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Stabilization of the wave equations with localized Kelvin–Voigt type damping under optimal geometric conditions

Stabilisation d'une équation des ondes avec un amortissement de type Kelvin–Voigt localisé sous conditions géométriques optimales

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ABSTRACT

The purpose of this note is to investigate the stabilization of the wave equation with Kelvin–Voigt damping in a bounded domain. Damping is localized via a non-smooth coefficient in a suitable subdomain. We prove a polynomial stability result in any space dimension, provided that the damping region satisfies some geometric conditions. The main novelty of this note is that the geometric situations covered here are richer than that considered in [25], [22], [16] and include in particular an example where the damping region is not localized in a neighborhood of the whole or a part of the boundary.

RÉSUMÉ

Nous nous intéressons à l'étude de la stabilisation d'une équation des ondes avec un amortissement de type Kelvin–Voigt dans un domaine borné. L'amortissement est localisé via un coefficient singulier dans une partie du domaine. Nous montrons un résultat de stabilisation polynomiale en toute dimension d'espace dès que la région d'amortissement satisfait certaines conditions géométriques. La principale nouveauté de cette note est que les situations géométriques couvertes ici sont plus riches que celles considérées dans [25], [22], [16] et incluent notamment un exemple où la région d'amortissement n'est pas localisée dans un voisinage de la totalité ou d'une partie de la frontière.

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1. Introduction

Local viscoelastic damping is a natural phenomenon of bodies arising from a solid that have one part made of viscoelastic material, and the other made of elastic material. Let $\Omega \subset \mathbb{R}^N$ be a nonempty bounded open set with boundary Γ of class C^2. We consider the wave equation with locally distributed Kelvin–Voigt-type damping given in the following equation:

$$\begin{cases} \rho(x)u_{tt}(x,t) = \operatorname{div}(a(x)\nabla u + b(x)\nabla u_t) & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0 & \text{on } \Gamma \times \mathbb{R}^+, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where we assume that the coefficient functions $\rho, a, b \in L^\infty(\Omega)$ and $\rho(x) \geq \rho_0 > 0$, $a(x) \geq a_0 > 0$ and $b(x) \geq 0$ for all $x \in \Omega$.

