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# On the summability of divergent power series satisfying singular PDEs ☆

## *Sur la sommabilité de séries entières divergentes satisfaisant des équations aux dérivées partielles avec singularités*

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ABSTRACT

The aim of this note is to apply the Borel–Laplace summation method studied by H. Chen, Z. Luo and C. Zhang (Summability of formal solutions of singular PDEs by means of two-dimensional Borel–Laplace method, preprint) to the divergent power series solutions to two families of nonlinear PDEs. The first one contains particularly a two-dimensional version of the so-called Euler equation (ODE), while the second is called totally characteristic type PDE by H. Chen and H. Tahara (On the holomorphic solution of non-linear totally characteristic equations, Math. Nachr. 219 (2000) 85–96).



RÉSUMÉ

Le but de cette Note est d'appliquer la méthode de sommation de Borel–Laplace étudiée par H. Chen, Z. Luo et C. Zhang (Summability of formal solutions of singular PDEs by means of two-dimensional Borel–Laplace method, preprint) aux solutions séries entières de deux familles d'EDP non linéaires. La première contient particulièrement une version bidimensionnelle de ce qu'on appelle équation d'Éuler, alors que la seconde famille d'EDP est dite de type totalement caractéristique par H. Chen et H. Tahara (On the holomorphic solution of non-linear totally characteristic equations, Math. Nachr. 219 (2000) 85–96).



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## Version française abrégée

L'équation différentielle $x^2 y' - y + x = 0$ admet pour solution formelle la série dite d'Euler $\hat{E}(x) = \sum_{n\ge 0} n! x^{n+1}$. Borel-sommable dans toutes les directions sauf l'axe réel positif $[0, +\infty)$ ; celle-ci joue un rôle de première importance dans la théorie analytique des équations différentielles à singularités irrégulières [1,14]. L'équation aux dérivée partielles (2) ci-dessous, variante bidimensionnelle de la précédente équation, est satisfaite terme à terme par la série entière $\hat{F}(x_1, x_2) = \sum_{n_1,n_2\ge 0}(n_1 + n_2)! x_1^{n_1+1} x_2^{n_2+1}$, qui donne lieu à la série d'Euler en considérant $\hat{F}_{x_1}(0, x_2)$ ou $\hat{F}_{x_2}(x_1, 0)$. Voir aussi (9) pour une expression de $\hat{F}$ au moyen du taux d'accroisement de $\hat{E}$ évalué en deux points distincts.

L'étude de l'équation (2) avec sa solution formelle $\hat{F}$ sera développée dans le § 3. On verra, en effet, que cette dernière est Borel-sommable au sens de la Définition 2.1 dans toutes les directions d'argument $(d_1, d_2) \in [0, 2\pi]^2$ tel que $|d_1 - d_2| < \pi$. Les multiplicateurs de Stokes ainsi associés seront décrits dans le Théorème 3.1. On pourrait s'étonner du fait que les directions singulières $(d_1, d_2) \in [0, 2\pi]^2$, avec $|d_1 - d_2| \ge \pi$, occupent un quart du tore $\mathbb{R}^2/(2\pi\mathbb{Z})^2$ des directions issues de l'origine du plan $\mathbb{C}^2$. Ce phénomène est pourtant inévitable, vu l'expression de la somme de Borel de $\hat{F}$ fournie dans le Théorème 3.2.

Par ailleurs, l'équation (2) fait partie d'une famille d'équations linéaires comme indiqué dans (11), lesquelles pourront ensuite être rendues non linéaires sous la forme générale (12) ; voir § 4.1-4.2. Nous nous intéresserons enfin, dans le § 4.3, à l'étude du problème de Cauchy pour (1), avec $n = 2$ et $u(0, x) = 0$, sous certaines hypothèses de généricité relatives à la singularité en $(x_1, x_2) = (0, 0) \in \mathbb{C}^2$ ; voir (13).

Nous verrons que les solutions formelles des équations évoquées ci-dessus sont toutes Borel-sommables dans des directions convenablement choisies ; voir Théorèmes 4.1, 4.2, et 4.3. Pour ce faire, nous allons convertir chacune de ces équations aux dérivées partielles en une équation mélangeant les produits de convolution et les différentielles partielles dans le plan de Borel. Le problème de la sommabilité sera alors ramené à une analyse d'existence de solutions à croissance au plus exponentielle à l'infini dans des secteurs dans le plan complexe bidimensionnel. Une telle analyse sera rendue possible grâce à une version adaptée au cas de deux variables de la norme de Nagumo initialement introduite dans [10, Définition 4.2].

