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Analysis and computation of probability density functions for a 1-D impulsively controlled diffusion process



Analyse et calcul des fonctions de densité de probabilité pour un processus de diffusion en dimension 1 contrôlé par impulsion

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ABSTRACT

This paper proposes appropriate boundary conditions to be equipped with Kolmogorov's Forward Equation that governs a stationary probability density function for a 1-D impulsively controlled diffusion process and derives an exact probability density function. The boundary conditions are verified numerically with a Monte Carlo approach. A finitevolume method for solving the equation is also presented and its accuracy is investigated through numerical experiments.

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RÉSUMÉ

Cette Note propose des conditions aux limites appropriées pour l'équation de Kolmogorov antérograde gouvernant une fonction de densité de probabilité stationnaire d'un processus de diffusion contrôlé par impulsion, en dimension 1. Nous obtenons une fonction de densité de probabilité exacte. La condition aux limites est vérifiée numériquement pour l'approche de Monte Carlo. Nous présentons également une méthode de volumes finis pour résoudre l'équation et nous étudions sa précision au moyen de simulations numériques.

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1. Introduction

The stochastic impulse control problem aims at controlling a stochastic system, such as a diffusion process, through impulsive interventions so that a performance index is maximized or minimized [4,5,15,22,23]. Usually, stochastic impulse

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control problems lead to threshold-type optimal intervention strategies, such that the state variables are instantaneously transported from a threshold to another threshold. Such problems are encountered in a variety of research topics where decision-making plays central roles, such as asset and portfolio management [6,9], bank operation [2], consumption and investment [16], cash management [1], and recently in animal population management [24]. A major approach for finding stochastic impulse control problems is solving the associated Hamilton–Jacobi–Bellman Quasi-Variational Inequality (HJBQVI) [18]. Although many studies so far have focused on the HJBQVIs and the associated optimal controls themselves, less attention has been paid to how the controlled dynamics behaves. As an example of population dynamics management, both the optimal control and the resulting population dynamics are important from practical viewpoints [7,24]. This is the main motivation of this paper.

Stochastic dynamics can be effectively characterized by probability density functions (PDFs) from statistical viewpoints. The Kolmogorov Forward Equations (KFEs) [10,19] serve as the governing differential equations of the PDFs, on which appropriate boundary conditions must be imposed. The zero-flux boundary condition [7] and the homogeneous Dirichlet condition are appropriate when the boundaries are reflecting and absorbing, respectively. However, how to specify boundary conditions for the impulsively-controlled cases where the process is controlled at boundaries, as commonly encountered in stochastic impulse control problems, is not a trivial issue. We approach this issue both theoretically and numerically in this paper.

This paper presents appropriate boundary conditions for the KFE associated with a model stochastic impulse control problem and derives an exact PDF. The boundary conditions and the exact PDF are verified with a standard Monte Carlo method. In addition, a finite-volume method (FVM) for solving the KFE based on the fitting technique [25] is proposed. Its accuracy is determined through numerical experiments.

The remainder of this paper is structured as follows. Section 2 introduces the model problem. Section 3 proposes an appropriate boundary condition for PDFs associated with the model problem and derives an exact PDF. Section 3 validates them through a standard Monte Carlo method as well. Section 4 proposes and verifies the numerical method for PDFs associated with the model problem based on FVM. Section 5 concludes this paper and proposes some future tasks.

2. Model stochastic impulse control problem

A model stochastic impulse control problem is introduced. For the sake of brevity of explanation, we consider a population management problem in an infinite period following Yaegashi et al. (2018) [24]. The decision-maker, the manager of the population, can impulsively reduce the population through a countermeasure if it is taken in a much shorter timescale than that of the dynamics. The population in the habitat at the time *t* is denoted as $X_t \ge 0$, whose evolution is governed by Itô's SDE,

$$\begin{cases} dX_t = X_t \left(\mu dt + \sigma dB_t \right), \tau_i \le t < \tau_{i+1} \\ X_{\tau_i} = X_{\tau_{i-}} - \zeta_i, \end{cases} \qquad (1)$$