In 1988, F. Huang proved that when the Kelvin–Voigt damping $\operatorname{div}(b(x)\nabla u_t)$ is globally distributed, *i.e.* $b(x) \geq b_0 > 0$ for almost every x in Ω, the corresponding semigroup of system (1) is not only exponentially stable, but also is analytic (see [8]). Thus, Kelvin–Voigt damping is stronger than the viscous damping $b(x)u_t$ in this case. Indeed, in [10], it was proved that the semigroup corresponding to the system of wave equations with global viscous damping is exponentially stable but not analytic. However, this result is still true if viscous damping is localized; via a smooth or a non-smooth damping coefficient, in a suitable subdomain satisfying the Geometric Control Condition (GCC in short) introduced by C. Bardos, G. Lebeau, and J. Rauch in [2] (see also [10]). Nevertheless, when viscoelastic damping is distributed locally, the situation is more delicate and such comparison between viscous and viscoelastic damping is not valid anymore. In fact, in 1998, K. Liu and Z. Liu considered a one-dimensional wave equation with Kelvin–Voigt damping distributed locally on any subinterval of the region occupied by the beam, where the damping coefficient is the characteristic function of the subinterval. They proved that the semigroup associated with the equation for the transversal motion of the beam is exponentially stable, although the semigroup associated with the equation for the longitudinal motion of the beam is not (see [13]). This shows that Kelvin–Voigt damping does not obey the GCC. This surprising result, due to the discontinuity of the materials and the unboundedness of viscoelastic damping, motivated the study of elastic systems with local Kelvin–Voigt damping. Later, in the one-dimensional case, it was found that the smoothness of the damping coefficient at the interface is a critical factor for the stability and the regularity of the solutions (see [7,14,15,17,18,23]). However, there are only a small number of publications on the corresponding N-dimensional case. In 2006, K. Liu and B. Rao considered this problem in the N-dimensional space where the damping region is a neighborhood (in Ω) of the entire boundary Γ (see [16]). They proved that the energy of the system goes exponentially to zero as t goes to infinity for all usual initial data by assuming that the damping coefficient b satisfies $b \in C^{1,1}(\overline{\Omega})$, $\Delta b \in L^\infty(\Omega)$ and $|\nabla b(x)|^2 \leq M_0 b(x)$ for almost every x in Ω, where M_0 is a positive constant. Also in [19], under the same assumption on b, S. Nicaise and C. Pignotti established the exponential stability of the wave equation with local Kelvin–Voigt damping localized around a part of the boundary and an extra boundary damping with time delay where they added an appropriate geometric condition (section 3.2 (Q4)). Later on, M. Cavalcanti, V. Cavalcanti and L. Tebou showed the exponential decay of the energy of a wave equation with two types of locally distributed mechanisms; a frictional damping and a Kelvin–Voigt-type damping where the location of each damping is such that none of them alone is able to exponentially stabilize the system (see [6]). Under an appropriate geometric condition (PMGC) on a subset ω of $\Omega \subset \mathbb{R}^N$ where the dissipation is effective, they proved that the energy of the system decays polynomially as type $\frac{1}{t}$ in the absence of regularity of the Kelvin–Voigt damping coefficient b. However, they established exponential stability when this coefficient is smooth. In [1], K. Ammari, F. Hassine, and L. Robbianio considered a wave equation with Kelvin–Voigt damping localized in a subdomain ω far away from the boundary without geometric conditions. They established a logarithmic energy decay rate for smooth initial data. On the other hand, in [22] L. Tebou studied the stabilization of the wave equation with Kelvin–Voigt damping. He established polynomial energy decay of type $\frac{1}{t}$ provided that the damping region is localized in a neighborhood of a part of the boundary and verifies the Piecewise Multiplier Geometric Condition (PMGC in short) introduced by K. Liu [12]. Moreover, Q. Zhang in [25] considered the wave equation with Kelvin–Voigt damping in a nonempty bounded convex domain Ω with partition $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ where the viscoelastic damping is localized in Ω_1. Under the condition that the damping coefficient b is non-smooth, she established a polynomial energy decay rate of type $\frac{1}{t}$ for smooth initial data in the following two cases: (1) the damping region Ω_1 is a neighborhood of the entire boundary Γ of Ω; (2) $\Omega \subset \mathbb{R}^N$ ($N = 2$ or 3), $\partial\Omega_1$ and $\partial\Omega_2$ are either convex curvilinear polygons or curved plane polyhedra, the damping region Ω_1 is a neighborhood of a part $\Gamma_1 \neq \emptyset$ of the boundary Γ and $m(x) \cdot \nu_2 \leq 0$ where $m(x) = x - x_0$ for x_0 fixed in $\mathbb{R}^N (N = 2, 3)$ for all $x \in \Gamma_2 = \Gamma \setminus \Gamma_1$. So, several important geometric situations are not covered by the previous papers (see for instance Fig. 1-c, Fig. 2, Fig. 3) and the problem of the energy decay rate is still open. So, our aim is to answer this open problem.

In this note, we consider the stabilization of the wave equation with Kelvin–Voigt damping in a bounded domain Ω of class C^2 with non-smooth damping coefficient. The system is given by (1). We establish a polynomial energy decay estimate of type $\frac{1}{t}$ for smooth initial data provided that the damping coefficient b satisfies the localization condition (LA) (see below) and the damping region ω satisfies one of the geometric conditions (A1) or (A2) (see below). The frequency domain approach and the piecewise multiplier method are used. To our knowledge, the result of Theorem 3.5 is new. Indeed, the geometric situations covered by this theorem are richer than that considered in [25], [22], [16] and include in particular an example of damping region far away from the boundary.

2. Well-posedness and strong stability

Let us define the energy space $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$ equipped with the following inner product

$$((u, v), (\tilde{u}, \tilde{v}))_{\mathcal{H}} = \int_\Omega (a\nabla u \cdot \nabla \tilde{u} + \rho v \tilde{v})\, dx.$$

Let (u, u_t) be a regular solution to the system (1). Its associated energy is defined by

$$E(t) = \int_\Omega (a|\nabla u|^2 + \rho |u_t|^2)\, dx. \tag{2}$$

A straightforward computation gives that

$$E'(t) = -\int_\Omega b(x)|\nabla u_t|^2\, dx \le 0. \tag{3}$$

Consequently, system (1) is dissipative in the sense that its energy is non-increasing with respect to t. Setting $U = (u, u_t)^\intercal$, system (1) may be recast as: $U' = AU$ in $(0, +\infty)$, $U(0) = (u_0, u_1)^\intercal$, where the unbounded operator $\mathcal{A} : D(\mathcal{A}) \longrightarrow \mathcal{H}$ is given by

$$D(\mathcal{A}) = \left\{(u, v) \in \mathcal{H} : \ v \in H_0^1(\Omega),\ \mathrm{div}(a\nabla u + b\nabla v) \in L^2(\Omega)\right\}, \quad A(u, v) = \left(v, \frac{1}{\rho}\mathrm{div}(a\nabla u + b\nabla v)\right).$$