Rappelons aussi que la notion de développement asymptotique du type Gevrey d'une série de deux ou plusieurs variables a été largement étudiée dans [15], [13], [17] et dans bien d'autres travaux. Dans [6], nous avons étudié plusieurs conditions équivalentes pour décrire une série Borel-summable dans une direction donnée dans le cas de deux variables. Comme le montrera le Théorème 2.1, nous allons privilèger le point de vue des fonctions ayant un développement asymptotique au sens de H. Majima [12]. On verra que, sur certains aspects, notre approche diffère de celle adoptée par [15] ; voir les discussions développées au § 2.2.

Les résultats annoncés dans la présente Note sont extraits de notre article [6].

## 1. Introduction

Let us consider the following non-linear singular PDE of the form

$$t\partial_t u = F(t, x, u, \nabla_x u), \qquad (t, x) \in \mathbb{C}_t \times \mathbb{C}_x^n, \tag{1}$$

where $u = u(t, x)$ is an unknown function, $\nabla_x$ is the usual derivative operator $(\partial_{x_1}, \cdots, \partial_{x_n})$, and $F$ is a function of $(t, x, u, v)$. Such equations can be found already in [2,16]; see also [7–9,11].

In [5,10], we dealt with the case of $n = 1$ under the condition that Eq. (1) admits $x = 0$ as a generic irregular singular point. It has been show that the formal solution to (1) is $k$-summable for $x$ and convergent for $t$. In this paper, we will study the case where $n = 2$; the main idea is to make use of the 2-dimensional Borel–Laplace summation method considered in [6]. Under some suitable conditions, it will be shown that the formal solution is Borel-summable for the variable $(x_1, x_2)$, while the variable $t$ remains near zero; see Theorem 4.3.

As $t = 0$ can be viewed as a Fuchsian type singular point in (1), we will start with some PDEs without the variable $t$. The first differential equation that we will study is the following one:

$$x_1^2\,\partial_{x_1} u + x_2^2\,\partial_{x_2} u - u + x_1 x_2 = 0\,, \tag{2}$$

which is a PDE version of the ODE satisfied by the Euler series; see §2. After having obtained the directions along which the formal solution to (2) is Borel-summable, we will consider the Stokes phenomenon, for which a link with the Euler series will be established.

Section 4 will be essentially devoted to two families of nonlinear PDEs, one of which is a special case of (1). In the one below, we will start with some general statements about the Borel-summability of power series of two variables.

## 2. Borel-summability of power series of two variables

Let $\tilde{\mathbb{C}}$ to denote the Riemann surface of the logarithm, which will be identified with $\mathbb{R}_{>0} \times \mathbb{R}$ via the map $(r, \varphi) \mapsto re^{i\varphi}$. Given $(d, \theta, \rho) \in \mathbb{R} \times \mathbb{R}^2_{>0}$, one writes $S^d(\theta; \rho) = \{x \in \tilde{\mathbb{C}} : |x| < \rho, |\arg(x) - d| < \theta/2\}$; by extension, $S^d(\theta) = S^d(\theta; \infty)$, which is an unbounded sector. Unless explicitly stated, we usually suppose that $\theta < 2\pi$, in such a way that each one of $S^d(\theta; \rho)$ and $S^d(\theta)$ can be viewed as one part of the complex plane $\mathbb{C}$. As usual, let $D(0; R)$ ($\bar{D}(0; R)$, resp.) be the open (closed, resp.) disc centered at 0 with radius $R > 0$. Furthermore, one defines $S^d_R(\theta) = D(0; R) \cup S^d(\theta)$.

In the case of two variables, we will write $\boldsymbol{d} = (d_1, d_2)$, $\boldsymbol{\theta} = (\theta_1, \theta_2)$, $\boldsymbol{\rho} = (\rho_1, \rho_2)$ and $\boldsymbol{R} = (R_1, R_2)$. This permits to define the domains $S^{\boldsymbol{d}}(\boldsymbol{\theta}; \boldsymbol{\rho})$, $S^{\boldsymbol{d}}(\boldsymbol{\theta})$ and $S^{\boldsymbol{d}}_{\boldsymbol{R}}(\boldsymbol{\theta})$ in an evident way. Furthermore, we will use the partial order $\boldsymbol{<}$ in $\mathbb{R}^2$ given by the relation $(a_1, a_2) \boldsymbol{<} (b_1, b_2) \Leftrightarrow a_1 < b_1$ and $a_2 < b_2$. By substituting $\leq$ for $<$ here, we obtain also the relation $\boldsymbol{\leq}$ in $\mathbb{R}^2$. Similarly, we can define the partial orders $\boldsymbol{>}$ and $\boldsymbol{\geq}$ in $\mathbb{R}^2$ respectively. Lastly, we will write $\boldsymbol{x}^{\boldsymbol{n}} = x_1^{n_1} x_2^{n_2}$ for all $\boldsymbol{x} = (x_1, x_2) \in \mathbb{C}^2$ and $\boldsymbol{n} = (n_1, n_2) \in \mathbb{Z}^2_{\geq 0}$.