where $\mu > 0$ is the drift that represents the growth rate of the population, $\sigma > 0$ with $\mu > \sigma^2$ is the volatility, B_t is the 1-D standard Brownian motion defined on a usual complete probability space [17], τ_i ($i = 0, 1, 2, \dots, \tau_0 = 0$) is the stopping time at which the countermeasure is taken, and $\zeta_i > 0$ represents the magnitude of the control at the time τ_i . The incurred cost at τ_i is the most common one:

$$K\left(\zeta_{i}\right) = k_{1}\zeta_{i} + k_{0},\tag{2}$$

where $k_1 > 0$ is the proportional cost and $k_0 > 0$ is the fixed cost. Equation (1) is of the geometric Brownian motion type widely utilized in the literature [15,22].

A performance index *J* represents the expected net profit of the decision-maker:

$$J(x;\eta) = \mathbb{E}\left[\int_{0}^{\infty} e^{-\delta s} \left(RX_{s}^{M} - \lambda X_{s}^{m}\right) ds - \sum_{i=0}^{\infty} e^{-\delta \tau_{i}} K(\zeta_{i}) \chi_{\tau_{i}}\right],$$
(3)

where $\delta > 0$ is the discount rate, R > 0, $\lambda > 0$, 0 < M < 1 and m > 1 are constants, and χ_S is the indicator function for the subset *S*. In the right-hand side of (3), the term RX_s^M represents the ecological utility provided by the existence of the population, the term $-\lambda X_s^m$ represents the disutility by the existence of the population, and the last summation term is the cost for taking the countermeasure. In what follows, the management policy is expressed as

$$\eta = (\tau_i, \zeta_i)_{i>0} \tag{4}$$

and is called an admissible control if it satisfies the conditions stated in Definition 2.1 in Onishi and Tsujimura (2006) [15] and $X_t \ge 0$. Let \mathfrak{A} be the set of the admissible controls.

Under the QVI controls (Definition 3.2 in Cadenillas (1999) [4]), the following threshold-type management policy with the threshold values \bar{x} and \underline{x} ($\bar{x} > \underline{x} > 0$) is optimal [24]:



Fig. 1. The sample path of X_t (blue) with the thresholds $\bar{x} = 2$ (red), $\underline{x} = 1$ (pink) and the magnitude of population control ζ_i (green).

- 1) if $X_{t-} < \bar{x}$, then no countermeasure is taken and only if $X_{t-} = \bar{x}$, the countermeasure is immediately taken and X_{t-} is reduced to \underline{x} ($X_t = \underline{x}$);
- 2) if $X_{0-} > \bar{x}$, then X_{0-} is immediately reduced to $\underline{x} (X_0 = \underline{x})$ by the countermeasure, and follows (A).

The above-mentioned policy means that the process X_t is confined in $[0, \bar{x}]$ if $X_{0-} \leq \bar{x}$. Hence, we can assume the range of X_t as $[0, \bar{x}]$. A sample path of X_t following the optimal management policy is plotted in Fig. 1. Hereafter, we assume that the thresholds values \bar{x} and \underline{x} are already found; our focus is the dynamics under the optimal management policy.

3. Exact solution to the KFE

3.1. Derivation of the exact PDF

An appropriate boundary condition for the KFE associated with the model problem is proposed and its exact solution, the PDF, is derived. Since the process X_t approaches a stationary state under the optimal management policy, the KFE is presented in the time-independent form. Then, the 1-D KFE to be considered here is [19]:

$$\frac{\mathrm{d}F}{\mathrm{d}x} = 0, \quad 0 < x < \bar{x}, \ x \neq \underline{x} \tag{5}$$

with the flux

$$F = \mu x p - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\sigma^2 x^2}{2} p \right),\tag{6}$$

where p = p(x) is the PDF of $X_t = x$. The point $x = \underline{x}$ is excluded from the domain of the KFE because the conservation law of the probability does not hold in the conventional sense at this point since the process is immediately transported from $x = \overline{x}$ to $x = \underline{x}$ irreversibly.

Equation (5) is formally solved as

$$F = rxp - Dx^2 \frac{\mathrm{d}p}{\mathrm{d}x} = \mathrm{const} \tag{7}$$

with $r = \mu - \sigma^2 > 0$ and $D = 0.5\sigma^2$. Equation (7) is a linear ordinary differential equation (ODE) and is analytically solved as

$$p(x) = Cx^{\frac{r}{D}} + \frac{F}{D+r}x^{-1}$$
(8)

where *C* is a constant for integration.

