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Quasi-invariant Gaussian measures for the cubic nonlinear Schrödinger equation with third-order dispersion

Mesures gaussiennes quasi invariantes pour l'équation de Schrödinger non linéaire cubique avec dispersion d'ordre trois

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ABSTRACT

In this paper, we consider the cubic nonlinear Schrödinger equation with third-order dispersion on the circle. In the non-resonant case, we prove that the mean-zero Gaussian measures on Sobolev spaces $H^s(\mathbb{T})$, $s > \frac{3}{4}$, are quasi-invariant under the flow. In establishing the result, we apply gauge transformations to remove the resonant part of the dynamics and use invariance of the Gaussian measures under these gauge transformations.

RÉSUMÉ

Dans cet article, nous considérons l'équation de Schrödinger non linéaire cubique avec dispersion d'ordre trois sur le cercle. Dans le cas non résonant, nous prouvons que les mesures gaussiennes de moyenne nulle sur les espaces de Sobolev $H^s(\mathbb{T})$, $s > \frac{3}{4}$, sont quasi invariantes par le flot. En établissant le résultat, nous appliquons des transformations de gauge pour éliminer la partie résonante de la dynamique, et nous utilisons l'invariance des mesures gaussiennes par rapport à ces transformations de gauge.

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1. Introduction

1.1. Cubic nonlinear Schrödinger equation with third-order dispersion

The main goal of this work is to extend the result of our previous paper [30] to the more involved case of lower-order dispersion. Namely, we consider the following cubic nonlinear Schrödinger equation with third-order dispersion (3NLS) on $\mathbb{T}$:

$$\begin{cases} \mathrm{i}\,\partial_t u - \mathrm{i}\,\partial_x^3 u - \beta\,\partial_x^2 u = |u|^2 u \\ u|_{t=0} = u_0, \end{cases} \qquad (x,t) \in \mathbb{T} \times \mathbb{R}, \tag{1.1}$$

where u is a complex-valued function on $\mathbb{T} \times \mathbb{R}$, with $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ and $\beta \in \mathbb{R}$. Equation (1.1) appears as a mathematical model for nonlinear pulse propagation phenomena in various fields of physics, in particular in nonlinear optics [19,1]. Equation (1.1) without third-order dispersion corresponds to the standard cubic nonlinear Schrödinger equation (NLS) and it has been studied extensively from both theoretical and applied points of view. In recent years, there has been an increasing interest in the cubic 3NLS (1.1) with third-order dispersion in nonlinear optics [32,23,24].

While the equation (1.1) conserves the following Hamiltonian:

$$H(u) = -\frac{1}{2}\operatorname{Im}\int_{\mathbb{T}} \partial_x^2 u\, \overline{\partial_x u}\,\mathrm{d}x + \frac{\beta}{2}\int_{\mathbb{T}} |\partial_x u|^2\,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{T}} |u|^4\,\mathrm{d}x,$$

the leading order term is sign-indefinite and hence it does not play an important role in the well-posedness theory of (1.1). On the other hand, the conservation of the mass $M(u)$ defined by

$$M(u) = \int_{\mathbb{T}} |u|^2\,\mathrm{d}x,$$

combined with local well-posedness in $L^2(\mathbb{T})$, yields the following global well-posedness of (1.1) in $L^2(\mathbb{T})$.

Proposition 1.1. *The cubic 3NLS (1.1) is globally well posed in $H^s(\mathbb{T})$ for $s \geq 0$.*

The proof of local well-posedness in $L^2(\mathbb{T})$ follows from the Fourier restriction norm method with the periodic Strichartz estimate. See [25]. We point out that Proposition 1.1 is sharp since (1.1) is ill posed below $L^2(\mathbb{T})$ in the sense of the non-existence of solutions [18,26].

In studying 3NLS (1.1) with the cubic nonlinearity, the following phase function $\phi(\bar{n})$ plays an important role:

$$\begin{aligned} \phi(\bar{n}) = \phi(n, n_1, n_2, n_3) &:= (n^3 - \beta n^2) - (n_1^3 - \beta n_1^2) + (n_2^3 - \beta n_2^2) - (n_3^3 - \beta n_3^2) \\ &= 3\,(n - n_1)\,(n - n_3)\,\big(n_1 + n_3 - \tfrac{2}{3}\beta\big), \end{aligned} \tag{1.2}$$

where the last equality holds under $n = n_1 - n_2 + n_3$. Note that when $\frac{2\beta}{3} \notin \mathbb{Z}$, the last factor never vanishes. On the other hand, when $\frac{2\beta}{3} \in \mathbb{Z}$, the last factor is identically 0 for $n_3 = -n_1 + \frac{2\beta}{3}$, $n_1 \in \mathbb{Z}$. We refer to the first case ($\frac{2\beta}{3} \notin \mathbb{Z}$) and to the second case ($\frac{2\beta}{3} \in \mathbb{Z}$) as the non-resonant case and the resonant case, respectively. In the following, we focus on the non-resonant case.