*2.1.* On the one hand, given $S = S^{\boldsymbol{d}}(\boldsymbol{\theta}; \boldsymbol{\rho})$, we will denote by $\mathcal{A}_{\mathbf{1}}(S)$ the space of analytic functions admitting a strong asymptotic expansion as $\boldsymbol{x} = (x_1, x_2) \to (0, 0)$ in $S$ in the sense of H. Majima [12]. On the other hand, if $S = S^{\boldsymbol{d}}_{\boldsymbol{R}}(\boldsymbol{\theta})$, we will denote by $\mathcal{E}_{\mathbf{1}}(S)$ the set of analytic functions $\phi$ in $S$ with an exponential growth of order at most one at infinity. This is to say, for any $(\boldsymbol{R}', \boldsymbol{\theta}') \in \mathbb{R}^2_{>0} \times \mathbb{R}^2_{>0}$ with $\boldsymbol{R}' \boldsymbol{<} \boldsymbol{R}$ and $\boldsymbol{\theta}' \boldsymbol{<} \boldsymbol{\theta}$, one can find $(C, \nu) \in \mathbb{R}^2_{>0}$ such that $|\phi(\boldsymbol{\xi})| \leq C \exp(\nu |\boldsymbol{\xi}|)$ for all $\boldsymbol{\xi} = (\xi_1, \xi_2) \in S^{\boldsymbol{d}}_{\boldsymbol{R}'}(\boldsymbol{\theta}')$, where $|\boldsymbol{\xi}| = |\xi_1| + |\xi_2|$. Thus, with each $\boldsymbol{d} \in \mathbb{R}^2$, we can associate the inductive limit spaces $\mathcal{A}^{\boldsymbol{d}}_{\mathbf{1}}$ and $\mathcal{E}^{\boldsymbol{d}}_{\mathbf{1}}$ in the following way:

$$\mathcal{A}^{\boldsymbol{d}}_{\mathbf{1}} = \cup_{\boldsymbol{\theta} > \boldsymbol{\pi}, \boldsymbol{\rho} > \mathbf{0}} \mathcal{A}_{\mathbf{1}}(S^{\boldsymbol{d}}(\boldsymbol{\theta}; \boldsymbol{\rho})), \quad \mathcal{E}^{\boldsymbol{d}}_{\mathbf{1}} = \cup_{\boldsymbol{\theta} > \mathbf{0}, \boldsymbol{R} > \mathbf{0}} \mathcal{E}_{\mathbf{1}}(S^{\boldsymbol{d}}_{\boldsymbol{R}}(\boldsymbol{\theta})),$$

where $\boldsymbol{\pi} = (\pi, \pi)$ and $\mathbf{0} = (0, 0)$. Furthermore, we introduce the following generalized space $\tilde{\mathcal{E}}^{\boldsymbol{d}}_{\mathbf{1}}$:

$$\tilde{\mathcal{E}}^{\boldsymbol{d}}_{\mathbf{1}} = \left(\mathbb{C}\, \delta_{\mathbf{0}}\right) \oplus \left(\mathcal{E}^{d_1}_1\, \delta_{0;1}\right) \oplus \left(\mathcal{E}^{d_2}_1\, \delta_{0;2}\right) \oplus \mathcal{E}^{\boldsymbol{d}}_{\mathbf{1}}.$$

Here $\delta_{\mathbf{0}}$, $\delta_{0;1}$ and $\delta_{0;2}$ are the Dirac operators related to $\mathbf{0} = (0, 0) \in \mathbb{C}^2 = \mathbb{C}_{\xi_1} \times \mathbb{C}_{\xi_2}$, $0 \in \mathbb{C}_{\xi_1}$ and $0 \in \mathbb{C}_{\xi_2}$, respectively; $\mathcal{E}^{d_1}_1$ and $\mathcal{E}^{d_2}_1$ denote the spaces of analytic functions with an exponential growth of order at most one at infinity in some sector containing the directions $d_1$ and $d_2$, respectively.