Based on the above-mentioned formal discussion, the exact PDF is constructed. We divide the domain $(0, \bar{x})$ into the sub-domains $(0, \underline{x})$ and (\underline{x}, \bar{x}) . In each sub-domain, p is constructed as

$$p(x) = \begin{cases} C_1 x^{\frac{1}{D}} + \frac{F_1}{D+r} x^{-1} & 0 < x < \underline{x} \\ C_2 x^{\frac{r}{D}} + \frac{F_2}{D+r} x^{-1} & \underline{x} < x < \overline{x} \end{cases}$$
(9)

with the four unknowns C_1 , C_2 , F_1 and F_2 , which are determined as follows. The state $X_t = 0$ is absorbing, which leads to p(0) = 0. The PDF is assumed to be continuous over $[0, \bar{x}]$, which is a natural requirement for the PDFs of diffusion

processes. In addition, the state $X_t = \bar{x}$ is also absorbing in the sense that the process is immediately transported from $x = \bar{x}$ to $x = \underline{x}$ irreversibly. Furthermore, PDFs should have the mass 1 over $[0, \bar{x}]$. In summary, the following four conditions should be supplemented to the KFE to completely specify the unknowns C_1 , C_2 , F_1 and F_2 :

$$p(0) = 0 \quad (\text{absorbing at } X_t = 0), \tag{10}$$

$$p(\underline{x} - 0) = p(\underline{x} + 0) \quad \text{(continuity of } p\text{)},\tag{11}$$

$$p(\bar{x}) = 0$$
 (absorbing at $X_t = \bar{x}$), (12)

and

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$$\int_{0}^{x} p(x) dx = 1 \quad \text{(normalization condition)}. \tag{13}$$

Obviously, the boundary condition (12) is a non-trivial condition. An absorbing boundary (12) should be imposed at $X_t = \bar{x}$ because X_t is immediately transported from $x = \bar{x}$ to $x = \underline{x}$ irreversibly.

The unknowns C_1 , C_2 , F_1 , and F_2 are found as follows. From the boundary condition (10), the condition $F_1 = 0$ holds. From the boundary condition (12),

$$C_2 \bar{x}^{\frac{r}{D}} + \frac{F_2}{D+r} \bar{x}^{-1} = 0, \tag{14}$$

which leads to

$$F_2 = kC_2 \quad \text{with } k = -(D+r)\bar{x}^{\frac{r}{D}+1}.$$
 (15)

From the boundary condition (11),

$$C_{1}\underline{x}^{\frac{r}{D}} = C_{2}\underline{x}^{\frac{r}{D}} + \frac{F_{2}}{D+r}\underline{x}^{-1},$$
(16)

which leads to

$$C_1 = lC_2 \quad \text{with } l = 1 + \frac{k}{D+r} \underline{x}^{-(\frac{r}{D}+1)}.$$
 (17)

From the normalization condition (13),

$$\int_{0}^{\frac{x}{D}} C_1 x^{\frac{r}{D}} dx + \int_{\frac{x}{2}}^{x} C_2 x^{\frac{r}{D}} dx + \frac{F_2}{D+r} x^{-1} dx = 1,$$
(18)

which leads to

$$C_2 = \frac{1}{h} \quad \text{with } h = \frac{1}{D+r} \left\{ Dl\underline{x}^{\frac{r}{D}+1} + D\left(\bar{x}^{\frac{r}{D}+1} - \underline{x}^{\frac{r}{D}+1}\right) + k\ln\left(\frac{\bar{x}}{\underline{x}}\right) \right\}.$$
(19)

The four unknowns in (9) are completely determined from (15), (17), and (19) as

$$p(x) = \begin{cases} \frac{\left\{\frac{x^{-\left(\frac{r}{D}+1\right)} - \bar{x}^{-\left(\frac{r}{D}+1\right)}\right\}}{\ln\left(\frac{\bar{x}}{\bar{x}}\right)} x^{\frac{r}{D}} & 0 \le x < \underline{x} \\ \frac{\left\{x^{-\left(\frac{r}{D}+1\right)} - \bar{x}^{-\left(\frac{r}{D}+1\right)}\right\}}{\ln\left(\frac{\bar{x}}{\bar{x}}\right)} x^{\frac{r}{D}} & \underline{x} \le x \le \bar{x}. \end{cases}$$
(20)

Consequently, we arrive at the following unique solvability result of the KFE (5).