1.2. Transport property of the Gaussian measures on periodic functions

Given $s > \frac{1}{2}$, let μ_s be the mean-zero Gaussian measure on $L^2(\mathbb{T})$ with the covariance operator $2(\mathrm{Id} - \Delta)^{-s}$, formally written as[1]

$$\mathrm{d}\mu_s = Z_s^{-1} \mathrm{e}^{-\frac{1}{2}\|u\|_{H^s}^2}\,\mathrm{d}u = \prod_{n \in \mathbb{Z}} Z_{s,n}^{-1}\,\mathrm{e}^{-\frac{1}{2}\langle n \rangle^{2s} |\widehat{u}_n|^2}\,\mathrm{d}\widehat{u}_n. \tag{1.3}$$

More concretely, we can define μ_s as the induced probability measure under the map[2]

$$\omega \in \Omega \mapsto u^\omega(x) = u(x;\omega) = \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{\langle n \rangle^s} \mathrm{e}^{\mathrm{i}nx}, \tag{1.4}$$

[1] Given a function f on $\mathbb{T}$, we use both $\widehat{f}_n$ and $\widehat{f}(n)$ to denote the Fourier coefficient of f at frequency n.

[2] In the following, we drop the harmless factor of 2π when it does not play any important role.

where $\langle\cdot\rangle = (1+|\cdot|^2)^{\frac{1}{2}}$ and $\{g_n\}_{n\in\mathbb{Z}}$ is a sequence of independent standard complex-valued Gaussian random variables (i.e. $\mathrm{Var}(g_n)=2$) on a probability space $(\Omega,\mathcal{F},P)$. It is easy to see that u^ω in (1.4) lies in $H^\sigma(\mathbb{T})\setminus H^{s-\frac{1}{2}}(\mathbb{T})$ for $\sigma < s-\frac{1}{2}$, almost surely. Namely, μ_s is a Gaussian probability measure on $H^\sigma(\mathbb{T})$, $\sigma < s-\frac{1}{2}$. Moreover, for the same range of σ, the triplet (H^s, H^σ, μ_s) forms an abstract Wiener space. See [15,21].

Our main goal is to study the transport property of the Gaussian measures μ_s on Sobolev spaces under the dynamics of (1.1). We first recall the following definition of quasi-invariant measures. Given a measure space (X,μ), we say that μ is *quasi-invariant* under a transformation $T: X\to X$ if the transported measure $T_*\mu = \mu\circ T^{-1}$ and μ are equivalent, i.e. mutually absolutely continuous with respect to each other.

We now state our main result.

Theorem 1.2. *Let $\frac{2}{3}\beta \notin \mathbb{Z}$. Then, for $s > \frac{3}{4}$, the Gaussian measure μ_s is quasi-invariant under the flow of the cubic 3NLS (1.1).*

In probability theory, the transport property of Gaussian measures under linear and nonlinear transformations has been studied extensively. See, for example, [9,20,34,11,12,5,2]. On the other hand, in the field of Hamiltonian PDEs, Gaussian measures naturally appear in the construction of invariant measures associated with conservation laws such as Gibbs measures, starting with the seminal work of Bourgain [7,8]. See [30,4] for the references therein. In [36], the third author initiated the study of transport properties of Gaussian measures under the flow of a Hamiltonian PDE, where two methods were presented in establishing the quasi-invariance of the Gaussian measures μ_s as stated in Theorem 1.2. See also the subsequent work [30,31,28] on the transport property of the Gaussian measures under nonlinear Hamiltonian PDEs.

• **Method 1:** The first method is to reduce an equation under consideration so that one can apply a general criterion on the quasi-invariance of a Gaussian measure on an abstract Wiener space under a nonlinear transformation due to Ramer [34]. Essentially speaking, this result states that μ_s is quasi-invariant if the nonlinear part is $(d+\varepsilon)$-smoother than the linear part for an evolution equation posed on $\mathbb{T}^d$. Namely, the given nonlinear dynamics is basically a compact perturbation of the linear dynamics.