Like in the case of one variable, the spaces $\tilde{\mathcal{E}}^{\boldsymbol{d}}_{\mathbf{1}}$ and $\mathcal{A}^{\boldsymbol{d}}_{\mathbf{1}}$ may be viewed as differential algebras.

**Theorem 2.1.** *Given $\boldsymbol{d} = (d_1, d_2) \in \mathbb{R}^2$, the following Laplace integral*

$$\mathcal{L}^{\boldsymbol{d}}_{\mathbf{1}} \phi(\boldsymbol{x}) = \int_0^{\infty e^{\mathrm{i} d_1}} \int_0^{\infty e^{\mathrm{i} d_2}} \phi(\xi_1, \xi_2)\, \mathrm{e}^{-\xi_1/x_1 - \xi_2/x_2}\, \mathrm{d}\xi_2\, \mathrm{d}\xi_1$$

*gives an isomorphism of differential algebras from $\tilde{\mathcal{E}}^{\boldsymbol{d}}_{\mathbf{1}}$ onto $\mathcal{A}^{\boldsymbol{d}}_{\mathbf{1}}$.*

If $\hat{f}(\boldsymbol{x}) = \sum_{\boldsymbol{n} \geq \mathbf{0}} a_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}} \in \mathbb{C}[[\boldsymbol{x}]]$, one defines its (formal) Borel-transform as follows ($\boldsymbol{\xi} = (\xi_1, \xi_2)$):

$$\hat{\mathcal{B}}_{\mathbf{1}} \hat{f}(\boldsymbol{\xi}) = a_{\mathbf{0}}\, \delta_{\mathbf{0}} + \sum_{n_1 \geq 0} \frac{a_{n_1+1,0}}{n_1!} \xi_1^{n_1}\, \delta_{0;1} + \sum_{n_2 \geq 0} \frac{a_{0,n_2+1}}{n_2!} \xi_2^{n_2}\, \delta_{0;2} + \sum_{\boldsymbol{n} \geq \mathbf{0}} \frac{a_{\boldsymbol{n}+\mathbf{1}}}{\boldsymbol{n}!} \boldsymbol{\xi}^{\boldsymbol{n}}. \tag{3}$$

**Definition 2.1.** *Let $\boldsymbol{d} = (d_1, d_2) \in \mathbb{R}^2$. A given power series $\hat{f}(\boldsymbol{x}) \in \mathbb{C}[[\boldsymbol{x}]]$ will be called Borel-summable in the direction of argument $\boldsymbol{d}$ if $\hat{\mathcal{B}}_{\mathbf{1}} \hat{f} \in \tilde{\mathcal{E}}^{\boldsymbol{d}}_{\mathbf{1}}$. In this case, we write $\hat{f} \in \mathbb{C}\{\boldsymbol{x}\}^{\boldsymbol{d}}_{\mathbf{1}}$.*

In view of Theorem 2.1, one finds that the composite map $\mathcal{L}^{\boldsymbol{d}} \circ \hat{\mathcal{B}}_{\mathbf{1}}$ gives an isomorphism of differential algebras from $\mathbb{C}\{\boldsymbol{x}\}^{\boldsymbol{d}}_{\mathbf{1}}$ onto $\mathcal{A}^{\boldsymbol{d}}_{\mathbf{1}}$. So, $\mathcal{L}^{\boldsymbol{d}} \circ \hat{\mathcal{B}}_{\mathbf{1}} \hat{f}$ can be called *Borel-sum* of $\hat{f} \in \mathbb{C}\{\boldsymbol{x}\}^{\boldsymbol{d}}_{\mathbf{1}}$ in $\boldsymbol{d}$.

*2.2.* The above-introduced spaces $\mathcal{A}_{\mathbf{1}}(S^{\boldsymbol{d}}(\boldsymbol{\theta}; \boldsymbol{\rho}))$ are already used in [15] with regard to the Borel-summability of power series of two or more variables; see also [4,13] and [17]. However, instead of our spaces $\mathcal{E}_{\mathbf{1}}(S^{\boldsymbol{d}}_{\boldsymbol{R}}(\boldsymbol{\theta}))$, one finds the spaces of type $\mathcal{E}_{\mathbf{1}}(S^{\boldsymbol{d}}_{\mathbf{0}}(\boldsymbol{\theta}))$ in [15], where $S^{\boldsymbol{d}}_{\mathbf{0}}(\boldsymbol{\theta}) = S^{\boldsymbol{d}}(\boldsymbol{\theta})$ denotes an unbounded sector containing $\boldsymbol{d}$. This difference may be significant for some situations, as it will be shown in the below.