Proposition 3.1. The KFE (5) subject to the boundary conditions (10), (11), and (12) with the normalization condition (13) admits a unique solution (20) belonging to $C([0, \bar{x}]) \cap C^2((0, \underline{x}) \cup (\underline{x}, \bar{x}))$.

Proof of Proposition 3.1. The existence of the solution is a direct consequence of the discussion above, and the uniqueness and the regularity follow from the construction method. \Box



Fig. 2. Histogram generated with the Monte Carlo approach (green) and the calculated PDF (20) (black).

Now the PDF is available, and therefore statistical moments of X_t can be evaluated. For example, the *n*th moment of X_t is calculated as

$$\int_{0}^{n} x^{n} p(x) dx = \frac{DC_{1}}{r + (n+1)D} \underline{x}^{\frac{r}{D} + n+1} + \frac{DC_{2}}{r + (n+1)D} \left(\bar{x}^{\frac{r}{D} + n+1} - \underline{x}^{\frac{r}{D} + n+1} \right) + \frac{F_{2}}{n(D+r)} \left(\bar{x}^{n} - \underline{x}^{n} \right).$$
(21)

A remark on the conservation property of the exact PDF is finally pointed out. By the irreversible nature of the transportation of the process X_t from $x = \bar{x}$ to $x = \underline{x}$, the conservation law to be held at $x = \underline{x}$ is

$$F_{\underline{x}-} + F_{\overline{x}-} = F_{\underline{x}+}.$$

The right-hand side of (22) is F_2 , while its left-hand side is $F_1 + F_2 = 0 + F_2$. Therefore, the exact PDF satisfies the conservation property at $x = \underline{x}$.

3.2. Validation

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The derived exact PDF (20) is validated with a numerical result of a standard Monte Carlo method based on the Mersenne twister [13] and Box Muller method [3]. The SDE (1) with the optimal management policy is directly discretized with the Euler–Maruyama method [8].

The exact PDF and the histogram generated with the Monte Carlo approach is compared for the parameter values $\mu = 0.17$, $\sigma = 0.2$, $\underline{x} = 1$, and $\overline{x} = 2$. Note that the condition $\mu > \sigma^2$ assumed in the mathematical model is satisfied. The number of sample paths generated is 10^8 and the interval for the histogram is $\Delta x = 10^{-3}$. Fig. 2 compares the exact and numerical results, demonstrating their good agreement. The numerical results imply the validity of the conditions (10), (11), (12), and (13). Furthermore, the first and second moments are 1.272 and 1.706 with the Monte Carlo approach, while those by the formula (21) are 1.273 and 1.708, respectively, again demonstrating a good agreement between the theoretical and numerical results.

4. Finite volume method

An FVM for solving the KFE is presented. For the sake of brevity of implementation, a time-dependent counterpart of the KFE (5)

$$\frac{\partial p}{\partial t} + \frac{\partial F}{\partial x} = 0 \tag{23}$$

is considered here, where (5) corresponds to the steady state of (23).

Recall that we have two types of boundaries: the inner boundary $x = \underline{x}$ and the ordinary boundaries x = 0 and $x = \overline{x}$. The computational domain $[0, \overline{x}]$ is divided into N + 1 cells and N + 1 nodes x_i as $0 = x_0 < x_1 < ... < x_{M-1} < x_M < x_{M+1}... < x_{N-1} < x_N = \overline{x}$, so that $x_M = \underline{x}$ (Fig. 3). Nodes are located at the center of the cells except at the ordinary boundary. The cells located at the ordinary boundary ($x = 0, \overline{x}$ and the cell numbers are 0 and N respectively) are set as the half size of the other cells, and the nodes are located at the ordinary boundary (not the center of the cells). For the sake of brevity, we assume the uniform discretization where the length between the nodes Δx is uniform: $\Delta x = \overline{x}/N$. The time increment is denoted by Δt . The PDF p approximated at the node i and the time step n is denoted as p_i^n .