• **Method 2:** This method was introduced in [36] by the third author to go beyond Ramer's general argument in studying concrete examples of evolution equations. It is based on combining both PDE techniques and probabilistic techniques in an intricate manner. In particular, the crucial step in this second method is to establish an effective energy estimate (with smoothing) for the (modified) H^s-functional. See, for example, Proposition 5.1 in [36] and Proposition 6.1 in [30].

We refer readers to [29] for a brief introduction of the subject and an overview of these two methods. We point out that, in applying either method, it is essential to exhibit nonlinear smoothing for given dynamics. We also remark that the second method in general performs better than the first method. See [36,30,31]. See also Remark 1.4.

In [30], we studied the transport property of the Gaussian measure μ_s under the following cubic fourth-order nonlinear Schrödinger equation (4NLS) on $\mathbb{T}$:

$$\mathrm{i}\,\partial_t u - \partial_x^4 u = |u|^2 u. \tag{1.5}$$

Our main tool to show nonlinear smoothing in this context was normal form reductions analogous to the approach employed in [3,22,17]. In [3], Babin–Ilyin–Titi introduced a normal form approach for constructing solutions to dispersive PDEs. It turned out that this approach has various applications such as establishing unconditional uniqueness [22,17,10] and exhibiting nonlinear smoothing [13]. In applying the first method, we performed a normal form reduction to the (renormalized) equation and proved the quasi-invariance of μ_s under (1.5) for $s>1$. On the other hand, in applying the second method, we performed a normal form reduction to the equation satisfied by the (modified) H^s-energy functional and proved quasi-invariance for $s > \frac{3}{4}$ (which was later improved to the optimal range of regularity $s > \frac{1}{2}$ via an infinite iteration of normal form reductions in [28]).

Now, let us turn our attention to the cubic 3NLS (1.1). Let us first proceed as in [30] and transform the equation. It is crucial that the Gaussian measure μ_s is quasi-invariant under the transformations we consider in the following. (In fact, μ_s is invariant under these transformations. See Lemma 2.2.) Hence, it suffices to prove the quasi-invariance of μ_s under the resulting dynamics. Given $t\in\mathbb{R}$, we define a gauge transformation $\mathcal{G}_t$ on $L^2(\mathbb{T})$ by setting

$$\mathcal{G}_t[f] := \mathrm{e}^{2\mathrm{i}t \fint |f|^2} f, \tag{1.6}$$

where $\fint_{\mathbb{T}} f(x)\,\mathrm{d}x := \frac{1}{2\pi}\int_{\mathbb{T}} f(x)\,\mathrm{d}x$. Given a function $u\in C(\mathbb{R};L^2(\mathbb{T}))$, we define $\mathcal{G}$ by setting

$$\mathcal{G}[u](t) := \mathcal{G}_t[u(t)].$$

Note that $\mathcal{G}$ is invertible and its inverse is given by $\mathcal{G}^{-1}[u](t) = \mathcal{G}_{-t}[u(t)]$.

Let $u\in C(\mathbb{R};L^2(\mathbb{T}))$ be a solution to (1.1). Define $\mathbf{u}$ by

$$\mathbf{u}(t) := \mathcal{G}[u](t) = \mathrm{e}^{2\mathrm{i}t\fint |u(t)|^2} u(t). \tag{1.7}$$

Then, it follows from the mass conservation that $\mathbf{u}$ is a solution to the following renormalized 3NLS:

$$\mathrm{i}\,\partial_t \mathbf{u} - \mathrm{i}\,\partial_x^3 \mathbf{u} - \beta\,\partial_x^2 \mathbf{u} = \bigg(|\mathbf{u}|^2 - 2 \fint_{\mathbb{T}} |\mathbf{u}|^2\,\mathrm{d}x\bigg)\mathbf{u}. \tag{1.8}$$

