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Eigenvalue and gap estimates of isometric immersions for the Dirichlet-to-Neumann operator acting on p-forms

Estimations de valeurs propres et du gap de l'opérateur de Dirichlet-à-Neumann agissant sur les p-formes pour les immersions isométriques

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A B S T R A C T

In this paper, we study the first eigenvalue of the Dirichlet-to-Neumann operator acting on differential forms of a Riemannian manifold with boundary isometrically immersed in some Euclidean space. We give a lower bound of the integral energy of p-forms in terms of its first eigenvalue associated with $(p-1)$-forms. We also find a lower bound for the gap between two consecutive first eigenvalues in terms of the curvature of the boundary.

R É S U M É

Dans cet article, nous étudions la première valeur propre de l'opérateur de Dirichlet-à-Neumann agissant sur les formes différentielles d'une variété riemannienne à bord plongée isométriquement dans un espace euclidien. Nous obtenons une borne inférieure de l'énergie des p-formes en termes de sa première valeur propre associée aux $(p-1)$-formes. Nous trouvons aussi une borne inférieure pour l'écart entre deux premières valeurs propres consécutives par rapport à la courbure de la frontière.

1. Introduction

Let (M,g) be an $(n+1)$-dimensional compact oriented Riemmanian manifold with smooth boundary ∂M isometrically immersed in some Euclidean space $\mathbb{R}^d$.

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In [4], P. Guerini and A. Savo studied the first eigenvalue of the Hodge Laplacian acting on p-differential forms of a manifold with boundary, as well as the gap for consecutive values of the degree p. For all $p = 0, \ldots, n+1$, the Laplacian Δ_p (often denoted Δ when no confusion is possible) is defined on the space of p-forms of M, $\Lambda^p(M)$, by:

$$\Delta_p \omega := (\mathrm{d}\delta + \delta \mathrm{d})\omega, \ \forall \omega \in \Lambda^p(M),$$

where d is the exterior derivative and δ the co-differential. They considered the absolute boundary condition problem:

$$\begin{cases} \Delta\omega = \mu\omega \\ J^* i_{\tilde{\nu}}\omega = J^* i_{\tilde{\nu}} \mathrm{d}\omega = 0, \end{cases}$$

where $J : \partial M \hookrightarrow M$ is the natural inclusion and $\tilde{\nu}$ is the inward unit normal vector field at each point of ∂M. Among other things, they obtained (see [4]) a general lower bound for the integral energy of a co-closed p-form on M and a lower bound for the gap $\mu''_{1,p} - \mu'_{1,p}$ in terms of the curvature term W_p of Bochner formula and the shape operator of the immersion T_p (see section 2). Here, $\mu''_{1,p}$ (resp. $\mu'_{1,p}$) is the first positive eigenvalue of the absolute problem restricted to co-closed (resp. closed) p-forms. In this paper, we apply the method developed in [4] to the Dirichlet-to-Neumann operator defined by Raulot and Savo in [6].

2. Definitions and basic facts

First, let us recall some facts about the Dirichlet-to-Neumann operator.

Originally, the Dirichlet-to-Neumann operator T, also called Steklov operator, acts on smooth functions on ∂M. It is defined, for all $f \in C^\infty(\partial M)$, by:

$$Tf := -\frac{\partial \hat{f}}{\partial \tilde{\nu}}$$

where $\hat{f}$ is the harmonic extension of f on M. The Steklov operator has been widely studied, initially because of its applications. In fact, if we consider a steady-state distribution of temperature in a body for given temperature values on the body's surface, then the resulting heat flux is a Steklov operator. It is also used to solve inverse boundary problems such as electrical impedance tomography problems. Especially, its first eigenvalues have already been estimated, for example, in [1] and [2].

We extend this operator on functions to an operator $T^{[p]}$ acting on the bundle of p-forms, $\Lambda^p(\partial M)$ for $0 \le p \le n$. The following definition is the one developed in [6].

