac{1 - I(z)\overline{I(w)}}{(1 - \langle z, w\rangle)^n} = (1 - I(z)\overline{I(w)})C(z,w)$$

is the reproducing kernel for $I^*(H^2)$, that is,

$$g(z) = \int_{S_d} g(w)K(z,w)\,\mathrm{d}\Sigma_d(w), \quad z \in B_d,$$

for all $g \in I^*(H^2)$. Indeed, $C(z,w)$ is the reproducing kernel for $H^2(B_d)$; hence, $I(z)C(z,w)\overline{I(w)}$ is the reproducing kernel for $IH^2(B_d)$. Therefore, the difference $C(z,w) - I(z)C(z,w)\overline{I(w)}$ is the reproducing kernel for $H^2(B_d) \ominus IH^2(B_d)$.

Put $K_w(z) = K(z,w)$ and define

$$(U_\alpha K_w)(\zeta) \stackrel{\text{def}}{=} \frac{1 - \alpha\overline{I(w)}}{(1 - \langle \zeta, w\rangle)^n} = (1 - \alpha\overline{I(w)})C(\zeta,w), \quad \zeta \in S_d.$$

Theorem 3.1. *For each $\alpha \in \mathbb{T}$, U_α has a unique extension to a unitary operator from $I^*(H^2)$ onto $L^2(\sigma_\alpha)$.*

Proof. Fix an $\alpha \in \mathbb{T}$. Since $K(z,w)$ is the reproducing kernel function for $I^*(H^2)$, the linear span of the family $\{K_w\}_{w\in B_d}$ is dense in $I^*(H^2)$. Therefore, if the required extension exists, then it is unique.

Now, we claim that $(U_\alpha K_w, U_\alpha K_z)_{L^2(\sigma_\alpha)} = (K_w, K_z)_{H^2}$ for $z, w \in B_d$. Indeed, applying Proposition 2.2, we obtain

$$\begin{aligned}
(U_\alpha K_w, U_\alpha K_z)_{L^2(\sigma_\alpha)} &= \int_{S_d} (1 - \alpha\overline{I(w)})C(\zeta,w)(1 - \overline{\alpha}I(z))C(z,\zeta)\,\mathrm{d}\sigma_\alpha(\zeta) \\
&= (1 - \alpha\overline{I(w)})(1 - \overline{\alpha}I(z)) \int_{S_d} C(\zeta,w)C(z,\zeta)\,\mathrm{d}\sigma_\alpha(\zeta) \\
&= (1 - I(z)\overline{I(w)})C(z,w) \\
&= K(z,w) = (K_w, K_z)_{H^2}.
\end{aligned}$$

So, U_α extends to an isometric embedding of $I^*(H^2)$ into $L^2(\sigma_\alpha)$. Hence, to finish the proof, it remains to observe that the linear span of the family $\{C(\zeta, z)\}_{z\in B_d}$ is dense in $L^2(\sigma_\alpha)$ by Lemma 2.8. □

3.2. About the proof of Theorem 1.2

In this section, we use standard facts of the function theory in B_d without explicit references. In particular, we identify the Hardy space $H^p(B_d)$, $p > 0$, and the space $H^p(S_d)$ of the corresponding boundary values. For a measure $\mu \in M(S_d)$, its Cauchy transform μ_+ is defined as

$$\mu_+(z) = \int_{S_d} C(z,\zeta)\,\mathrm{d}\mu(\zeta), \quad z \in B_d.$$

Also, put

$$\mu_-(z) = \int_{S_d} (C(\zeta,z)-1)\,\mathrm{d}\mu(\zeta), \quad z \in B_d.$$

Observe that $\mu_+(z) + \mu_-(z) = P[\mu](z)$, $z \in B_d$, for all $\mu \in PM(S_d)$.

Next, we claim that

$$(U_\alpha^* f)(z) = (1-\overline{\alpha} I(z))(f\sigma_\alpha)_+(z), \quad z \in B_d, \tag{1}$$

for $f \in L^2(\sigma_\alpha)$, $\alpha \in \mathbb{T}$.

Indeed, the definition of U_α and Proposition 2.2 imply the above equality for $f(\zeta) = (1-\alpha\overline{I(w)})C(\zeta, w)$ with $w \in B_d$. By Lemma 2.8, the linear span of the family

$$\left\{(1-\alpha\overline{I(w)})C(\zeta,w)\right\}_{w\in B_d}$$

is dense in $L^2(\sigma_\alpha)$. So, the claim is proved.