Let $\mathbf{N}(\mathbf{u}) = (|\mathbf{u}|^2 - 2\fint_{\mathbb{T}} |\mathbf{u}|^2\,\mathrm{d}x)\mathbf{u}$ be the renormalized nonlinearity in (1.8). Then, we have

$$\begin{aligned}\mathbf{N}(\mathbf{u}) &= \sum_{n\in\mathbb{Z}} \mathrm{e}^{inx} \sum_{\substack{n=n_1-n_2+n_3\\ n\neq n_1,n_3\\ n_1+n_3\neq\frac{2\beta}{3}}} \widehat{\mathbf{u}}_{n_1}\overline{\widehat{\mathbf{u}}_{n_2}}\widehat{\mathbf{u}}_{n_3} - \sum_{n\in\mathbb{Z}} \mathrm{e}^{inx}|\widehat{\mathbf{u}}_n|^2\widehat{\mathbf{u}}_n\\ &\quad + \sum_{n\in\mathbb{Z}} \mathrm{e}^{inx} \sum_{\substack{n=n_1-n_2+n_3\\ n\neq n_1,n_3\\ n_1+n_3=\frac{2\beta}{3}}} \widehat{\mathbf{u}}_{n_1}\overline{\widehat{\mathbf{u}}_{n_2}}\widehat{\mathbf{u}}_{n_3}\\ &=: \mathbf{N}_1(\mathbf{u}) + \mathbf{N}_2(\mathbf{u}) + \mathbf{N}_3(\mathbf{u}).\end{aligned} \tag{1.9}$$

In view of (1.2), the first term corresponds to the non-resonant contribution, while the second and third terms correspond to the resonant contribution. Moreover, under the non-resonant assumption: $\frac{2\beta}{3} \notin \mathbb{Z}$, we have $\mathbf{N}_3(\mathbf{u}) \equiv 0$. See Remark 1.7 for more on the renormalized equation (1.8).

At this point, we can introduce the interaction representation v of $\mathbf{u}$ as in [30] by

$$v(t) = S(-t)\mathbf{u}(t), \tag{1.10}$$

where $S(t) = \mathrm{e}^{t(\partial_x^3 - \mathrm{i}\beta\,\partial_x^2)}$ denotes the linear solution map for 3NLS (1.1). Under the non-resonant assumption ($\frac{2}{3}\beta \notin \mathbb{Z}$), this reduces (1.8) to the following equation for $\{\widehat{v}_n\}_{n\in\mathbb{Z}}$[3]:

$$\begin{aligned}\partial_t \widehat{v}_n &= -i \sum_{\Gamma(n)} \mathrm{e}^{\mathrm{i}t\phi(\bar{n})}\widehat{v}_{n_1}\overline{\widehat{v}_{n_2}}\widehat{v}_{n_3} + \mathrm{i}|\widehat{v}_n|^2\widehat{v}_n\\ &=: \widehat{\mathcal{N}_0(v)}(n) + \widehat{\mathcal{R}_0(v)}(n),\end{aligned} \tag{1.11}$$

where the phase function $\phi(\bar{n})$ is as in (1.2) and the plane $\Gamma(n)$ is given by

$$\Gamma(n) = \big\{(n_1,n_2,n_3) \in \mathbb{Z}^3 : n = n_1 - n_2 + n_3,\ n \neq n_1, n_3,\ \text{and}\ n_1 + n_3 \neq \tfrac{2\beta}{3}\big\}. \tag{1.12}$$

In view of (1.2), we refer to the first term $\mathcal{N}_0(v)$ and the second term $\mathcal{R}_0(v)$ on the right-hand side of (1.11) as the non-resonant and resonant terms, respectively. On the one hand we do not have any smoothing on $\mathcal{R}_0(v)$ under a time integration. On the other hand, Lemma 2.6 below on the phase function $\phi(\bar{n})$ shows that there is a smoothing on the non-resonant term $\mathcal{N}_0(v)$ under a time integration. Hence by applying a normal form reduction as in [30], we can exhibit $(1+\varepsilon)$-smoothing on the nonlinear part if $s > 1$. See Lemma 2.4. Then, by invoking Ramer's result (Proposition 2.3), we conclude on the quasi-invariance of μ_s under (1.11) (and hence under (1.1); see Lemma 2.2) for $s > 1$. We first point out that the regularity $s > 1$ is optimal with respect to this argument (namely, applying Ramer's result in a straightforward manner) due to the resonant part $\mathcal{R}_0(v)$. See Remark 5.4 in [30]. Moreover, due to a weaker dispersion for 3NLS (1.1) as compared to 4NLS (1.5), the second method applied to (1.11) based on an energy estimate does not work for any $s \in \mathbb{R}$. Hence, a new idea is needed to go below $s = 1$.