The semi-discretized KFE (23) in the cell i (except at the boundary cells, 0, M and N) is

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{1}{\Delta x} \left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} \right).$$
(24)



Fig. 3. Schematic diagrams of nodes, cells $(i = 0, \dots, M, \dots N)$ and fluxes $F_{i+\frac{1}{2}}$. The flux $F_{N-\frac{1}{2}}$ flows into the cell M irreversibly.

Yoshioka and Unami (2013) [25] employed the fitting technique [14] for constructing fluxes on the cell interfaces. In this technique, fluxes on the cell interfaces is evaluated by using an exact solution to a two-point boundary value problem, leading to a stable discretization complying with the TVD property [21]. Based on Yoshioka and Unami (2013) [25], $F_{i+\frac{1}{2}}$ is set as:

$$F_{i+\frac{1}{2}} = \frac{-V_{i+\frac{1}{2}}e^{\text{Pe}}}{1-e^{\text{Pe}}}p_i + \frac{V_{i+\frac{1}{2}}}{1-e^{\text{Pe}}}p_{i+1} = \alpha_i p_i + \beta_i p_{i+1}$$
(25)

with

$$Pe = \frac{V_{i+\frac{1}{2}}\Delta x}{\varepsilon_{i+\frac{1}{2}}}, \quad V_{i+\frac{1}{2}} = rx_{i+\frac{1}{2}} = \left(\mu - \sigma^2\right) \left(\frac{x_i + x_{i+1}}{2}\right), \quad \varepsilon_{i+\frac{1}{2}} = Dx_{i+\frac{1}{2}}^2 = \frac{\sigma^2}{2} \left(\frac{x_i + x_{i+1}}{2}\right)^2. \tag{26}$$

Substituting (25) into (24) and applying the conventional θ method [20] yields:

$$\frac{p_i^{n+1} - p_i^n}{\Delta t} = -\frac{\theta}{\Delta x} \left(\alpha_i p_i^{n+1} + \beta_i p_{i+1}^{n+1} - \alpha_{i-1} p_{i-1}^{n+1} - \beta_{i-1} p_i^{n+1} \right) - \frac{(1-\theta)}{\Delta x} \left(\alpha_i p_i^n + \beta_i p_{i+1}^n - \alpha_{i-1} p_{i-1}^n - \beta_{i-1} p_i^n \right),$$
(27)

which can be rewritten as

$$-\frac{\Delta t\theta}{\Delta x}\alpha_{i-1}p_{i-1}^{n+1} + \left(1 + \frac{\Delta t\theta}{\Delta x}\alpha_{i} - \frac{\Delta t\theta}{\Delta x}\beta_{i-1}\right)p_{i}^{n+1} + \frac{\Delta t\theta}{\Delta x}\beta_{i}p_{i+1}^{n+1}$$
$$= \frac{\Delta t\left(1-\theta\right)}{\Delta x}\alpha_{i-1}p_{i-1}^{n} + \left(1 - \frac{\Delta t\left(1-\theta\right)}{\Delta x}\alpha_{i} + \frac{\Delta t\left(1-\theta\right)}{\Delta x}\beta_{i-1}\right)p_{i}^{n} - \frac{\Delta t\left(1-\theta\right)}{\Delta x}\beta_{i}p_{i+1}^{n},$$
(28)

except at i = 0, M, N. At the ordinary boundary x = 0, the formal zero-flux boundary condition

$$F_{-\frac{1}{2}} = 0$$
 (29)

with (24) and (25) is specified. At the ordinary boundary $x = \bar{x}$, the condition

$$p_N^{n+1} = 0 \tag{30}$$

is directly specified. Notice that the boundary condition (29) is consistent with the theoretical one (20) at least for the exact solution in Proposition 3.1. At the inner boundary ($x = \underline{x}$ and the cell number is M), since the flux $F_{N-\frac{1}{2}}$ flows into the cell M ($x = \underline{x}$) irreversibly, the equation

