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Polynomial birth–death processes and the second conjecture of Valent

Processus d'apparition–disparition polynomial et la seconde conjecture de Valent

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ABSTRACT

The conjecture of Valent about the type of Jacobi matrices with polynomially growing weights is proved.

RÉSUMÉ

Nous démontrons la conjecture de G. Valent sur les matrices de type Jacobi avec des poids à croissance polynomiale.

1. Introduction

In 1998, G. Valent in [8] conjectured the order and type of certain indeterminate Stieltjes moment problems associated with birth and death processes having polynomial birth and death rates of degree $p \geq 3$. His conjecture says that the order of the birth–death processes, with rates λ_n, μ_n being the polynomials of degree p,

$$\lambda_n = n^p + Cn^{p-1} + \dots,$$
$$\mu_n = n^p + Dn^{p-1} + \dots,$$

are subject to the condition $1 < C - D < p - 1$ is $1/p$, and its type with respect to that order is

$$p \int_0^1 (1 - x^p)^{-2/p} \, dx, \tag{1}$$

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respectively. The conjecture was formulated based on explicitly solvable examples for $p = 3$ and $p = 4$ found by Valent and his collaborators, see [4], [5].

This note is to announce a proof of this conjecture and give a sketch of it. We are going to establish the following assertion. Let $p > 1$, and

$$J_p = \begin{pmatrix} 0 & 1^p & 0 & \cdots & \cdots \\ 1^p & 0 & 2^p & 0 & \cdots \\ 0 & 2^p & 0 & 3^p & \cdots \\ \cdots & 0 & 3^p & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Then the Hamburger moment problem corresponding to the Jacobi matrix J_p is well known (see, e.g., [1, Chapter 1]) to be indeterminate, hence the corresponding Nevanlinna entire matrix-function is well defined. It follows from the log-concavity of the sequence $\rho_j = j^p$ that the order of entries of the Nevanlinna matrix equals $1/p$, see [2, Theorem 4.11].

Theorem 1. *The entries of the Nevanlinna matrix corresponding to J_p have type*

$$p \int_0^1 (1 - x^p)^{-2/p}\, \mathrm{d}x$$

with respect to their order $1/p$.

The part of the Valent conjecture pertaining to the order was first proved by Romanov in [7] as an application of a general estimate of the order for canonical systems. In [3], Berg and Szwarc noticed that the order and type in the Valent conjecture coincide with those of the Nevanlinna matrix for J_p, thus giving another proof of the order conjecture and establishing that the assertion of Theorem 1 would imply (1). As far as the type is concerned, the only known result so far has been the following estimate from [3],

$$\frac{\pi}{\sin(\pi/p)} \le t_p \le \frac{\pi}{\sin(\pi/p)\cos(\pi/p)}.$$

Our proof of Theorem 1 uses the following assertion due to Berg and Szwarc (a minor misprint in the formulation is corrected). The sign $\prec$ in the summation index throughout means $<$ or $\le$ depends on the parity of the number n involved.

Theorem 2 *([3], Theorem 1.11). Let t_p, $p > 1$, be the type of the Hamburger problem with Jacobi matrix J_p, and let*

$$s(n) = \sum_{1 \le x_1 \le x_2 < x_3 \le x_4 < \ldots \prec x_n} (x_1 x_2 \cdots x_n)^{-p}.$$

Then

$$t_p = \frac{p}{e} \limsup_{n\to\infty} \left(n\left((s(2n)\right)^{\frac{1}{2pn}} \right). \tag{2}$$

Thus Theorem 1 is equivalent to the following assertion.

Theorem 3. *Let $p > 1$ be a real number, and let*

$$k_n = n(s(n))^{\frac{1}{np}}. \tag{3}$$

Then

$$k_n \underset{n\to\infty}{\longrightarrow} \frac{e}{p} B\left(\frac{1}{2p}, 1 - \frac{1}{p}\right) = 2e \int_0^1 (1 - x^p)^{-2/p}\, \mathrm{d}x.$$

Here and throughout the paper B is the Euler beta-function. Before proceeding to the proof of Theorem 3, let us explain where (2) comes from, referring to [3] for details. One can represent the Nevanlinna matrix, $M(z)$, $z \in \mathbb{C}$, corresponding to J_p as the product $\ldots M_n \ldots M_2 M_1$ of 2×2 elementary monodromy matrices, M_j, of the form $M_j = I + zR_j$, where R_j is an upper or lower-triagonal matrix, depending on the parity of j, explicitly calculated in terms of ρ_j. On developing this product in the powers of z and taking into account the triangle structure, we end up with an explicit expression for Taylor coefficients of the matrix elements of $M(z)$. Theorem 2 is just an expression for the type of those functions via their Taylor coefficients.