To overcome this problem, we introduce another gauge transformation, which is the main new idea of this paper. Given $t \in \mathbb{R}$, we define a gauge transformation $\mathcal{J}_t$ on $L^2(\mathbb{T})$ by setting

$$\mathcal{J}_t[f] := \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}t|\widehat{f}_n|^2}\widehat{f}_n\,\mathrm{e}^{\mathrm{i}nx}. \tag{1.13}$$

Define u by

$$\mathrm{u}(t) = \mathcal{J}_t[\mathbf{u}(t)]. \tag{1.14}$$

First, by noting that $|\widehat{\mathrm{u}}_n(t)|^2 = |\widehat{\mathbf{u}}_n(t)|^2$, it follows from (1.9) and (1.14) that

[3] The non-resonant part $\mathcal{N}_0(v)$ is non-autonomous. For simplicity of notation, however, we drop the t-dependence. A similar comment applies to multilinear terms appearing in the following.

$$\partial_t\big(|\widehat{\mathrm{u}}_n|^2\big) = 2\,\mathrm{Re}(\partial_t\widehat{\mathbf{u}}_n\overline{\widehat{\mathbf{u}}_n})$$
$$= 2\,\mathrm{Im}\Bigg(\sum_{\Gamma(n)} \mathrm{e}^{\mathrm{i}t\psi(\bar{n})}\widehat{\mathrm{u}}_{n_1}\overline{\widehat{\mathrm{u}}_{n_2}}\widehat{\mathrm{u}}_{n_3}\overline{\widehat{\mathrm{u}}_n}\Bigg), \tag{1.15}$$

where the (time-dependent) phase function $\psi(\bar{n})$ is defined by

$$\psi(\bar{n}) = \psi(n,n_1,n_2,n_3)(\mathrm{u}) := -|\widehat{\mathrm{u}}_n|^2 + |\widehat{\mathrm{u}}_{n_1}|^2 - |\widehat{\mathrm{u}}_{n_2}|^2 + |\widehat{\mathrm{u}}_{n_3}|^2.$$

Then, we see that u satisfies the following equation:

$$\mathrm{i}\,\partial_t\mathrm{u} - \mathrm{i}\,\partial_x^3\mathrm{u} - \beta\,\partial_x^2\mathrm{u} = \mathrm{N}_1(\mathrm{u}) + \mathrm{N}_2(\mathrm{u}), \tag{1.16}$$

where the nonlinearities $\mathrm{N}_1(\mathrm{u})$ and $\mathrm{N}_2(\mathrm{u})$ are given by

$$\mathrm{N}_1(\mathrm{u}) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}nx} \sum_{\Gamma(n)} \mathrm{e}^{\mathrm{i}t\psi(\bar{n})}\,\widehat{\mathrm{u}}_{n_1}\overline{\widehat{\mathrm{u}}_{n_2}}\widehat{\mathrm{u}}_{n_3},$$
$$\mathrm{N}_2(\mathrm{u}) = 2t\sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}nx}\widehat{\mathrm{u}}_n\,\mathrm{Im}\Bigg(\sum_{\Gamma(n)} \mathrm{e}^{\mathrm{i}t\psi(\bar{n})}\,\widehat{\mathrm{u}}_{n_1}\overline{\widehat{\mathrm{u}}_{n_2}}\widehat{\mathrm{u}}_{n_3}\overline{\widehat{\mathrm{u}}_n}\Bigg).$$

Finally, we consider the interaction representation w of u given by

$$w(t) = S(-t)\,\mathrm{u}(t). \tag{1.17}$$

Then, the equation (1.16) is reduced to the following equation for $\{\widehat{w}_n\}_{n\in\mathbb{Z}}$:

$$\partial_t\widehat{w}_n = -\mathrm{i}\sum_{\Gamma(n)} \mathrm{e}^{\mathrm{i}t(\phi(\bar{n})+\psi(\bar{n}))}\,\widehat{w}_{n_1}\overline{\widehat{w}_{n_2}}\widehat{w}_{n_3}$$
$$\qquad - 2\mathrm{i}t\,\widehat{w}_n\,\mathrm{Im}\Bigg(\sum_{\Gamma(n)} \mathrm{e}^{\mathrm{i}t(\phi(\bar{n})+\psi(\bar{n}))}\,\widehat{w}_{n_1}\overline{\widehat{w}_{n_2}}\widehat{w}_{n_3}\overline{\widehat{w}_n}\Bigg)$$
$$=: \widehat{\mathcal{N}_1(w)}(n) + \widehat{\mathcal{N}_2(w)}(n), \tag{1.18}$$

where $\phi(\bar{n})$ is as in (1.2) and $\psi(\bar{n})$ is now expressed in terms of w:

$$\psi(\bar{n}) = \psi(n,n_1,n_2,n_3)(w) := -|\widehat{w}_n|^2 + |\widehat{w}_{n_1}|^2 - |\widehat{w}_{n_2}|^2 + |\widehat{w}_{n_3}|^2. \tag{1.19}$$