$$\frac{\mathrm{d}p_M}{\mathrm{d}t} = -\frac{1}{\Delta x} \left(F_{M+\frac{1}{2}} - F_{M-\frac{1}{2}} - F_{N-\frac{1}{2}} \right) \tag{31}$$

is specified. Following the discretization for the other cells, equation (31) is discretized as

$$-\frac{\Delta t\theta}{\Delta x}\alpha_{M-1}p_{M-1}^{n+1} + \left(1 + \frac{\Delta t\theta}{\Delta x}\alpha_M - \frac{\Delta t\theta}{\Delta x}\beta_{M-1}\right)p_M^{n+1} + \frac{\Delta t\theta}{\Delta x}\beta_M p_{M+1}^{n+1} - \frac{\Delta t\theta}{\Delta x}\alpha_{N-1}p_{N-1}^{n+1}$$
$$= \frac{\Delta t\left(1-\theta\right)}{\Delta x}\alpha_{M-1}p_{M-1}^n + \left(1 - \frac{\Delta t\left(1-\theta\right)}{\Delta x}\alpha_M + \frac{\Delta t\left(1-\theta\right)}{\Delta x}\beta_{M-1}\right)p_M^n$$
$$- \frac{\Delta t\left(1-\theta\right)}{\Delta x}\beta_M p_{M+1}^n + \frac{(1-\theta)}{\Delta x}\alpha_{N-1}p_{N-1}^n.$$
(32)

The system of linear equations assembling (28), (30), and (32) is solved with the standard Gauss–Seidel method until the convergence condition

$$\max_{i} \left| p_{i}^{n+1} - p_{i}^{n} \right| < \omega \tag{33}$$

is satisfied where $\omega = 10^{-10}$ in this paper.

Theoretically, the present FVM is mass-conservative, as shown in the proposition below if the system (32) is numerically solved at each time step without further introducing numerical error.

Proposition 4.1. Assume an initial condition such that

$$\Delta x \sum_{i=0}^{N} p_i^0 = 1.$$
(34)

Then, the FVM satisfies

$$\Delta x \sum_{i=0}^{N} p_i^n = 1 \quad (n \ge 0).$$
(35)

Proof of Proposition 4.1. The flux $F_{i+\frac{1}{2}}$ evaluated at the time step *n* is $F_{i+\frac{1}{2}}^n$. The proof is by an induction argument. Assume that (35) holds true for some $n \ge 1$. Then, by (24) and (31), we have

$$\Delta x \sum_{i=0}^{N} p_i^{n+1} = \Delta x \sum_{i=0}^{N} p_i^n - \Delta t \theta \left\{ \sum_{i=0, i \neq M}^{N-1} \left(F_{i+\frac{1}{2}}^{n+1} - F_{i-\frac{1}{2}}^{n+1} \right) + \left(F_{M+\frac{1}{2}}^{n+1} - F_{M-\frac{1}{2}}^{n+1} - F_{N+\frac{1}{2}}^{n+1} \right) \right\} - \Delta t (1-\theta) \left\{ \sum_{i=0, i \neq M}^{N-1} \left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n \right) + \left(F_{M+\frac{1}{2}}^n - F_{M-\frac{1}{2}}^n - F_{N+\frac{1}{2}}^n \right) \right\}.$$
(36)

Because of

$$\sum_{i=0,i\neq M}^{N-1} \left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n \right) + \left(F_{M+\frac{1}{2}}^n - F_{M-\frac{1}{2}}^n - F_{N+\frac{1}{2}}^n \right) = F_{-\frac{1}{2}}^n = 0$$
(37)

and

$$\sum_{i=0,i\neq M}^{N-1} \left(F_{i+\frac{1}{2}}^{n+1} - F_{i-\frac{1}{2}}^{n+1} \right) + \left(F_{M+\frac{1}{2}}^{n+1} - F_{M-\frac{1}{2}}^{n+1} - F_{N+\frac{1}{2}}^{n+1} \right) = F_{-\frac{1}{2}}^{n+1} = 0,$$
(38)

the equation (36) gives

$$\Delta x \sum_{i=0}^{N} p_i^{n+1} = \Delta x \sum_{i=0}^{N} p_i^n,$$
(39)

which is the desired equality. The proof is thus completed. \Box

In addition, we can prove a non-negativity preserving property under some assumptions as in the previous FVM [12]. This property is of practical importance, since its combination with Proposition 4.1 shows that the PDFs generated by the FVM are non-negative and mass-conservative as the exact one.