If one considers the partial differential operator $L = x_2^2\, \partial_{x_2}\, (x_1^2\, \partial_{x_1} + 1) + 1$, then the following series

$$\hat{f}(x_1, x_2) = \sum_{n_2 \geq n_1 \geq 0} \frac{(-1)^{n_2} (n_2!)^2}{(n_2 - n_1)!} x_1^{n_1+1} x_2^{n_2+1} \tag{4}$$

will satisfy $L u = x_1 x_2$. Letting $\phi = \hat{\mathcal{B}} \hat{f}$, one finds that $\phi(\boldsymbol{\xi}) = \frac{1}{1+\xi_2(1+\xi_1)}$, so $\phi$ is analytic at $\mathbf{0} \in \mathbb{C}^2$ and can be analytically continued in any sector $S^{\boldsymbol{d}}_{\mathbf{0}}(\boldsymbol{\theta})$ such that $|d_j \pm \theta_j/2| < \pi/2$ for $j = 1$ and 2. Thus, the series $\hat{f}$ given by (4) could be Borel-summable in such directions $\boldsymbol{d}$ in the sense of [15]. What may be surprising is that $\phi$ can not be analytically continued in any domain of the form $S^{\boldsymbol{d}}_{\boldsymbol{R}}(\boldsymbol{\theta})$ with $\boldsymbol{R} \in \mathbb{R}^2_{>0}$; this means $\hat{f} \notin \mathbb{C}\{\boldsymbol{x}\}^{\boldsymbol{d}}_{\mathbf{1}}$ for all $\boldsymbol{d} \in \mathbb{R}^2$, by considering Definition 2.1 here.

In [6], we introduced intermediate spaces situated between the inductive spaces $\mathcal{E}^{\boldsymbol{d}}_{\mathbf{1}}$ and $\cup_{\boldsymbol{\theta} > \mathbf{0}} \mathcal{E}_{\mathbf{1}}(S^{\boldsymbol{d}}_{\mathbf{0}}(\boldsymbol{\theta}))$, and this defines what we call $(1, 2)$ or $(2, 1)$-iteratively Borel summable series in a given direction $\boldsymbol{d}$. In this setting, the above series $\hat{f}$ is $(2, 1)$-iteratively Borel summable, but not $(1, 2)$-iteratively Borel summable in the direction $(0, 0)$, contrarily to the statement of [15, Def. 5.3 & Prop. 5.4].

*2.3.* Let $S = S_{\boldsymbol{R}}^{\boldsymbol{d}}(\boldsymbol{\theta})$, and consider $\tilde{\mathcal{E}}_{\mathbf{1}}(S)$. By generalizing the Nagumo norm considered in [10, Definition 4.2], one can get a family of Banach spaces $(\tilde{\mathcal{E}}_{\mathbf{1};\mu,n}(S), \|\ \|_{S;\mu,n})$, with $\mu \in \mathbb{R}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 0}$. Let $m \in \mathbb{N}$, $D = D(0;\rho_1) \times \ldots \times D(0;\rho_m)$ with $\rho_j > 0$, and let $\Phi$ be an analytic function defined in $S \times D$. By using the results in [3, §3], one can prove that $\Phi \in \tilde{\mathcal{E}}_{\mathbf{1};\mu,n}(S)\{\boldsymbol{z}\}_{\boldsymbol{\rho}}$, with $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_m)$ if the map $\boldsymbol{z} \mapsto \Phi(., \boldsymbol{z})$ is an analytic mapping from $D$ to $\tilde{\mathcal{E}}_{\mathbf{1};\mu,n}(S)$. Now, we define $\tilde{\mathcal{E}}^{\boldsymbol{d}}_{\mathbf{1};\mu,n}\{\boldsymbol{z}\} = \cup \tilde{\mathcal{E}}_{\mathbf{1};\mu,n}(S)\{\boldsymbol{z}\}_{\boldsymbol{\rho}}$ by taking the union for all $S = S_{\boldsymbol{R}}^{\boldsymbol{d}}(\boldsymbol{\theta})$ with $\boldsymbol{\theta} > \mathbf{0}$, $\boldsymbol{R} > \mathbf{0}$ and for all $\rho_1 > 0$, ..., and $\rho_m > 0$, where $(\rho_1, \ldots, \rho_m) = \boldsymbol{\rho}$. By Theorem 2.1, $\mathcal{A}_{\mathbf{1}}^{\boldsymbol{d}}$ can be found from $\tilde{\mathcal{E}}_{\mathbf{1}}^{\boldsymbol{d}}$ via the Laplace integral $\mathcal{L}_{\mathbf{1}}^{\boldsymbol{d}}$. Thus, we define