For ω a p-form on ∂M, there exists a unique p-form (see [8]) $\hat{\omega}$ on M such that:

$$\begin{cases} \Delta\hat{\omega} = 0 \text{ on } M, \\ J^*\hat{\omega} = \omega, \ i_{\tilde{\nu}}\hat{\omega} = 0. \end{cases} \tag{1}$$

If we let:

$$T^{[p]}\omega = -i_{\tilde{\nu}} \mathrm{d}\hat{\omega}, \tag{2}$$

then $T^{[p]} : \Lambda^p(\partial M) \to \Lambda^p(\partial M)$ defines a pseudo-differential linear operator which is elliptic, self-adjoint and positive. It possesses a discrete spectrum denoted by:

$$0 \le \nu_{1,p}(M) \le \nu_{2,p}(M) \le \ldots \nearrow \infty.$$

Moreover, one has the variational characterization for the eigenvalues:

$$\nu_{1,p}(M) = \inf\left\{ \frac{\int_M \|\mathrm{d}\omega\|^2 + \|\delta\omega\|^2}{\int_{\partial M} \|\omega\|^2} \mid \omega \in \Lambda^p(M), \ i_{\tilde{\nu}}\omega = 0 \text{ on } \partial M \right\}. \tag{3}$$

We note that $\nu_{1,p}$ could be zero and its multiplicity is given by the p-th Betti number (see [6]). In particular, one has $\nu_{1,0} = 0$, with multiplicity one and associated eigenfunction given by the constants. Thus, the Dirichlet-to-Neumann operator is closely related to the shape of the boundary and encodes some of its curvature properties.

The main result in this paper is Theorem 4, which extends the previously mentioned estimates to arbitrary compact manifolds with boundary immersed in $\mathbb{R}^n$, provided that a suitable curvature condition holds. Namely, let M be a $(n+1)$ dimensional compact oriented Riemannian manifold with smooth boundary ∂M, and fix $x \in \partial M$ and $p = 0, \ldots, n$. We denote:

$$\sigma_p(x) := \text{ sum of the } p \text{ smallest principal curvatures of } \partial M,$$

$$\sigma_p := \inf_{x \in \partial M} \sigma_p(x).$$

Then, one has:

$$\nu_{1,p}(M) \geq \nu_{1,p-1}(M) + \frac{\sigma_p}{p},$$

provided that $W_p \geq T_p$, where W_p is the curvature term in the Bochner–Weitzenbock formula (see Proposition 1) and T_p is a symmetric endomorphism acting on p-forms and defined in (6) below.

This estimate depends itself on the estimate of Theorem 3, which is of its own independent interest and extends to the Dirichlet-to-Neumann case a corresponding result obtained for the Hodge Laplacian in [4].

3. Gap of isometric immersions

From now on, we consider an isometric immersion of the Riemannian manifold M^{n+1} into some Euclidean space $\mathbb{R}^d$, $d \geq n+1$.

If ν is a vector normal to M, we introduce the shape operator S_ν characterized by:

$$\langle S_\nu(X), Y\rangle = \langle L(X,Y), \nu\rangle\,, \tag{4}$$

for all $X, Y \in TM$, where L is the second fundamental form of the immersion.

Then we can extend S_ν to a self-adjoint endomorphism of $\Lambda^p(M)$, denoted by S_ν^p and given by:

$$S_\nu^p(\omega)\left(X_1,\ldots,X_p\right) = \sum_{i=1}^p \omega\left(X_1,\ldots,S_\nu(X_i),\ldots,X_p\right) \tag{5}$$

If $(\nu_1,\ldots,\nu_m)$ is an orthonormal basis of the normal bundle of M at any fixed point (i.e. $m = d-n-1$), then

$$T_p = \sum_{\alpha=1}^m \left(S_{\nu_\alpha}^p\right)^2 \tag{6}$$

defines a self-adjoint nonnegative endomorphism of $\Lambda^p(M)$, which does not depend on the orthonormal basis chosen. In particular, for all $\omega \in \Lambda^p(M)$:

$$\left\langle T_p(\omega), \omega\right\rangle = \sum_{\alpha=1}^m \|S_{\nu_\alpha}^p(\omega)\|^2. \tag{7}$$

Last but not least are the very usefull Bochner–Weitzenbock and Reilly formulas, which can be found in [3] and [5] respectively.