Let us mention a wide context of the Valent conjecture. It belongs to the theory of indeterminate moment problems [1]. According to M. Riesz's theorem, the Nevanlinna matrices corresponding to indeterminate problems have minimal exponential type (with respect to the order 1). This leads to the question about their order and respective type. The examples where the order and especially the type are known are few and isolated. Apart from those already mentioned, most of them come from explicitly solvable orthogonal polynomials within the q-Askey scheme [6] and have order zero. The main difficulty is the high instability of the indeterminate problems. In particular, spectral estimates obtained by the variational approach are apparently not precise enough to calculate the type.

2. Sketch of the proof

Definition 1. Two sequences, x_n and y_n of positive reals are said to be equal, denoting $x_n \approx y_n$, if $\ln(x_n/y_n) = o(n)$ as $n \to \infty$.

We are going to calculate the limit of k_n by successively replacing the sequence $s(n)$ by equivalent sequences in (3). It is clear that it suffices to establish the limit for an equivalent sequence.

Step 1 – Cutting the tails. For A large enough, the sequence

$$s'_n = \sum_{1 \le x_1 \le x_2 < x_3 \le \ldots \prec x_n < T(n)} (x_1 x_2 \cdots x_n)^{-p}, \; T(n) = n^A, \tag{4}$$

satisfies $s(n) \approx s'_n$. In particular, for any $\alpha > 1$, the sequence

$$S_n(\alpha) = \sum_{1 \le x_1 < x_2 \le x_3 < \ldots \prec x_n < T'(n)} (x_1 x_2 \cdots x_n)^{-p}, \; T'(n) :\, = \alpha^{\lceil A \log_\alpha n \rceil}$$

satisfies $S_n(\alpha) \approx s(n)$.

The proof of this fact, although not very complicated, is rather lengthy and will not be discussed here.

Step 2 – "dyadization". We are going to estimate the sum in (4) over the simplex by plunging the net $1, \alpha, \alpha^2, \alpha^3, \ldots$, $\alpha > 1$, over each summation index x_j and estimating it by the closest element of the net from the left/right. The point is that on the scale we are interested in, which is described by the equivalence $\approx$, the change in the limit will be small in the dyadization parameter $\alpha - 1$.

To do this, for given $\alpha > 1$, define the function P by $P(n) = \alpha^k$, $n \in [\alpha^k, \alpha^{k+1})$. Let

$$s'_n(\alpha) = \sum_{1 \le x_1 < x_2 \le x_3 < \ldots \prec x_n < T'(n)} (P(x_1) P(x_2) \cdots P(x_n))^{-p}, \tag{5}$$

$$k'_n(\alpha) = n(s'_n(\alpha))^{\frac{1}{np}}, \; K_n(\alpha) = n(S_n(\alpha))^{\frac{1}{np}}.$$

Then we obviously have

$$S_n(\alpha) \le s'_n(\alpha) \le S_n(\alpha) \alpha^{np}.$$

Thus, $1 \le k'_n(\alpha)/K_n(\alpha) \le \alpha$. The theorem will now be established if we show that

$$k'_n(\alpha) \underset{n \to \infty}{\longrightarrow} \frac{e}{p} B\left(\frac{1}{2p}, 1 - \frac{1}{p}\right)(1 + O(\alpha - 1)) \tag{6}$$

since the limit of $K_n(\alpha)$ is independent of α by step 1.

Step 3 – The dyadized sum (5) is calculated explicitly. For a given $\alpha > 1$ define $l(n) = \lfloor A \log_\alpha n \rfloor$. We will drop the prime sign for notation convenience, writing $s_n(\alpha) = s'_n(\alpha)$ from now on. Each $P(x_i)$ in (5) is one of the numbers $1, \alpha, \ldots, \alpha^{l(n)}$. Let $c_i = \#\{y : P(y) = \alpha^i\}$, and let $H(a_0, a_1, \ldots, a_l)$ be the number of tuples $(x_1, x_2, \ldots, x_n)$ such that $1 \le x_1 \le x_2 < \ldots \prec x_n < T'(n)$ and $\#\{j : P(x_j) = \alpha^i\} = a_i$ for all i, $0 \le i \le l$. Then

$$s_n(\alpha) = \sum_{(a_0, a_1, \ldots, a_l) :\ \sum a_i = n} H(a_0, a_1, \ldots, a_l) \alpha^{p(-a_1 - 2a_2 - \ldots - l a_l)}. \tag{7}$$

