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Superconductivity and the Aharonov-Bohm effect

Supraconductivité et effet Aharonov-Bohm

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ABSTRACT

We consider the influence of the Aharonov–Bohm magnetic potential on the onset of superconductivity within the Ginzburg–Landau model. As the flux of the magnetic potential varies, we obtain a relation with the Little–Parks effect.

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RÉSUMÉ

Nous considérons l'influence du potentiel magnétique d'Aharonov–Bohm sur le démarrage de la supraconductivité dans le modèle de Ginzburg–Landau. Lorsque le flux du potentiel magnétique varie, nous obtenons une relation avec l'effet Little–Parks.

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1. Introduction

We are interested in the analysis of the Ginzburg-Landau functional,

$$\mathcal{E}[\psi,\mathfrak{A}] = \int_{\Omega} \left(|(\nabla - i\mathfrak{A})\psi|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \right) dx + \int_{\Omega} |\operatorname{curl}(\mathfrak{A} - \mathbf{F})|^2 dx,$$
(1.1)

where $\Omega = \{x \in \mathbb{R}^2 : |x| < R\}$ is a disc of radius R and $\kappa \in (0, +\infty)$ is a characteristic parameter of the material of the sample occupying Ω ; κ depends on the temperature in the following manner, $\kappa \approx T_c - T$, where T_c is the critical temperature of the sample. That κ is of positive sign signifies that the sample is cooled down below its critical temperature. Here we use the notation curl $\mathbf{a} = \partial_{x_1} a_2 - \partial_{x_2} a_1$ for $\mathbf{a} = (a_1, a_2)$. The energy, $\mathcal{E}[\psi, \mathfrak{A}]$, is defined for $(\psi, \mathfrak{A}) \in \mathcal{H}_F = H_F^1(\Omega; \mathbb{C}) \times (H^1(\Omega; \mathbb{R}^2) + \mathbf{F})$, where $H_F^1(\Omega; \mathbb{C}) = \{u \in L^2(\Omega; \mathbb{C})\} : (\nabla - \mathbf{i}F)u \in L^2(\Omega)\}$ is the magnetic Sobolev space. As a consequence of the diamagnetic inequality, for $\psi \in H_F^1(\Omega; \mathbb{C})$, $|\psi|$ is in $H^1(\Omega; \mathbb{C}) \hookrightarrow L^4(\Omega)$. Note that, if $\mathbf{F} \in H^1(\Omega; \mathbb{R}^2)$, the space \mathcal{H}_F becomes the usual

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variational space, $H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$. However, we are going to inspect the functional for $\mathbf{F} \notin H^1(\Omega; \mathbb{R}^2)$. Namely, we assume that $\mathbf{F} = h\mathbf{F}_{AB}$, where h > 0 and \mathbf{F}_{AB} is defined as follows:

$$\mathbf{F}_{AB}(x) = \left(\frac{-x_2}{2|x|^2}, \frac{x_1}{2|x|^2}\right) \qquad \left(x = (x_1, x_2) \in \mathbb{R}^2\right).$$
(1.2)

We scale the Ginzburg–Landau functional in (1.1) properly by writing $\mathfrak{A} = h\mathbf{A}$. Hence,

$$\mathcal{E}(\psi, \mathbf{A}) = \int_{\Omega} \left(|(\nabla - ih\mathbf{A})\psi|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \right) dx + h^2 \int_{\Omega} |\operatorname{curl} (\mathbf{A} - \mathbf{F}_{AB})|^2 dx$$
(1.3)

is defined on the natural variational space

$$\mathcal{H} = H^{1}_{\mathbf{F}_{AB}}(\Omega; \mathbb{C}) \times (H^{1}(\Omega; \mathbb{R}^{2}) + \mathbf{F}_{AB}).$$
(1.4)

A critical point $(\psi, \mathbf{A})_{\kappa,h}$ of the functional is a weak solution to the corresponding Euler–Lagrange equations (named Ginzburg–Landau equations in this context):

$$\begin{cases} -(\nabla - ih\mathbf{A})^{2}\psi = \kappa^{2}(1 - |\psi|^{2})\psi & \text{in }\Omega, \\ -\nabla^{\perp}(\operatorname{curl}(\mathbf{A} - \mathbf{F}_{AB})) = \frac{1}{h}\operatorname{Im}(\overline{\psi}(\nabla - ih\mathbf{A})\psi) & \text{in }\Omega, \\ \nu \cdot (\nabla - ih\mathbf{A})\psi = 0 & \text{on }\partial\Omega, \\ \operatorname{curl}(\mathbf{A} - \mathbf{F}_{AB}) = 0 & \text{on }\partial\Omega, \end{cases}$$
(1.5)

where ν is the outward unit normal vector on $\partial\Omega$, and the operator $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$ is the Hodge gradient. Note that the boundary condition in (1.5) actually reads curl **A** = 0 on $\partial\Omega$, because on $\partial\Omega$ curl **F**_{AB} vanishes.