Proposition 1. *If ω is a p-form, then:*

$$\langle \Delta\omega, \omega\rangle = \|\nabla\omega\|^2 + \frac{1}{2}\Delta\|\omega\|^2 + \left\langle W_p\omega, \omega\right\rangle.$$

Here, W_p is a symmetric endomorphism acting on $\Lambda^p(M)$, called the Bochner curvature term. For $p=1$, one has $W_1 =$ Ric, the Ricci tensor; hence W_1 is non-negative provided that M has non-negative Ricci curvature. Moreover, from the work of Gallot and Mayer (in [3]), we know that if γ is a lower bound of the eigenvalues of the Riemann curvature operator, then:

$$W_p \geq p(n+1-p)\gamma. \tag{8}$$

We deduce from this that, if the curvature operator of M is non-negative, then also $W_p \geq 0$ for all degrees p.

Proposition 2. *Let ω be a p-form on M, then:*

$$\int_M \|\mathrm{d}\omega\|^2 + \|\delta\omega\|^2 = \int_M \|\nabla\omega\|^2 + \left\langle W_p\omega, \omega\right\rangle + 2\int_{\partial M} \left\langle i_{\tilde\nu}\omega, \delta^{\partial M}(J^*\omega)\right\rangle + \int_{\partial M} B(\omega,\omega),$$

where

$$B(\omega,\omega) = \left\langle S^p(J^*\omega), (J^*\omega)\right\rangle + nH\|i_{\tilde\nu}\omega\|^2 - \left\langle S^{p-1}(i_{\tilde\nu}\omega), i_{\tilde\nu}\omega\right\rangle,$$

$\delta^{\partial M}$ is the co-derivative on ∂M and ∇ the Levi-Civita connection on M; S denotes the shape operator of the immersion of ∂M in M.

On the immersed manifold M, we focus on the family of all vector fields that are the orthogonal projection of unit parallel vector fields on the ambient Euclidean space $\mathbb{R}^d$. This family is naturally parametrized by the sphere S^{d-1}. Its typical elements will be denoted by $\bar{V}$.

At any point of M, we can split:

$$\bar{V} = V + V^{\perp}, \tag{9}$$

where $V \in TM$ is the orthogonal projection of $\bar{V}$ onto TM and $V^{\perp} \in TM^{\perp}$. Hence, any $\bar{V} \in S^{d-1}$ gives rise to a vector field V on M.

Remark 1. As in [4], it can be easily proved that, if $\omega, \phi \in \Lambda^p(M)$, then at any point of M, we have:

$$\int_{S^{d-1}} \langle i_V \omega, i_V \phi \rangle \, \mathrm{d}\bar{V} = c_d p \, \langle \omega, \phi \rangle \, . \tag{10}$$

Here, $c_d = \frac{|S^{d-1}|}{d}$.

In order to prove our main result, we wish to integrate some inequalities with respect to $\bar{V}$ and the canonical measure of S^{d-1}, which will be denoted by $\mathrm{d}\bar{V}$. To this scope, we complete lemma 4.8 in [4].

Lemma 1. Let $\omega \in \Lambda^p(M)$, $p = 1, \ldots, n+1$. At any point of M:

$$\int_{S^{d-1}} \|i_V \omega\|^2 \mathrm{d}\bar{V} = c_d p \|\omega\|^2,$$

$$\int_{S^{d-1}} \|\mathrm{d}i_V \omega\|^2 \mathrm{d}\bar{V} = c_d \left\{ \|\nabla\omega\|^2 + \left\langle T_p(\omega), \omega \right\rangle + (p-1)\|\mathrm{d}\omega\|^2 \right\},$$

$$\int_{S^{d-1}} \|\delta i_V \omega\|^2 \mathrm{d}\bar{V} = c_d (p-1) \|\delta\omega\|^2.$$