To compute $H(a_0, a_1, \ldots, a_l)$, consider the underlying combinatorics. The integers from 1 to $T'(n)$ are split into $l+1$ groups of respective sizes $c_0, c_1, \ldots, c_l$, and according to the definition of H, there are a_i numbers chosen from the i-th group. With $a_0, a_1, \ldots, a_l$ fixed, the choices from different groups are independent, so $H(a_0, a_1, \ldots, a_n)$ is a product of numbers of ways to choose the a_i numbers, y_j, in such a way that $y_1 \prec y_2 \ldots \prec y_{a_i}$. Consider the i-th group. Without loss of generality, one can shift the enumeration of y_j's so that they become integers from the interval $[1, c_i]$. There exists a bijection between the set of sequences $\{y_j\}_1^{a_i}$ subject to $y_1 \prec y_2 \ldots \prec y_{a_i}$ and the set of strictly increasing sequences $x_1' < x_2' < \ldots < x_{a_i}' \le w_i + c_i$ with w_i equal to either $[a_i/2]$, or $[a_i/2]-1$. This transformation has the form

$$(y_1, y_2 \ldots, y_{a_i}) \to (y_1 + m_1, y_2 + m_2, \ldots, y_{a_i} + m_{a_i})$$

with nonnegative integers m_j. Thus the number of choices for i-th group is equal to the number of strictly increasing sequences $x_1' < x_2' < \ldots < x_{a_i}' \le w_i + c_i$, $w_i = m_{a_i}$, and the latter number is $\binom{c_i+w_i}{a_i}$. Thus,

$$H(a_0, a_1, \ldots, a_l) = \prod_{i=0}^{l} \binom{c_i + w_i}{a_i}.$$

Step 4 – Understanding the binomial coefficients for noninteger values of the arguments via the corresponding Γ-functions, one can replace w_i by $a_i/2$ in the formula for H, replace the sum in (7) by the maximum of the summand, and replace the maximum over the integer simplex $(a_0, a_1, \ldots, a_l)\colon \sum a_i = n$ by the maximum over the real one. Namely, let $\Pi_n = \{(x_0, \ldots, x_l) \in \mathbb{R}^{l+1} \colon 0 \le x_j \le 2c_j,\ \sum x_j = n\}$, and

$$s_n^{(3)}(\alpha) = \max_{\Pi_n} \prod_{i=0}^{l} \frac{\alpha^{-ix_i p}}{(c_i + \frac{x_i}{2} + 1) B(a_i + 1, c_i - \frac{x_i}{2} + 1)}.$$

Then $s_n^{(3)}(\alpha) \approx s_n(\alpha)$. This is possible because each of the three mentioned steps changes $s_n(\alpha)$ by at most a factor of $\mathrm{e}^{O(\ln^2 n)}$, which suffices for the equivalence.

Step 5 – Applying the Strling formula and finding the maximum of the resulting expression by the calculus, we obtain that $s_n^{(3)}(\alpha) \approx s_n^{(4)}(\alpha)$ with

$$s_n^{(4)}(\alpha) = \exp\left(\lambda n + \sum^{l} c_i \ln \frac{\sqrt{D_i} + 1}{\sqrt{D_i} - 1}\right), \tag{8}$$
$$D_i \colon = 1 + 4\alpha^{2ip} \mathrm{e}^{2\lambda},$$

λ being the unique solution to the equation

$$\sum^{l} \frac{2c_i}{\sqrt{4\alpha^{2ip}\,\mathrm{e}^{2\lambda} + 1}} = n. \tag{9}$$

Step 6 – The solution λ has the following asymptotics as s $n \to \infty$,

$$\limsup_{n\to\infty} \left|\lambda - \left(-p\ln n + p \ln\left[\frac{\alpha - 1}{\ln\alpha} J(p)\right]\right)\right| = O(\alpha - 1), \tag{10}$$
$$J(p) = 2^{1-1/p} \int_0^{\infty} \frac{\mathrm{d}u}{\sqrt{1 + u^{2p}}}.$$

It is at this stage that the required beta-function arises in the form of $J(p)$ as the constant term in the right-hand side of (10). It comes from the sum in (9) turning into the Riemann sum for the integral $J(p)$.

Step 7 – Plugging this asymptotics in (8) and summing by parts in the second term of (8) after some calculations, we obtain

$$\limsup_{n\to\infty} n \left(s_n^{(4)}\right)^{\frac{1}{np}} = \frac{\alpha - 1}{\ln\alpha} e\, J(p)\, q(\alpha),$$
$$q(\alpha) = 1 + O(\alpha - 1).$$

The right-hand side of this formula is easily seen to be of the form (6) and this completes the proof.

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