Noting that due to the fact that $b(x) \ge 0$, the operator $\mathcal{A}$ is m-dissipative in $\mathcal{H}$ and generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$. So, system (1) is well-posed in $\mathcal{H}$ (see [16]).

In addition, if $\omega \neq \emptyset$ and b satisfies the following localization assumption

$$\exists b_0 > 0 : \ b(x) \ge b_0 \ \ \forall x \in \omega, \tag{LA}$$

then system (1) is strongly stable (see [1], Theorem 2.2) *i.e.*

$$\lim_{t\to+\infty} \|e^{t\mathcal{A}}(u_0, u_1)\| = 0, \quad \forall\, (u_0, u_1) \in \mathcal{H}.$$

So, our aim is to study the energy decay rate.

3. Polynomial energy decay rate

Q. Zhang proved in [24] that system (1) is not uniformly (exponentially) stable in any geometry. So, it is natural to hope for a polynomial stability under some considerations that represent the main goal of this note.

Before stating our results, we recall the Geometric Control Condition (GCC in short) introduced by J. Rauch and M. Taylor in [21] for manifolds without boundaries and by C. Bardos, G. Lebeau and J. Rauch in [2] (see also [10]) for domains with boundaries.

Definition 3.1. For a subset ω of Ω and $T > 0$, we shall say that (ω, T) satisfies the Geometric Control Condition if every geodesic traveling at speed one in Ω meets ω in time $t < T$.

We also introduce the following geometric condition:

Definition 3.2. For a subset ω of Ω, we shall say that ω satisfies Strictly the Geometric Control Condition (SGCC in short) if there exists an open subset $\tilde{\omega}$ included strictly in ω (*i.e.* $\overline{\tilde{\omega}} \subset \omega$) and satisfying the GCC.

For the study of the energy decay rate, we need the following geometric assumptions:

(A1) the open subset ω verifies the GCC and $\mathrm{meas}(\overline{\omega} \cap \Gamma) > 0$,
(A2) the open subset ω verifies the SGCC.

Remark 3.3. It is easy to see that, if ω verifies the SGCC, then it verifies the GCC. The converse of this implication is false (see Fig. 1-c).

There are several geometries that verify the previous assumptions. For example:

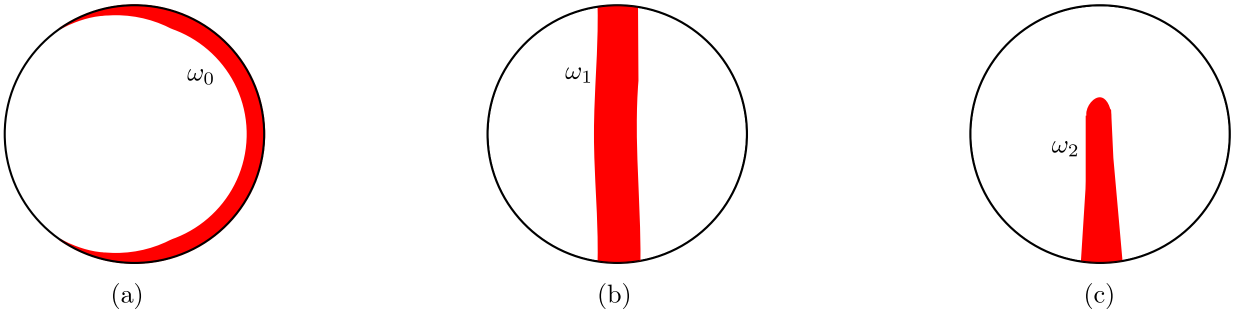

Fig. 1. Elastic–viscoelastic waves interaction model satisfying the assumption (A1).

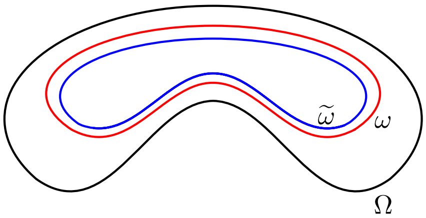

Fig. 2. A model satisfying assumption (A2).