Proposition 4.2. Assume $p_i^0 \ge 0$ ($0 \le i \le N$) and that Δt is chosen sufficiently small. Then, $p_i^n \ge 0$ ($0 \le i \le N, n \ge 1$).



Fig. 4. The PDF calculated by the numerical method with the number of cells 100 (red circles) and the exact PDF (20) (black line).

Proof of Proposition 4.2. For the sake of brevity, we prove the non-negativity for Euler's scheme ($\theta = 0$), but the other cases can be proven following [12].

Firstly, we have

$$V_{i+\frac{1}{2}} = \left(\mu - \sigma^2\right) \left(\frac{x_i + x_{i+1}}{2}\right) \ge 0, \qquad Pe = \frac{V_{i+\frac{1}{2}} \Delta x}{\varepsilon_{i+\frac{1}{2}}} \ge 0, \qquad 1 - e^{Pe} < 0 \quad \text{for } 0 \le i \le N - 1, \tag{40}$$

and

$$\frac{-V_{i+\frac{1}{2}}e^{\text{Pe}}}{1-e^{\text{Pe}}} = \alpha_i > 0, \qquad \frac{V_{i+\frac{1}{2}}e^{\text{Pe}}}{1-e^{\text{Pe}}} = \beta_i < 0 \quad \text{for } 0 \le i \le N-1.$$
(41)

Then, choose one Δt such that

$$1 \ge \frac{\Delta t}{\Delta x} \left(\alpha_i - \beta_{i-1} \right) \left(1 \le i \le N - 1 \right). \tag{42}$$

From the FVM scheme, we obtain

$$p_i^{n+1} = \frac{\Delta t}{\Delta x} \alpha_{i-1} p_{i-1}^n + \left\{ 1 - \frac{\Delta t}{\Delta x} \left(\alpha_i - \beta_{i-1} \right) \right\} p_i^n - \frac{\Delta t}{\Delta x} \beta_i p_{i+1}^n \quad \text{for } 1 \le i \le N-1 \text{ except } i = M,$$
(43)

$$p_M^{n+1} = \frac{\Delta t}{\Delta x} \alpha_{M-1} p_{M-1}^n + \left\{ 1 - \frac{\Delta t}{\Delta x} \left(\alpha_M - \beta_{M-1} \right) \right\} p_M^n - \frac{\Delta t}{\Delta x} \beta_M p_{M+1}^n + \frac{\Delta t}{\Delta x} \alpha_{N-1} p_{N-1}^n \quad \text{for } i = M, \tag{44}$$

and

$$p_0^{n+1} = \left(1 - \frac{\Delta t}{\Delta x}\alpha_i\right)p_0^n - \frac{\Delta t}{\Delta x}\beta_0 p_1^n \quad \text{for } i = 0.$$
(45)

By (42), all the coefficients of p_i^n are non-negative in (43), (44), and (45). The fact means that if $p_i^n \ge 0$ ($0 \le i \le N$), then $p_i^{n+1} \ge 0$ ($0 \le i \le N$). The proof is thus completed. \Box

We found that the estimation of the order of accuracy of the FVM is a difficult task due to the non-local relationship (31) and the degenerate coefficients (26). Hence, we carried out numerical experiments to determine its accuracy. Nevertheless, we can estimate a formal accuracy of the FVM except at the inner boundary i = M and the boundaries i = 0, N (see Remark 4.2). Fig. 4 compares the PDF with the present FVM with 100 cells (red circles) and the exact PDF (20) (black line). The used model parameter values are $\mu = 0.17$, $\sigma = 0.2$, $\underline{x} = 1$, and $\overline{x} = 2$. The condition $\mu - \sigma^2 > 0$ is satisfied in this case. Fig. 4 demonstrates that the FVM successfully computes the exact PDF.