Proof. (1) The first and the last equations can be found in [4]. They are proved using Remark 1 and the Cartan formula:

$$L_V \omega = \mathrm{d}i_V \omega + i_V \mathrm{d}\omega. \tag{11}$$

(2) Now, remarking that, for any vector field V, which is the gradient of a smooth function, one has $\delta i_V = -i_V \delta$. Indeed, the projected fields V are gradients of restrictions to M of suitable distance functions to hyperplanes in R^{n+1}. Thus,

$$\delta i_V = -i_V \delta. \tag{12}$$

Using Remark 1, we obtain:

$$\int_{S^{d-1}} \|\delta i_V \omega\|^2 \mathrm{d}\bar{V} = \int_{S^{d-1}} \|i_V \delta\omega\|^2 \mathrm{d}\bar{V}$$
$$= c_d \, (p-1) \|\delta\omega\|^2. \quad \square$$

Remark 2. We have:

$$\nabla_{e_{i_j}} V = \nabla_{e_{i_j}} \left(\sum_{k=1}^{d} \left\langle \bar{V}, e_k \right\rangle e_k \right)$$
$$= \sum_{k=1}^{n} \left(e_j \left\langle \bar{V}, e_k \right\rangle \right) e_k + \sum_{k=1}^{n} \left\langle \bar{V}, e_k \right\rangle \nabla_{e_{i_j}} e_k.$$

Let $\tilde{\nabla}$ be the Levi-Civita connection on $\mathbb{R}^d$. By its compatibility with the metric and the fact that $\bar{V}$ is parallel, we get:

$$\nabla_{e_{i_j}} V = \left(\sum_{k=1}^{n} \left\langle \bar{V}, \tilde{\nabla}_{e_j} e_k \right\rangle e_k \right).$$

Moreover, if we choose $(e_1, \ldots, e_n)$ to be geodesic at $x \in M$,

$$\tilde{\nabla}_{e_i} e_j = \nabla_{e_i} e_j + \left(\nabla_{e_i} e_j\right)^{\perp} = \left(\nabla_{e_i} e_j\right)^{\perp}.$$

Thus,

$$\begin{aligned}\left\langle \bar{V}, \tilde{\nabla}_{e_j} e_k \right\rangle &= \left\langle \bar{V}, \left(\tilde{\nabla}_{e_j} e_k\right)^{\perp} \right\rangle \\ &= \left\langle V^{\perp}, \left(\tilde{\nabla}_{e_j} e_k\right)^{\perp} \right\rangle = \left\langle V^{\perp}, \left(\tilde{\nabla}_{e_j} e_k\right) \right\rangle \\ &= \left\langle S_{V^{\perp}} e_j, e_k \right\rangle\end{aligned}$$

Eventually, we obtain:

$$\nabla_{e_{i_j}} V = S_{V^{\perp}} e_j.$$

Lemma 2. Let $\omega \in \Lambda^p(M)$ such that $i_{\tilde{\nu}}\omega = 0$ on ∂M, with $\tilde{\nu}$ the unit inner vector field normal to the boundary. Then, for all $\bar{V} \in S^{d-1}$:

$$\nu_{1,p-1}(M) \int_{\partial M} \|i_V \omega\|^2 \leq \int_M \|\mathrm{d} i_V \omega\|^2 + \|\delta i_V \omega\|^2. \tag{13}$$

Proof. If $p \geq 1$ and ω is tangential, then $i_{\tilde{\nu}} i_V \omega = -i_V i_{\tilde{\nu}} \omega = 0$. So, $i_V \omega$ is a tangential $(p-1)$-form. Hence, for all $\bar{V} \in S^{d-1}$, $i_V \omega$ is a relevant choice as a test form for $\nu_{1,p-1}(M)$. We apply the min–max principle and get (13). □

Integrating these inequality on S^{d-1} and using Lemma 1, we obtain the following theorem:

Theorem 3. *Let $M^{n+1} \to \mathbb{R}^d$ an isometric immersion, with M a Riemannian compact and oriented manifold with smooth boundary ∂M. Let also ω be a p-form on M, $p = 1, \ldots, n+1$, satisfying $i_{\tilde{\nu}}\omega = 0$ on ∂M. Then:*

$$\int_M \left\{ \|\nabla \omega\|^2 + \left\langle T_p \omega, \omega \right\rangle + (p-1)\left(\|\mathrm{d}\omega\|^2 + \|\delta\omega\|^2\right) \right\} \geq p\, \nu_{1,p-1}(M) \int_{\partial M} \|\omega\|^2.$$

The inequality is sharp for any harmonic extension on B^{n+1} of an eigenform associated with $\nu_{1,p}(B^{n+1})$ for $\frac{n+3}{2} \leq p \leq n$.