By using the additional gauge transformation $\mathcal{J}_t$, we removed the resonant part at the expense of introducing the second term $\mathcal{N}_2(w)$ in (1.18). While this second term looks more complicated, it can be handled essentially in the same manner as the non-resonant term $\mathcal{N}_1(w)$ by noting that $\widehat{\mathcal{N}_2(w)}(n)$ is basically $\widehat{\mathcal{N}_1(w)}(n)$ with two extra (harmless) factors of $\widehat{w}_n$. See Lemma 2.5. We also note that the phase function $\psi(\bar{n})$ in (1.19) depends on the time variable t, which introduces extra terms in the normal form reduction step. See (2.9) and (2.10) below. The main point is, however, that there is no resonant contribution in (1.18) and, as a result, we can show $(1+\varepsilon)$-smoothing on the nonlinear part for $\frac{3}{4} < s < 1$ (Lemma 2.5) and apply Ramer's result to conclude on the quasi-invariance of the Gaussian measure μ_s.

Given $t,\tau\in\mathbb{R}$, let $\Phi(t): L^2\to L^2$ be the solution map for (1.1) and $\Psi_0(t,\tau)$ and $\Psi_1(t,\tau): L^2\to L^2$ be the solution maps for (1.11) and (1.18), respectively, sending initial data at time τ to solutions at time t.[4] When $\tau=0$, we denote $\Psi_0(t,0)$ and $\Psi_1(t,0)$ by $\Psi_0(t)$ and $\Psi_1(t)$ for simplicity. Then, it follows from (1.7), (1.10), (1.14), and (1.17) that

$$\Phi(t) = \mathcal{G}_t^{-1}\circ S(t)\circ\Psi_0(t) \qquad\text{and}\qquad \Phi(t) = \mathcal{G}_t^{-1}\circ\mathcal{J}_t^{-1}\circ S(t)\circ\Psi_1(t).$$

As we pointed out above, the Gaussian measure μ_s is invariant under $S(t)$, $\mathcal{G}_t$, and $\mathcal{J}_t$ (Lemma 2.2) and hence it suffices to prove the quasi-invariance of μ_s under $\Psi_0(t)$ or $\Psi_1(t)$. In applying Ramer's result, we view $\Psi_0(t)$ and $\Psi_1(t)$ as the identity plus a perturbation. By writing

$$\Psi_0(t) = \mathrm{Id} + K_0(t) \qquad\text{and}\qquad \Psi_1(t) = \mathrm{Id} + K_1(t), \tag{1.20}$$

we show that $K_0(t)(u_0)$ and $K_1(t)(u_0)$ are $(1+\varepsilon)$-smoother than the random initial data u_0 distributed according to μ_s in appropriate ranges of regularities (Lemmas 2.4 and 2.5).

We conclude this introduction with several remarks.

[4] Note that (1.11) and (1.18) are non-autonomous. As in [30], this non-autonomy does not play an essential role in the remaining part of the paper.

Remark 1.3. Dispersion is essential in establishing the quasi-invariance of μ_s in Theorem 1.2. In [28], the first and third authors with Sosoe studied the transport property of μ_s under the following dispersionless model on $\mathbb{T}$:

$$\mathrm{i}\,\partial_t u = |u|^2 u \tag{1.21}$$

In particular, they showed that μ_s is not quasi-invariant under the dynamics of (1.21).