The computational accuracy of the FVM is further investigated. The number of cells for numerical computation is changed as 100, 200, 400, 800, and 1600, and l^2 and l^{∞} norms are calculated. According to Table 1, both l^2 and l^{∞} norms monotonically decrease as the number of cells increases. According to Table 2, the FVM is convergent and has first-order accuracy in both l^2 and l^{∞} norms. The conservative property of the scheme is also checked, and the results imply that the mass conservation errors under the employed computational conditions are at most $O(10^{-8})$, which is acceptable considering the convergence condition (33). In addition, non-negativity of the numerical solutions was also satisfied in our numerical experiments.

Table 1					
l^2	norm	and	l∞	norm.	

Number of cells	l ² norm	l^{∞} norm
100	$3.93 imes10^{-4}$	$1.00 imes 10^{-3}$
200	$1.39 imes 10^{-4}$	$2.50 imes 10^{-4}$
400	$4.95 imes 10^{-5}$	$6.45 imes 10^{-5}$
800	$1.90 imes10^{-6}$	$2.04 imes10^{-5}$
1600	$1.10 imes 10^{-6}$	9.38×10^{-6}

Table 2Convergent rate (CR) with l^2 and l^∞ norms.

Compared norms	CR with l^2 norm	CR with l^{∞} norm
100/200	1.50	2.00
200/400	1.49	1.96
400/800	1.38	1.66
800/1600	0.791	1.12

Remark 4.1. One may directly specify the boundary condition $p_0^{n+1} = 0$ instead of the zero-flux condition (29). In such a case, actually the mass-conservation property of the FVM is also achieved with a reasonable accuracy. This is because both the boundary conditions are consistent in this problem. This would be because of the fact that, as it is straightforwardly checked, the exact solution (20) satisfies both p = 0 and F = 0 at x = 0 in the point-wise sense.

Remark 4.2. We can estimate a formal accuracy of the FVM by considering its built-in numerical diffusion in the conventional finite-volume framework (see, e.g., Chapter 5.4 of [11]). The flux F' with the central difference is given as

$$F'_{i+\frac{1}{2}} = r\left(\frac{x_i + x_{i+1}}{2}\right) \frac{p_i + p_{i+1}}{2} - Dx_{i+\frac{1}{2}}^2 \frac{p_{i+1} - p_i}{\Delta x} = \left(\frac{x_i + x_{i+1}}{2}\right) \left\{\frac{r}{2} + \frac{D}{\Delta x}\left(\frac{x_i + x_{i+1}}{2}\right)\right\} p_i + \left(\frac{x_i + x_{i+1}}{2}\right) \left\{\frac{r}{2} - \frac{D}{\Delta x}\left(\frac{x_i + x_{i+1}}{2}\right)\right\} p_{i+1}.$$
(46)

Then, we have

$$F_{i+\frac{1}{2}} - F'_{i+\frac{1}{2}} = Q \left(p_{i+1} - p_i \right), \tag{47}$$

where

$$Q = -\left(\frac{x_i + x_{i+1}}{2}\right) \frac{\frac{r}{2}\left(1 + e^{Pe}\right) + \frac{D}{\Delta x}\left(\frac{x_i + x_{i+1}}{2}\right)\left(1 - e^{Pe}\right)}{1 - e^{Pe}},$$
(48)

which is the numerical diffusion coefficient introduced by the present FVM. We see $Q = O(\Delta x)$ if Pe is sufficiently large, and $Q \rightarrow 0$ if Pe approaches 0, implying that the numerical diffusion has at most the order of the mesh size. The result obtained here, although it is based on a formal argument, is consistent with our results of the numerical experiment. A full theoretical analysis of the present FVM for our KFE, and possibly those for a generalized KFE, will be carried out in our future research.

5. Conclusions

This paper proposed an appropriate boundary conditions for the KFE associated with a model stochastic impulse control problem and derived an exact PDF. The validity of the boundary conditions was numerically checked with a standard Monte Carlo method. In addition, an FVM for directly solving the KFE was presented and its accuracy was determined through numerical experiments. The results obtained in this paper suggest that the boundary conditions for the KFE are appropriate.

One of the future tasks is to handle time-dependent problems in which thresholds may change as time elapses. Such a situation will be encountered when the population dynamics is driven by time-dependent coefficients. In addition, extension of the proposed numerical method to multi-dimensional problems is also an important task.

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