A critical point $(\psi, \mathbf{A})_{\kappa,h}$ is said to be trivial if $\psi = 0$; it is said to be a minimizer if it minimizes the functional in (1.3) in the variational space \mathcal{H} (see (1.4)). Our main result, Theorem 1.1 below, involves a spectral constant $\lambda_{AB}(1) > 0$ introduced in (2.1) below.

Theorem 1.1.

A. There exists a constant $c_* \in (0, 1)$ such that the following is true.

- i. If $n \in \mathbb{N}$ is odd and $0 < \kappa < \sqrt{\lambda_{AB}(1)}$, then every minimizer $(\psi, \mathbf{A})_{\kappa,h=n}$ of the functional in (1.3) satisfies $\psi \neq 0$.
- ii. If $n \in \mathbb{N}$ is even and $0 < \kappa < c_* \sqrt{\lambda_{AB}(1)}$, then every critical point $(\psi, \mathbf{A})_{\kappa,h=n}$ of the functional in (1.3) satisfies $\psi \equiv 0$.
- B. Given any $\kappa > \sqrt{\lambda_{AB}(1)}$ and h > 0, any minimizer $(\psi, \mathbf{A})_{\kappa,h}$ of the functional in (1.3) satisfies $\psi \neq 0$.

Remark 1.2. The parameter *h* is chosen in this paper so that h/2 is the flux of the applied magnetic potential \mathbf{F}_{AB} . Theorem 1.1 then exhibits a regime where the flux destroys the superconducting properties. This is consistent with the results in [5].

Remark 1.3. Theorem 1.1 is consistent with the Little–Parks experiment [10] and displays the analogy between the Aharonov–Bohm magnetic potential and non-simply connected domains. Also, Theorem 1.1 displays a situation where the breakdown of superconductivity does not occur under high magnetic fields, in contrast to [6].

Remark 1.4. The constant c_* is explicitly constructed, modulo various Sobolev inequalities, but most probably, it is not the optimal one.

Remark 1.5. The proof of Theorem 1.1 relies on the periodicity of the principal eigenvalue of the Aharonov–Bohm Hamiltonian (see Proposition 2.1 below). Such periodicity results are quite common in the Aharonov–Bohm setting, for example in domains with holes [7], annuli and annulus-like domains with Dirichlet condition [8]. The proof of Theorem 1.1 carries over in these situations as well, with due modifications, including the formulation of the GL functional in non-simply connected domains, and the replacement of the flux condition by circulation conditions for the Aharonov–Bohm potential around the holes of the domain.

2. The eigenvalue problem

A key element to prove Theorem 1.1 is a remarkable observation regarding the principal eigenvalue, $\lambda_{AB}(h)$, of the magnetic Laplacian $-(\nabla - ih\mathbf{F}_{AB})^2$ with Neumann boundary condition, in $L^2(\Omega; \mathbb{C})$, defined via the Friedrichs extension theorem [2], with form domain being the magnetic Sobolev space $H^1_{h\mathbf{F}}(\Omega; \mathbb{C})$ (see [9, Prop. 2.1]). By the min-max principle,

$$\lambda_{AB}(h) = \inf_{\substack{u \in H_{hF}^1(\Omega; \mathbb{C}) \\ \|u\|_{L^2(B_R)} = 1}} \int_{\Omega} |(\nabla - ih\mathbf{F}_{AB})u|^2 \, \mathrm{d}x \,.$$

$$(2.1)$$

Proposition 2.1. The function $h \mapsto \lambda_{AB}(h)$ is periodic, with period 2, achieves its minimum at h = 0, and its maximum at h = 1; in fact, $\lambda_{AB}(0) = 0$, $\lambda_{AB}(1) \ge \frac{1}{4R^2}$, and $\lambda_{AB}(h) = 0$ if and only if h is an odd integer.

Remark 2.2. Proposition 2.1 shows a strong analogy with non-simply connected domains [7,8]. Furthermore, it is an example where *strong* diamagnetism fails (see [3,5]) and is in fact related to the Little–Parks effect (see [5]), as displayed in the main result, Theorem 1.1.