Remark 1.4. (i) We point out that the regularity restriction $s > \frac{3}{4}$ for the cubic 4NLS (1.5) in [30] was optimal in a straightforward application of the second method based on an energy estimate of the form:

$$\frac{\mathrm{d}}{\mathrm{d}t} E(u) \le C(\|u\|_{L^2}) \|u\|^{2-\theta}_{H^{s-\frac{1}{2}-\varepsilon}} \tag{1.22}$$

for some $\theta > 0$ and for any solution u to (1.5).[5] Here, $E(u) = \|u\|^2_{H^s} + R(u)$ denotes a modified H^s-energy with a suitable correction term $R(u)$ obtained via a normal form reduction applied to (the evolution equation satisfied by) $\|u\|^2_{H^s}$. Note that, in (1.22), we are allowed to place only (at most) two factors of u in the H^σ-norm with $\sigma = s - \frac{1}{2} - \varepsilon$ and need to place other factors in the conserved (weaker) L^2-norm. In particular, the regularity restriction $s > \frac{3}{4}$ (i.e. $\sigma > \frac{1}{4}$) comes from the following estimate [30, (6.14)]:

$$\Bigg\| \sum_{(m_1,m_2,m_3)\in\Gamma(n_1)} \widehat{u}_{m_1} \overline{\widehat{u}_{m_2}} \widehat{u}_{m_3} \Bigg\|_{\ell^\infty_{n_1}} \lesssim \|u\|^3_{H^{\frac{1}{6}}} \lesssim \|u\|^{1+\theta}_{L^2} \|u\|^{2-\theta}_{H^\sigma}, \tag{1.23}$$

which holds for $\sigma > \frac{1}{4}$. We point out that, by applying the first method based on Ramer's result, we can place all the factors in the H^σ-norm. See Lemmas 2.4 and 2.5

We stress that this regularity restriction $s > \frac{3}{4}$ can not be removed unless one applies an infinite iteration of normal form reductions as in [28], since, if we stop applying normal form reductions within a finite number of steps, then we would need to apply (1.23) to estimate the contribution from the trilinear terms added at the very last step. The same restriction applies to the cubic 3NLS (1.1). Namely, even if we apply the second method based on an energy estimate to the transformed equation (1.18), we can expect, at best, the same regularity range $s > \frac{3}{4}$ as in Theorem 1.2, not yielding any improvement over our proof of Theorem 1.2 based on the first method.

(ii) If we apply the second gauge transformation (1.14) to the cubic 4NLS (1.5) and apply the first method based on Ramer's argument, we can prove the quasi-invariance of μ_s for $s > \frac{2}{3}$. While this is better than the regularity restriction $s > \frac{3}{4}$ in [30], this approach does not seem to yield an optimal result ($s > \frac{1}{2}$) as in [28] in view of (2.14) below. One way in this direction would be to apply an infinite iteration of normal form reductions at the level of the equation as in [17].

Remark 1.5. After the completion of this paper, the second method based on the energy estimate has been further developed in [33,16]. In a recent preprint [14], Forlano–Trenberth applied the approaches developed in [33,16] to study the cubic fractional NLS:

$$\mathrm{i}\,\partial_t u + (-\partial_x^2)^\alpha u = |u|^2 u \tag{1.24}$$

and showed that the Gaussian measure μ_s in (1.3) is quasi-invariant under the flow of (1.24) for

$$s > \begin{cases} \max\left(\frac{2}{3}, \frac{11}{6} - \alpha\right), & \text{if } \alpha \ge 1 \\ \frac{10\alpha+7}{12}, & \text{if } \frac{1}{2} < \alpha < 1. \end{cases}$$

In particular, this shows the quasi-invariance of μ_s under the standard NLS with the second-order dispersion for $s > \frac{5}{6}$.

Remark 1.6. In [27], the second author with Nakanishi and Takaoka studied the low-regularity well-posedness of the modified KdV equation on $\mathbb{T}$. In particular, the following "gauge" transformation was used in [27, Theorem 1.3] under an extra regularity assumption $|n|^{\frac{1}{2}} \widehat{u}_n(0) \in \ell^\infty_n$:

$$\widetilde{\mathcal{J}}_{\mathrm{mKdV}}[u](t) := \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}n \int_0^t |\widehat{u}_n(t')|^2 \,\mathrm{d}t'} \widehat{u}_n(t)\, \mathrm{e}^{\mathrm{i}nx}$$

to completely remove the resonant part of the dynamics. Note that this extra regularity assumption was needed to guarantee the boundedness of the transformation $\widetilde{\mathcal{J}}_{\mathrm{mKdV}}$. Instead of $\mathcal{J}_t$ in (1.13), one may be tempted to use an analogous "gauge" transformation $\widetilde{\mathcal{J}}$ defined by

[5] We point out that one can also close an argument by establishing an energy estimate (1.22) with $\theta = 0$. See [31].

$$\widetilde{\mathcal{J}}[\mathbf{u}](t) := \sum_{n\in\mathbb{Z}} e^{-i\int_0^t |\widehat{\mathbf{u}}_n(t')|^2\,dt'}\widehat{\mathbf{u}}_n(t)\,e^{inx} \tag{1.25}$$

(for solutions $\mathbf{u}$ to (1.8)), since it would produce a simpler equation than (1.16) and (1.18). Note that, in our smooth setting, we do not need an extra regularity assumption thanks to the global well-posedness in $L^2(\mathbb{T})$ stated in Proposition 1.1.