$$\mathcal{A}^{\boldsymbol{d}}_{\mathbf{1};\mu,n}\{\boldsymbol{z}\} = \left\{\mathcal{L}_{\mathbf{1}}^{\boldsymbol{d}}\Phi(., z) : \Phi(., z) \in \tilde{\mathcal{E}}^{\boldsymbol{d}}_{\mathbf{1};\mu,n}\{\boldsymbol{z}\}\right\}, \quad \mathcal{A}_{\mathbf{1}}^{\boldsymbol{d}}\{\boldsymbol{z}\} = \cup_{\mu \geq 0}\mathcal{A}^{\boldsymbol{d}}_{\mathbf{1};\mu,n}\{\boldsymbol{z}\}\ (n \geq 0)\,. \tag{5}$$

## 3. 2-dimensional Euler equation

Let $\hat{E}(x) = \sum_{n\geq 0} n!\, x^{n+1}$ be the so-called Euler's series that satisfies $x^2 y' - y + x = 0$. One can see that (2) is satisfied by the 2-dimensional Euler's series $\hat{F}(\boldsymbol{x}) = \sum_{n_1,n_2\geq 0}(n_1+n_2)!\, x_1^{n_1+1} x_2^{n_2+1}$.

*3.1.* Let $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, the quotient of the additive group $\mathbb{R}^2$ by $(2\pi\mathbb{Z})^2$, and set

$$\mathbb{D} = \{(d_1, d_2) \in (0, 2\pi)^2 : |d_1 - d_2| < \pi\}, \quad \mathbb{S} = \{\bar{\boldsymbol{d}} \in \mathbb{T}^2 : \boldsymbol{d} \in [0, 2\pi]^2 \setminus \mathbb{D}\}\,. \tag{6}$$

Given $\boldsymbol{d} \in \mathbb{D}$ and $\lambda > 0$, there exist $\boldsymbol{\theta} \in \mathbb{R}^2_{>0}$ and $\delta > 0$ such that $|1 - \xi_1 - \xi_2| \geq \delta(\lambda + |\boldsymbol{\xi}|)$ for all $\boldsymbol{\xi} = (\xi_1, \xi_2) \in S^{\boldsymbol{d}}(\boldsymbol{\theta})$. Consequently, $\hat{F} \in \mathbb{C}\{\boldsymbol{x}\}_{\mathbf{1}}^{\boldsymbol{d}}$ if, and only if, $\boldsymbol{d} \in \mathbb{D}$.

Let $\{j, k\} = \{1, 2\}$, $\boldsymbol{d} = (d_1, d_2) \in \mathbb{D}$, and define $\alpha_j = \max(0, d_k - \pi)$, $\beta_j = \min(2\pi, d_k + \pi)$. Since $\beta_j > \alpha_j + \pi$ and $d_k \in (\alpha_j, \alpha_j + \pi]$, one can choose $\epsilon \gtrsim 0$ in such a way that, letting $d_j^- = \alpha_j + \epsilon$ and $d_j^+ = \beta_j - \epsilon$, one has $d_k \in (d_j^-, d_j^- + \pi) \subset (d_j^-, d_j^+)$. Thus, if

$$U = \big((d_1 - \epsilon, d_1 + \epsilon) \times (d_2^-, d_2^+)\big) \cup \big((d_1^-, d_1^+) \times (d_2 - \epsilon, d_2 + \epsilon)\big)\,,$$

then $U \subset \mathbb{D}$ and $\hat{F} \in \mathbb{C}\{\boldsymbol{x}\}_{\mathbf{1}}^{\boldsymbol{\delta}}$ for $\boldsymbol{\delta} \in U$. Let $F$ be the Borel-sum of $\hat{F}$ associated with a given $\boldsymbol{\delta} \in U$. It follows that $F \in \mathcal{A}_{\mathbf{1}}(S)$ with $S = S_1 \times S_2'$ or $S = S_1' \times S_2$, where

$$S_j' = \{x_j \in \tilde{\mathbb{C}}^* : \arg(x_j) \in (d_j^- - \frac{\pi}{2}, d_j^+ + \frac{\pi}{2})\}, \quad S_k = \{x_k \in \tilde{\mathbb{C}}^* : \arg(x_k) \in (d_k - \epsilon - \frac{\pi}{2}, d_k + \epsilon + \frac{\pi}{2}\}\,. \tag{7}$$