Proof of Proposition 2.1. Consider the quadratic form $q_h(u) = \int_{\Omega} |(\nabla - ih \mathbf{F}_{AB})u|^2 dx$. Using the polar coordinates (r, θ) , we may express the quadratic form q_h and the L^2 -norm in Ω as follows:

$$q_h(u) = \int_0^R \int_0^{2\pi} r\left(|\partial_r u|^2 + \frac{1}{4r^2} |(2\partial_\theta - ih)u|^2 \right) d\theta dr \quad \text{and} \quad ||u||^2 = \int_0^R \int_0^{2\pi} r|u|^2 d\theta dr.$$
(2.2)

Performing the Fourier decomposition of u w.r.t. the θ -variable, $u = \sum_{n \in \mathbb{Z}} u_n(r) e^{-in\theta}$, we get

$$\|u\|^{2} = 2\pi \sum_{n \in \mathbb{Z}} \int_{0}^{R} |u_{n}(r)|^{2} r \, dr \quad \text{and} \quad q_{h}(u) = 2\pi \sum_{n \in \mathbb{Z}} \int_{0}^{R} \left(|\partial_{r} u_{n}|^{2} + \frac{1}{4r^{2}} |(2n-h)u_{n}|^{2} \right) r \, dr.$$

The operator $-\nabla_{hF_{AB}}^2$ is actually the direct sum of the fiber operators $\mathcal{L}_n = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{r^2}(n-\frac{h}{2})^2$ in the weighted space $L^2((0, R), r \, dr)$. The spectral theorem then yields that

$$\lambda_{\rm AB}(h) = \inf_{n \in \mathbb{Z}} \mu_1(h, \mathcal{L}_n) \,, \tag{2.3}$$

where $\mu_1(h, \mathcal{L}_n) = \inf \sigma(\mathcal{L}_n)$. The function $(h) = \inf_{n \in \mathbb{Z}} |n - \frac{h}{2}|$ is periodic in h, of period 2, attains its minimum at h = 0 and its maximum at h = 1; furthermore, (0) = 0 and $(1) = \frac{1}{2}$. By the min–max principle and (2.3), we get

$$\lambda_{AB}(h) = \inf\{\int_{0}^{R} \left(|\partial_{r}v|^{2} + \frac{(h)^{2}}{r^{2}} |v|^{2} \right) r \, \mathrm{d}r : \int_{0}^{R} |v|^{2} r \, \mathrm{d}r = 1 \}$$

hence a periodic function. Clearly, $\lambda_{AB}(0) = 0$; for h = 1, the lower bound $\lambda_{AB}(1) \ge \frac{1}{4R^2}$ follows from the min-max principle (and the inequality $\frac{1}{r^2} \ge \frac{1}{R^2}$). \Box

3. Proof of Theorem 1.1

Lemma 3.1. Every solution $(\psi, \mathbf{A})_{\kappa,h} \in \mathcal{H}$ to (1.5) satisfies, for all $\kappa, h > 0$,

(1) $\|(\nabla - ih\mathbf{A})\psi\|_{L^{2}(\Omega)} \le \kappa \|\psi\|_{L^{2}(\Omega)};$ (2) $\|\psi\|_{L^{\infty}(\Omega)} \le 1.$

Proof. Item (1) follows from the identity $\frac{d}{dt} \mathcal{E}(\psi + t\psi, \mathbf{A})\Big|_{t=0} = 0$, which yields

$$\mathcal{E}_{0}(\psi, \mathbf{A}) := \int_{\Omega} \left(|(\nabla - ih\mathbf{A})\psi|^{2} - \kappa^{2}|\psi|^{2} + \frac{\kappa^{2}}{2}|\psi|^{4} \right) dx = -\frac{\kappa^{2}}{2} \int_{\Omega} |\psi|^{4} dx \le 0.$$
(3.1)

For Item (2), we use the identity, $\operatorname{Re} \int_{\Omega} \left((\nabla - ih\mathbf{A})\psi \cdot \overline{(\nabla - ih\mathbf{A})\tilde{\psi}} + (|\psi|^2 - 1)\psi \overline{\tilde{\psi}} \right) dx = 0$ for $\tilde{\psi} = [|\psi| - 1]_+ \frac{\psi}{|\psi|}$. That $\tilde{\psi} \in H^1_{\mathsf{F}_{AB}}(\Omega; \mathbb{C})$ follows from $\psi \in H^1_{\mathsf{F}_{AB}}(\Omega; \mathbb{C})$ and the diamagnetic inequality, which yields $|\psi| \in H^1(\Omega)$. The rest of the proof is as [4, Prop. 10.3.1]. \Box

Remark 3.2. Performing a gauge transformation, we may restrict the analysis to the solutions to (1.5) that live in the space

$$\mathcal{H}_{0} = H^{1}_{\mathbf{F}_{AB}}(\Omega; \mathbb{C}) \times \left(\mathbf{F}_{AB} + \mathfrak{h}(\Omega)\right)$$

where $\mathfrak{h}(\Omega) = \{ \mathbf{u} \in H^1(\Omega; \mathbb{R}^2) : \text{ div } \mathbf{u} = 0 \text{ in } \Omega \& \nu \cdot \mathbf{u} = 0 \text{ on } \partial \Omega \}.$

Lemma 3.3. Let $\alpha \in (0, 1)$. There exists C > 0 such that, every solution $(\psi, \mathbf{A})_{\kappa,h} \in \mathcal{H}_0$ to (1.5) satisfies $\mathbf{A} - \mathbf{F}_{AB} \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^2)$ and $\|\mathbf{A} - \mathbf{F}_{AB}\|_{C^{0,\alpha}(\overline{\Omega})} \leq \frac{C\kappa}{h}$.