Proof. Integrating (13) on S^{d-1} gives:

$$\int_{S^{d-1}} \nu_{1,p-1}(M) \int_{\partial M} \|i_V \omega\|^2 \mathrm{d}\bar{V} \leq \int_{S^{d-1}} \int_M \|d i_V \omega\|^2 + \|\delta i_V \omega\|^2 \mathrm{d}\bar{V}.$$

Then, using Lemma 1 as well as Fubini's theorem, we get Theorem 3.

Concerning the sharpness, let $\omega \in \Lambda^p(S^n)$ an eigenform of unit L^2 norm associated with $\nu_{1,p}$ ad $\hat{\omega}$ its harmonic extension on the ball. Applying Theorem 3 to $\hat{\omega}$, one gets:

$$\int_{B^{n+1}} \left\{ \|\nabla \hat{\omega}\|^2 + \left\langle T_p \hat{\omega}, \hat{\omega} \right\rangle + (p-1)\left(\|\mathrm{d}\hat{\omega}\|^2 + \|\delta\hat{\omega}\|^2\right) \right\} \geq p\, \nu_{1,p-1}(M) \int_{S^n} \|\hat{\omega}\|^2. \tag{14}$$

Clearly, we have:

- $\int_{B^{n+1}} \|\mathrm{d}\hat{\omega}\|^2 + \|\delta\hat{\omega}\|^2 = \nu_{1,p}$ by the variational characterization;
- $\left\langle T_p \hat{\omega}, \hat{\omega} \right\rangle = 0$ since B^{n+1} is open in $\mathbb{R}^{n+1}$;
- $\int_{S^n} \|\hat{\omega}\|^2 = \int_{S^n} \|\omega\|^2 = 1$.

Now, by Reilly formula in Proposition 2 applied to $\hat{\omega}$, and the fact that in the given range $\nu_{1,p}(B^{n+1}) = p+1$.

$$\int_{B^{n+1}} \|\nabla \hat{\omega}\|^2 = (p+1) - \int_{S^n} \left\langle S^p \omega, \omega \right\rangle,$$

because $\hat{\omega}$ is tangential and $W_p = 0$. Since S^n is totally umbilical with constant mean curvature equals to 1, we obtain:

$$S^p\omega = p\omega.$$

Finally: $\int_{B^{n+1}} \|\nabla\hat{\omega}\|^2 = 1$. Next, by the results in [7]:

$$\nu_{1,p}(B^{n+1}) = \begin{cases} \frac{n+3}{n+1}p \text{ if } 1 \leq p \leq \frac{n+1}{2} \\ p+1 \text{ if } \frac{n+1}{2} \leq p \leq n. \end{cases}$$

Calculating both sides of (14) as a function of p, we see that the equality is attained for $\frac{n+3}{2} \leq p \leq n$. □

Theorem 4. *Let $M^{n+1} \to \mathbb{R}^d$ an isometric immersion with M having p-convex boundary. We also suppose that $W_p - T_p \geq 0$ at all points of M. For all $p = 1, \ldots, n$, one has:*

$$\nu_{1,p}(M) \geq \nu_{1,p-1}(M) + \frac{\sigma_p}{p}. \tag{15}$$

Equality is achieved when M is the unit Euclidean ball of $\mathbb{R}^{n+1}$ and $\frac{n+3}{2} \leq p \leq n$.