**Theorem 3.1.** *Let $T_j = \{z \in S_j' : z e^{2\pi i} \in S_j'\}$ for $j = 1$ and 2. The following relations hold respectively for $\boldsymbol{x} \in S_1 \times T_2$ and $\boldsymbol{x} \in T_1 \times S_2$:*

$$F(x_1, x_2 e^{2\pi i}) - F(x_1, x_2) = 2\pi i\, \frac{x_1 x_2}{x_1 - x_2}\, e^{-1/x_2}\,, \quad F(x_1 e^{2\pi i}, x_2) - F(x_1, x_2) = 2\pi i\, \frac{x_1 x_2}{x_2 - x_1}\, e^{-1/x_1}\,. \tag{8}$$

*3.2.* The above series $\hat{F}$ can be also expressed as follows:

$$\hat{F}(\boldsymbol{x}) = \sum_{n\geq 0} n! \sum_{k=0}^{n} x_1^{k+1} x_2^{n-k+1} = \frac{x_1 x_2}{x_1 - x_2} \sum_{n\geq 0} n!\,(x_1^{n+1} - x_2^{n+1}) = \frac{x_1 x_2}{x_1 - x_2}\big(\hat{E}(x_1) - \hat{E}(x_2)\big)\,. \tag{9}$$

In other words, if we denote by $\Delta$ the operator defined by the following relation:

$$\Delta f(x_1, x_2) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} \text{ if } x_1 \neq x_2, \quad \Delta f(x_1, x_2) = f'(x_1) \text{ if } x_1 = x_2, \tag{10}$$

then $\hat{F}(\boldsymbol{x}) = x_1 x_2\, \Delta f(\boldsymbol{x})$. One remembers that the Euler series $\hat{E}$ is Borel-summable in all directions excepted in $\mathbb{R}_{>0}$, and that its Borel-sum in $\mathbb{R}_-$ can be continued to be analytic in $S^{\pi}(3\pi)$; see [14].

**Theorem 3.2.** *Let $\boldsymbol{d} = (d_1, d_2) \in \mathbb{T}^2 \setminus \mathbb{S}$, conserve $S_j'$ and $S_k$ as in (7), and consider the Borel-sum $F$ of $\hat{F}$ in $\boldsymbol{d}$. Then $F(\boldsymbol{x}) = x_1 x_2\, \Delta E(\boldsymbol{x})$ for all $\boldsymbol{x} = (x_1, x_2) \in (S_1 \times S_2') \cup (S_1' \times S_2)$, where $\Delta E$ is obtained from (10) by replacing $f$ with the Borel-sum $E$ of $\hat{E}$ in $S^{\pi}(3\pi)$.*

## 4. Summability of formal solutions to linear or non-linear PDEs

Let $\mathbb{D}$ be as in (6), and choose $\arg t \in [0, 2\pi)$ for any $t \in \mathbb{C}^*$; given $(\beta, \gamma_1, \gamma_2) \in \mathbb{C}^{*3}$, define

$$\mathbb{D}_{\beta;\gamma_1,\gamma_2} = \left\{\boldsymbol{d} \in \mathbb{R}^2 : \boldsymbol{d} + (\arg\frac{\gamma_1}{\beta}, \arg\frac{\gamma_2}{\beta}) \in \mathbb{T}^2 \setminus \mathbb{S}\right\}.$$

*4.1.* In order to generalize the study of (2), consider the following linear PDE:

$$c_1(\boldsymbol{x})\, x_1^2\, \partial_{x_1} u + c_2(\boldsymbol{x})\, x_2^2\, \partial_{x_2} u - b(\boldsymbol{x})\, u + a(\boldsymbol{x}) = 0\,, \tag{11}$$

where $a$, $b$, $c_1$ and $c_2$ are some given functions of the variable $\boldsymbol{x} = (x_1, x_2)$.