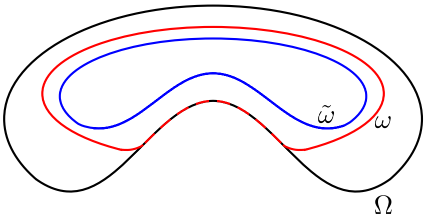

Fig. 3. A model satisfying both (A1) and (A2).

Remark 3.4. The PMGC introduced in [12] is a generalization of the Γ-condition introduced in [11] and is much more restrictive than the GCC. For example, in Fig. 1, we consider the case where Ω is a disk and we draw three different subsets in Ω. The Γ-condition is only satisfied by ω_0. The PMGC is satisfied by ω_0 and ω_1. However, ω_2 does not satisfy either the PMGC or the Γ-condition. Finally, the GCC is satisfied by the three different subsets of Ω.

Now, we are in position to state our main result.

Theorem 3.5 *(Polynomial decay rate). Assume that condition* (LA) *holds. Assume also that assumption* $(A1)$ *or assumption* $(A2)$ *holds. Then, for all initial data* $U_0 \in D(\mathcal{A})$*, there exists a constant* $C > 0$ *independent of* U_0 *such that the energy of system (1) satisfies the following estimation*

$$E(t,U) \leq C\,\frac{1}{t}\,||U_0||^2_{D(\mathcal{A})}, \quad \forall t > 0. \tag{4}$$

Remark 3.6. i) The result of Theorem 3.5 generalizes that of [16], [22], and [25]. Indeed, the geometric situations covered by this theorem are richer than that considered in the previous references. In addition, unlike the result of Theorem 4.1 in [25], our result holds for all $N \geq 2$ and for non-convex domains.

ii) It is unknown whether the polynomial decay rate obtained in (4) is optimal in the sense that, for any $\varepsilon > 0$, we can not expect the decay rate of type $\frac{1}{t^{1+\varepsilon}}$ for all initial data $U_0 \in D(\mathcal{A})$. From our point of view, the energy decay rate (4) is not optimal, and we conjecture an optimal decay of type $\frac{1}{t^2}$.

Ideas for proving Theorem 3.5 under the assumption (A1). For the proof of Theorem 3.5, we use a frequency-domain approach; namely, we use Theorem 2.4 of [5] (see also [3,4,17]) that we partially recall.

Theorem 3.7. *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a Hilbert space H with generator A such that $\mathrm{i}\mathbb{R} \subset \rho(A)$. Then for a fixed $\ell > 0$ the following conditions are equivalent*

$$\|(\mathrm{i}s - A)^{-1}\| = O(|s|^{\ell}),\ s \to \infty, \tag{5}$$

$$\|T(t)A^{-1}\| = O(t^{-1/\ell}),\ t \to \infty. \tag{6}$$

Since the resolvent of the operator $\mathcal{A}$ is not compact in the energy space $\mathcal{H}$ (see [16]) and $0 \in \rho(\mathcal{A})$, then to prove $\mathrm{i}\mathbb{R} \subset \rho(\mathcal{A})$ is equivalent to prove that $(\mathrm{i}\beta I - \mathcal{A})$ is bijective in the energy space $\mathcal{H}$ for all $\beta \in \mathbb{R}^\star$. This last is proven in [1] according to a unique continuation theorem and Fredholm's alternative. Then, the proof of Theorem 3.5 is reduced to show that condition (5) holds with $\ell = 2$. This is checked by using a contradiction argument. Indeed, assume that it does not hold, then there exist a sequence $\beta_n \in \mathbb{R}$ and a sequence $(u_n, v_n) \in D(A)$ such that

$$|\beta_n| \to +\infty, \quad ||(u_n, v_n)||_{\mathcal{H}} = 1, \tag{7}$$

$$\beta_n^2(\mathrm{i}\beta_n I - \mathcal{A})(u_n, v_n) = (f_n, g_n) \to 0 \text{ in } \mathcal{H}. \tag{8}$$

Our aim is to show that $||(u_n, v_n)||_H \to 0$. This condition permits to conclude a contradiction with (7). The proof is divided into several steps.