Proof. Let $\mathbf{u} = \mathbf{A} - \mathbf{F}_{AB}$. Since $(\psi, \mathbf{A}) \in \mathcal{H}_0$ is a solution to (1.5), so div $\mathbf{u} = 0$ and curl $\mathbf{u} \in H_0^1(\Omega)$. Hence (see [1, Lem. B.1]), $\mathbf{u} \in H^2(\Omega; \mathbb{R}^2)$ and

$$\|\mathbf{u}\|_{H^{2}(\Omega)} \leq C_{1} \|\operatorname{curl} \mathbf{u}\|_{H^{1}(\Omega)} \leq \frac{C_{1}}{h} \|\operatorname{Im}(\overline{\psi}(\nabla - ih\mathbf{A})\psi)\|_{L^{2}(\Omega)}$$

Using Lemma 3.1, we obtain that $\mathbf{A} - \mathbf{F}_{AB} \in H^2(\Omega; \mathbb{R}^2)$ and $\|\mathbf{A} - \mathbf{F}_{AB}\|_{H^2(\Omega)} \leq \frac{C_1 \kappa}{h}$. The Sobolev embedding theorem yields the estimate in $C^{0,\alpha}(\overline{\Omega}; \mathbb{R}^2)$ -norm. \Box

Proof of Theorem 1.1. 1. Assume that $\kappa > \sqrt{\lambda_{AB}(h)}$ and h > 0. Every minimizer $(\psi, \mathbf{A})_{\kappa,h}$ satisfies $\mathcal{E}(\psi, \mathbf{A}) \le \mathcal{E}(tu_h, \mathbf{F}_{AB})$ for any t > 0 and u_h a normalized ground state of the eigenvalue $\lambda_{AB}(h)$. Now

$$\mathcal{E}(tu_h, \mathbf{F}_{AB}) = t^2 \int_{\Omega} \left(\lambda_{AB}(h) - \kappa^2 + \frac{t^2 \kappa^2}{2} |u_h|^4 \right) dx$$

can be made negative when t is sufficiently small. Hence, every minimizer is non-trivial.

2. Assume that $0 < \kappa^2 < \lambda_{AB}(1)$. For even $n \in \mathbb{N}$, $\lambda_{AB}(n) = \lambda_{AB}(2) = 0$, and by Step 1, every minimizer $(\psi, \mathbf{A})_{\kappa, h=n}$ is non-trivial, that is $\psi \neq 0$.

3. For $\delta \in (0, 1)$, let $c(\delta) = \sqrt{\frac{1-\delta}{1+C^2\delta^{-1}}}$, where *C* is the constant from Lemma 3.3. This function is maximized for $\delta = \delta_*$ where $\delta_* = (1 + \sqrt{1+C^{-2}})^{-1/2}$. We set $c_* = c(\delta_*)$ and notice that $\delta_*, c_* \in (0, 1)$. Now, assume that $n \in \mathbb{N}$ is odd, $0 < \kappa^2 < c_*\lambda_{AB}(1)$ and $(\psi, \mathbf{A})_{\kappa,h=n}$ is a critical point of the functional in (1.5). By (3.1) and Cauchy's inequality,

$$0 \geq \mathcal{E}_0(\psi, \mathbf{A}) \geq (1 - \delta_*) \int\limits_{\Omega} |(\nabla - in\mathbf{F}_{AB})\psi|^2 \, dx - \delta_*^{-1} n^2 \int\limits_{\Omega} |\mathbf{A} - \mathbf{F}_{AB}|^2 |\psi|^2 \, dx - \int\limits_{\Omega} \kappa^2 |\psi|^2 \, dx \, .$$

Lemma 3.3 and the min-max principle now yield that

$$0 \ge \left((1 - \delta_*) \lambda_{AB}(n) - \kappa^2 - \delta_*^{-1} C^2 \kappa^2 \right) \int_{\Omega} |\psi|^2 \, \mathrm{d}x \, .$$

Our choice of δ_* , c_* and κ guarantees that $(1 - \delta_*)\lambda_{AB}(1) - \kappa^2 - \delta_*^{-1}C^2\kappa^2 < 0$. Since $n \in \mathbb{N}$ is odd, $\lambda_{AB}(n) = \lambda_{AB}(1)$ and we get then that $\int_{\Omega} |\psi|^2 dx = 0$. \Box

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