Proof of (ii)⇒(i) in Theorem 1.2. Let $f\sigma_\alpha \in PM(S_d)$. Put $\overline{G} = -(f\sigma_\alpha)_-$. Then $G \in H^p_0$, $0 < p < 1$. The property $f\sigma_\alpha \in PM(S_d)$ guarantees that

$$P[f\sigma_\alpha](z) = (f\sigma_\alpha)_+(z) - \overline{G}(z), \quad z \in B_d.$$

Since $f\sigma_\alpha$ is a singular measure, we have $(f\sigma_\alpha)_+(\zeta) = \overline{G}(\zeta)$ for Σ_d-a.e. $\zeta \in S_d$. Therefore, (1) and Theorem 3.1 imply that

$$(1-\overline{\alpha} I)\overline{G} = U_\alpha^* f \in H^2(S_d).$$

Also, for $0 < p < 1$, we have $I\overline{U_\alpha^* f} = I(1-\alpha\overline{I})G = (I-\alpha)G \in L^2(S_d) \cap H^p_0(S_d) = H^2_0(S_d)$. So, (ii) implies (i). □

About the proof of (i)⇒(ii) in Theorem 1.2. Let $F = U_\alpha^* f \in I_*(H^2)$. By (1), we have $(1-\overline{\alpha} I(z))^{-1} F(z) \in H^p$, $0 < p < 1$. By assumption, there exists $g \in H^2_0$ such that $F = I\overline{g}$. Put

$$G \stackrel{\text{def}}{=} \frac{g}{I-\alpha}.$$

Then $G \in H^p_0$ for sufficiently small $p > 0$. Since I is inner, we have

$$(1-\overline{\alpha} I(\zeta))^{-1} F(\zeta) = \overline{G(\zeta)} \quad \text{for } \Sigma_d\text{-a.e. } \zeta \in S_d.$$

Applying the Clark–Poltoratski theory in the unit disk and integrating by slices, we conclude that

$$\int_{S_d} \left| \frac{F(r\zeta)}{1-\overline{\alpha} I(r\zeta)} - \overline{G(r\zeta)} \right| \mathrm{d}\Sigma_d(\zeta) \le C < \infty$$

for all $0 < r < 1$. Hence, there exists a measure $\nu \in PM(S_d)$, $\nu \perp \Sigma_d$, such that

$$P[\nu] = (1-\overline{\alpha} I)^{-1} F - \overline{G}. \tag{2}$$

Now, using (1) and (2), observe that $\overline{f}\sigma_\alpha - \overline{\nu} \in A(B_d)^\perp$, thus $\overline{f}\sigma_\alpha - \overline{\nu}$ is a Henkin measure. Hence, by the Cole–Range theorem,

$$f\sigma_\alpha - \nu \ll \rho \tag{3}$$

for some representing measure ρ. By Corollary 2.6, $f\sigma_\alpha$ is totally singular; by Theorem 2.5, ν is also totally singular because ν is a singular pluriharmonic measure. So, $f\sigma_\alpha - \nu$ is a totally singular measure and (3) holds. Therefore, $f\sigma_\alpha = \nu \in PM(S_d)$; in particular, $f\sigma_\alpha$ is a pluriharmonic measure, as required. □

References

[1] D.N. Clark, One dimensional perturbations of restricted shifts, J. Anal. Math. 25 (1972) 169–191.
[2] B. Cole, R.M. Range, A-measures on complex manifolds and some applications, J. Funct. Anal. 11 (1972) 393–400.
[3] E.S. Dubtsov, Singular parts of pluriharmonic measures, Zap. Nauč. Semin. POMI 217 (1994), no. Issled. po Lineĭn. Oper. i Teor. Funktsiĭ. 22, 54–58, 219 (in Russian); translation in J. Math. Sci. (N.Y.) 85 (2) (1997) 1790–1793.
[4] M.T. Jury, Clark theory in the Drury–Arveson space, J. Funct. Anal. 266 (6) (2014) 3855–3893.
[5] A. Matheson, M. Stessin, Applications of spectral measures, in: Recent Advances in Operator-Related Function Theory, in: Contemp. Math., vol. 393, American Mathematical Society, Providence, RI, USA, 2006, pp. 15–27.
[6] A. Poltoratski, D. Sarason, Aleksandrov–Clark measures, in: Recent Advances in Operator-Related Function Theory, in: Contemp. Math., vol. 393, American Mathematical Society, Providence, RI, USA, 2006, pp. 1–14.
[7] W. Rudin, Function Theory in the Unit Ball of $\mathbf{C}^n$, Grundlehren der Mathematischen Wissenschaften, vol. 241, Springer-Verlag, New York, Berlin, 1980.
[8] E. Saksman, An elementary introduction to Clark measures, in: Topics in Complex Analysis and Operator Theory, Univ. Málaga, Málaga, Spain, 2007, pp. 85–136.
[9] B.V. Shabat, Introduction to complex analysis. Part II, in: Functions of Several Variables, in: Translations of Mathematical Monographs, vol. 110, American Mathematical Society, Providence, RI, USA, 1992, translated from the third (1985) Russian edition by J.S. Joel.