Step 1. (Local asymptotic estimation of $\beta_n v_n$). First, using (7) and (8), we have the following estimation

$$\int_\Omega |\beta_n u_n|^2 \,\mathrm{d}x = O(1). \tag{9}$$

Multiplying (8) by $U_n = (u_n, v_n)$ in $\mathcal{H}$, we get

$$\mathrm{Re}(\beta_n^2(\mathrm{i}\beta_n I - \mathcal{A})U_n, U_n)_{\mathcal{H}} = -\int_\Omega b|\beta_n \nabla v_n|^2 = o(1). \tag{10}$$

It follows from the localization assumption (LA) that

$$\int_\omega |\beta_n \nabla v_n|^2 \,\mathrm{d}x = o(1). \tag{11}$$

Since assumption (A1) holds, then using Poincaré's inequality and equation (11), we obtain

$$\int_\omega |\beta_n v_n|^2 \,\mathrm{d}x = o(1). \tag{12}$$

The aim of step 1 is achieved.

Now, writing equation (8) in a detailed form:

$$\mathrm{i}\beta_n u_n - v_n = \frac{f_n}{\beta_n^2} \to 0 \quad \text{in } H_0^1(\Omega), \tag{13}$$

$$\mathrm{i}\beta_n v_n - \frac{1}{\rho}\,\mathrm{div}(a\nabla u_n + b\nabla v_n) = \frac{g_n}{\beta_n^2} \to 0 \quad \text{in } L^2(\Omega). \tag{14}$$

Step 2. (Local asymptotic estimation of $\beta_n u_n$). Multiplying equation (13) by $\mathrm{i}\beta_n \overline{u}_n$, integrating over ω and using estimation (12), we get

$$\int_\omega |\beta_n u_n|^2 = o(1). \tag{15}$$

Step 3. (The multiplier φ_n). Now, for all $\beta_n \in \mathbb{R}$, let $\varphi_n \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution to the following system

$$\begin{cases} \rho\beta_n^2\varphi_n + \mathrm{div}(a\nabla\varphi_n) - \mathrm{i}\beta_n(\mathbb{1}_\omega b)(x)\,\varphi_n = u_n, & \text{in } \Omega,\\ \varphi_n = 0, & \text{on } \Gamma \end{cases} \tag{16}$$

where (u_n, v_n) is solution to (13)–(14). Since ω satisfies the GCC, then the wave equation with local viscous damping $(\mathbb{1}_\omega b)(x)\,\varphi_t$ is exponentially stable (see [10]) and, following Huang [9] and Pruss [20], the resolvent of its associated operator $\mathcal{A}_{\mathrm{aux}} : D(\mathcal{A}_{\mathrm{aux}}) \longrightarrow H_0^1(\Omega) \times L^2(\Omega)$ defined by:

$$D(\mathcal{A}_{\text{aux}}) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega), \quad \mathcal{A}_{\text{aux}}(\varphi, \psi) = \left(\psi, \frac{1}{\rho}(\operatorname{div}(a\nabla\varphi) - (\mathbb{1}_\omega b)(x)\psi)\right)$$

is uniformly bounded on the imaginary axis. This implies that there exists $M > 0$ independent of n such that

$$||\beta_n \varphi_n||_{L^2(\Omega)} + ||\nabla\varphi_n||_{L^2(\Omega)} \leq M||u_n||_{L^2(\Omega)}. \tag{17}$$

Step 4. (Global asymptotic estimations). Multiplying equations (13) and (14) by $\mathrm{i}\,\beta_n^3 \rho \overline{\varphi}_n$ and $\beta_n^2 \rho \overline{\varphi}_n$ respectively, applying Green's formula and adding the resulting equations, we get

$$-\int_\Omega |\beta_n u_n|^2 \,\mathrm{d}x + \int_\omega \mathrm{i}\,\beta_n^3 b \overline{\varphi}_n u_n \,\mathrm{d}x = o(1). \tag{18}$$

It follows, from (15) and (17), that

$$\int_\Omega |\beta_n u_n|^2 \,\mathrm{d}x = o(1). \tag{19}$$

This implies, by using the multiplier $\overline{u}_n$ that

$$\int_\Omega |\nabla u_n|^2 \,\mathrm{d}x = o(1). \tag{20}$$

Proof of Theorem 3.5. Adding estimations (19) and (20), we deduce that $||U_n||_{\mathcal{H}} = o(1)$, which gives the desired contradiction. The demonstration is thus achieved. □

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