Proof. Let ω an eigenform associated with $\nu_{1,p}(M)$ (which we suppose has unit L^2 norm on ∂M by normalization) and let $\hat{\omega}$ its harmonic extension. Then,

$$\begin{aligned} \nu_{1,p}(M) &= \nu_{1,p}(M) \int_{\partial M} \|\omega\|^2 \\ &= \nu_{1,p}(M) \int_{\partial M} \|\hat{\omega}\|^2 \\ &= \int_M \|\mathrm{d}\hat{\omega}\|^2 + \|\delta\hat{\omega}\|^2. \end{aligned}$$

Applying Theorem 3 to $\hat{\omega}$ leads to:

$$\begin{aligned} p\ \nu_{1,p-1}(M) \int_{\partial M} \|\hat{\omega}\|^2 &\leq \int_M \left\{ \|\nabla\hat{\omega}\|^2 + \langle T_p\hat{\omega}, \hat{\omega}\rangle + (p-1)\left(\|\mathrm{d}\hat{\omega}\|^2 + \|\delta\hat{\omega}\|^2\right) \right\} \\ &= \int_M \left\{ \|\nabla\hat{\omega}\|^2 + \langle T_p\hat{\omega}, \hat{\omega}\rangle \right\} + (p-1)\nu_{1,p}(M); \end{aligned}$$

which reads as

$$p\nu_{1,p-1}(M) - (p-1)\nu_{1,p}(M) \leq \int_M \left\{ \|\nabla\hat{\omega}\|^2 + \langle T_p\hat{\omega}, \hat{\omega}\rangle \right\}.$$

But, by Reilly's formula in Proposition 2 and since $\hat{\omega}$ is tangential, we have

$$\begin{aligned} \int_M \|\nabla\hat{\omega}\|^2 &= \int_M \left\{ \|\mathrm{d}\hat{\omega}\|^2 + \|\delta\hat{\omega}\|^2 \right\} - \int_M \langle W_p\hat{\omega}, \hat{\omega}\rangle - \int_{\partial M} \langle S_p\omega, \omega\rangle \\ &= \nu_{1,p} - \int_M \langle W_p\hat{\omega}, \hat{\omega}\rangle - \int_{\partial M} \langle S_p\omega, \omega\rangle, \end{aligned}$$

hence:

$$\nu_{1,p-1}(M) \leq \nu_{1,p}(M) + \int_M \frac{1}{p} \langle (T_p - W_p)\hat{\omega}, \hat{\omega}\rangle - \frac{1}{p} \int_{\partial M} \langle S_p\omega, \omega\rangle.$$

Now, by hypothesis, ∂M is p-convex, so that $\sigma_p \geq 0$, and since we assumed $W_p - T_p \geq 0$, we get:

$$\nu_{1,p}(M) \geq \nu_{1,p-1}(M) + \frac{\sigma_p}{p}.$$

Concerning the equality case, let $M = B^{n+1}$ the unit ball and its boundary S^n. As B^{n+1} is flat and open in $\mathbb{R}^{n+1}$, $W_p = 0$ and $T_p = 0$, so that $W_p - T_p = 0$ is satisfied. Moreover, the ball is convex, so it is p convex for all $p \geq 1$. Now, for all p, we have $\sigma_p = p$ and

- if $1 \leq p \leq \frac{n+1}{2}$:

$$\begin{cases} \nu_{1,p}(B^{n+1}) = \frac{n+3}{n+1} p \\ \nu_{1,p-1}(B^{n+1}) + \frac{\sigma_p}{p} = \frac{n+3}{n+1}(p-1) + 1; \end{cases}$$

- if $\frac{n+1}{2} \leq p \leq n$:
  - if $\frac{n-1}{2} \leq p-1 \leq \frac{n+1}{2}$,

$$\begin{cases} \nu_{1,p}(B^{n+1}) = p+1 \\ \nu_{1,p-1}(B^{n+1}) + \frac{\sigma_p}{p} = \frac{n+3}{n+1}(p-1) + 1; \end{cases}$$

  - if $\frac{n+1}{2} \leq p-1 \leq n-1$,

$$\begin{cases} \nu_{1,p}(B^{n+1}) = p+1 \\ \nu_{1,p-1}(B^{n+1}) + \frac{\sigma_p}{p} = p+1. \end{cases}$$

And we see, as announced, that the equality is obtained for $\frac{n+3}{2} \leq p \leq n$. □

We apply Theorem 4 to Euclidean domains, as done in [6] (see Theorem 4).