**Theorem 4.1.** *Let $\boldsymbol{d} \in \mathbb{R}^2$, and consider (11) with $a$, $b$, $c_1$ and $c_2 \in \mathcal{A}_{\mathbf{1}}^{\boldsymbol{d}}$. If $b(\mathbf{0})\, c_1(\mathbf{0})\, c_2(\mathbf{0}) \neq 0$ and $\boldsymbol{d} \in \mathbb{D}_{b(\mathbf{0});c_1(\mathbf{0}),c_2(\mathbf{0})}$, then (11) has a unique solution $u \in \mathcal{A}_{\mathbf{1}}^{\boldsymbol{d}}$.*

To prove Theorem 4.1, one can apply the Borel transform to rewrite (11) into a convolution equation:

$$(b(\mathbf{0}) - c_1(\mathbf{0})\,\xi_1 - c_2(\mathbf{0})\,\xi_2)\, U = A - B * U + C_1 * (\xi_1\, U) + C_2 * (\xi_2\, U)\,,$$

in which $A$, $B$, $C_1$, and $C_2$ are linked with the Borel-transforms of $a$, $b$, $c_1$, and $c_2$.

*4.2.* Let $\mathcal{A}_{\mathbf{1}}^{\boldsymbol{d}}\{\boldsymbol{z}\}$ be as in (5), and consider the following Cauchy problem:

$$c_1(\boldsymbol{x})\, x_1^2\, \partial_{x_1} u + c_2(\boldsymbol{x})\, x_2^2\, \partial_{x_2} u - G(\boldsymbol{x}, u) = 0, \quad u(\mathbf{0}) = 0\,. \tag{12}$$

In what follows, we will assume the following hypothesis for (12):

(H1) $\boldsymbol{d} = (d_1, d_2) \in \mathbb{R}^2$, $c_j \in \mathcal{A}_{\mathbf{1}}^{\boldsymbol{d}}$ with $c_j(\mathbf{0}) = \gamma_j \neq 0$ for $j = 1$ and 2.
(H2) $G \in \mathcal{A}_{\mathbf{1}}^{\boldsymbol{d}}\{u\}$, $G(\mathbf{0}, 0) = 0$ and $\partial_u G(\mathbf{0}, 0) = \beta \neq 0$.
(H3) The relation $\boldsymbol{d} \in \mathbb{D}_{\beta;\gamma_1,\gamma_2}$ holds.

**Theorem 4.2.** *Under (H1), (H2), and (H3), the Cauchy problem in (12) admits a unique solution $u \in \mathcal{A}_{\mathbf{1}}^{\boldsymbol{d}}$.*

*4.3.* Let $\delta_{\boldsymbol{x}} = (x_1\partial_{x_1}, x_2\partial_{x_2})$ be a Euler vector fields, and consider the following Cauchy problem:

$$t\partial_t\, u = c_1(\boldsymbol{x}) x_1^2 \partial_{x_1} u + c_2(\boldsymbol{x}) x_2^2 \partial_{x_2} u - G(\boldsymbol{x}, t, u, \delta_{\boldsymbol{x}} u), \quad u(\boldsymbol{x}, 0) = 0, \tag{13}$$

which is a special case of (1). We will conserve Hypothesis (H1) made in §4.2 and replace (H2) and (H3) with the following ones:

(H2a) $G \in \mathcal{A}_{\mathbf{1}}^{\boldsymbol{d}}\{t, u, v_1, v_2\}$, $G(\boldsymbol{x}, \mathbf{0}_4) = \partial_{v_1} G(\boldsymbol{x}, \mathbf{0}_4) = \partial_{v_2} G(\boldsymbol{x}, \mathbf{0}_4) = 0$ and $\partial_u G(\mathbf{0}_6) = \beta \notin \mathbb{Z}_{<0}$.
(H3a) The relation $\boldsymbol{d} \in \mathbb{D}_{1;c_1(\mathbf{0}),c_2(\mathbf{0})} \cap \left(\cap_{\ell\in\mathbb{N}^*} \mathbb{D}_{\beta+\ell;c_1(\mathbf{0}),c_2(\mathbf{0})}\right)$ holds.

**Theorem 4.3.** *Under (H1), (H2a), and (H3a), the Cauchy problem in (13) admits a unique solution $u \in \mathcal{A}_{\mathbf{1}}^{\boldsymbol{d}}\{t\}$, that is to say, there exists a unique sequence of functions $u_\ell \in \mathcal{A}_{\mathbf{1}}^{\boldsymbol{d}}$ such that the associated power series $u(\boldsymbol{x}, t) = \sum_{\ell=1}^{\infty} u_\ell(\boldsymbol{x})\, t^\ell$ converges into a solution of (13) in a domain of the form $S_{\boldsymbol{R}}^{\boldsymbol{d}}(\boldsymbol{\theta}) \times D(0; T)$ for some suitable $\boldsymbol{R} \in \mathbb{R}_{>0}^2$, $\boldsymbol{\theta} > (\pi, \pi)$ and $T > 0$.*

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