Corollary 5.

(1) *Assume that the Euclidean domain M is p-convex for some $p = 1, \ldots, n-1$. Then:*

$$\nu_{1,p-1}(M) \leq \nu_{1,p-1}(M) \leq \ldots \leq \nu_{1,n}(M). \tag{16}$$

(2) *If M is indeed convex, then the sequence $\{\nu_{1,p}(M)\}$ is non-decreasing with respect to the degree p:*

$$\nu_{1,0}(M) \leq \nu_{1,1}(M) \leq \ldots \leq \nu_{1,n}(M). \tag{17}$$

(3) *If M is strictly p-convex, so that $\sigma_p > 0$, then $\nu_{1,q}(M) < \nu_{1,q+1}(M)$ for all $q \geq p$. Thus, these inequalities in (1) and (2) are strict when starting from $\nu_{1,p}(M)$ and $\nu_{1,1}(M)$, respectively.*

Proof. Since M is an Euclidean domain of $\mathbb{R}^{n+2}$ (so that $T_p = W_p = 0$), we get by Theorem 4:

$$\nu_{1,p} \geq \nu_{1,p-1}. \tag{18}$$

As a p-convex domain is q-convex for all $q \in [p, n]$, (1) follows. Then (2) and (3) are consequences of (1). □

Corollary 6. *Let M be a p-convex spherical domain, isometrically immersed in $\mathbb{R}^{n+2}$. Then, for all $p \leq \frac{n+1}{2}$, we obtain an estimation of the p-gap:*

$$\nu_{1,p} - \nu_{1,p-1} \geq 0. \tag{19}$$

Proof. We have $T_p = p^2 \cdot Id$ and $W_p = p(n+1-p) \cdot Id$ so that:

$$W_p - T_p \geq 0 \Longleftrightarrow p \leq \frac{n+1}{2}.$$

So, the hypotheses of Theorem 4 are satisfied for all $p \leq \frac{n+1}{2}$. Moreover, in this case, we have, for all p, $\sigma_p \geq 0$. Applying Theorem 4, the claim follows. □

Remark 3. Note that $M = S_+^{n+1}$, the upper hemisphere, is a particular case of Corollary 6 for $p \leq \frac{n+1}{2}$. Indeed, its boundary is the equator so that $\sigma_p = 0$. Thus, for all $p \leq \frac{n+1}{2}$, we get an estimation of the p-gap of the hemisphere:

$$\nu_{1,p}\left(S_+^{n+1}\right) - \nu_{1,p-1}\left(S_+^{n+1}\right) \geq 0. \tag{20}$$

References

[1] J.F. Escobar, An isoperimetric inequality and the first Steklov eigenvalue, J. Funct. Anal. 165 (1) (1999) 101–116.
[2] J.F. Escobar, A comparison theorem for the first non-zero Steklov eigenvalue, J. Funct. Anal. 178 (1) (2000) 143–155.
[3] S. Gallot, D. Meyer, Opérateur de courbure et laplacien des formes différentielles d'une variété riemannienne, J. Math. Pures Appl. (9) 54 (3) (1975) 259–284.
[4] P. Guerini, A. Savo, Eigenvalue and gap estimates for the Laplacian acting on p-forms, Trans. Amer. Math. Soc. 356 (1) (2004) 319–344.
[5] S. Raulot, A. Savo, A Reilly formula and eigenvalue estimates for differential forms, J. Geom. Anal. 22 (3) (2011) 620–640.
[6] S. Raulot, A. Savo, On the first eigenvalue of the Dirichlet-to-Neumann operator on forms, J. Funct. Anal. 262 (3) (2012) 889–914.
[7] S. Raulot, A. Savo, On the spectrum of the Dirichlet-to-Neumann operator acting on forms of a Euclidean domain, J. Geom. Phys. 77 (2014) 1–12.
[8] G. Schwarz, Hodge Decomposition—A Method for Solving Boundary Value Problems, Lect. Notes Math., vol. 1607, Springer-Verlag, Berlin, 1995.