We point out that one crucial ingredient in the proof of Theorem 1.2 is the invariance[6] of the Gaussian measure μ_s under the gauge transformation $\mathcal{J}_t$ defined in (1.13). Note that the transformation $\widetilde{\mathcal{J}}$ in (1.25) depends on the evolution on $[0,t]$; namely, it is not a well-defined gauge transformation on the phase space $L^2(\mathbb{T})$. Hence, studying the transport property of μ_s under $\widetilde{\mathcal{J}}$ (such as quasi-invariance) would already require the understanding of the transport property of μ_s under (1.8) (in particular, in a time average manner that is highly non-trivial).[7] Therefore, while the transformation $\widetilde{\mathcal{J}}$ may be of use for the low-regularity well-posedness theory, it is not suitable for our analysis.

Remark 1.7. In [26], the second author with Miyaji considered the Cauchy problem for the renormalized 3NLS (1.8) in the non-resonant case: $\frac{2\beta}{3}\notin\mathbb{Z}$. By adapting the argument in [35], they proved the local well-posedness of (1.8) in $H^s(\mathbb{T})$, $s>-\frac{1}{6}$.

It is of interest to study the transport property of the Gaussian measure μ_s in the resonant case: $\frac{2\beta}{3}\in\mathbb{Z}$. In this case, we can write $\mathbf{N}_3(\mathbf{u})$ in (1.9) as

$$\mathbf{N}_3(\mathbf{u}) = \sum_{n\in\mathbb{Z}} e^{inx}\,\overline{\widehat{\mathbf{u}}\big(-n+\tfrac{2\beta}{3}\big)}\Bigg\{\sum_{\substack{n_1\in\mathbb{Z}\\ n\neq n_1,-n_1+\frac{2\beta}{3}}}\widehat{\mathbf{u}}(n_1)\,\widehat{\mathbf{u}}\big(-n_1+\tfrac{2\beta}{3}\big)\Bigg\}.$$

Since there is no dispersion to exploit on this term, we do not have a result analogous to Theorem 1.2 in the resonant case. We also point out that the well-posedness of the renormalized 3NLS (1.8) in negative Sobolev spaces is open in the resonant case.

In the following, various constants depend on the parameter $\beta\notin\frac{3}{2}\mathbb{Z}$, but we suppress its dependence since β is fixed. In view of the time reversibility of the equation, we only consider positive times.

2. Proof of Theorem 1.2

In this section, we present the proof of Theorem 1.2 under the non-resonant assumption $\frac{2}{3}\beta\notin\mathbb{Z}$. Our basic approach is to apply Ramer's result after exhibiting sufficient smoothing on the nonlinear part. As mentioned in Section 1, we first transform the original equation (1.1) to (1.11) or (1.18). We then perform a normal form reduction and establish nonlinear smoothing by exploiting the dispersion of the equation.

We first recall the precise statement of the main result in [34] for the readers' convenience. In the following, we use $HS(H)$ to denote the space of Hilbert–Schmidt operators on H and $GL(H)$ to denote the space of invertible linear operators on H with a bounded inverse.

Proposition 2.1 *(Ramer [34]). Let (H,E,μ) be an abstract Wiener space, where μ is the standard Gaussian measure on E. Suppose that $T=\mathrm{Id}+K:U\to E$ be a continuous (nonlinear) transformation from some open subset $U\subset E$ into E such that*

(i) *T is a homeomorphism of U onto an open subset of E;*
(ii) *we have $K(U)\subset H$ and $K:U\to H$ is continuous.*
(iii) *for each $x\in U$, the map $DK(x)$ is a Hilbert–Schmidt operator on H; moreover, $DK: x\in U\to DK(x)\in HS(H)$ is continuous;*
(iv) *$\mathrm{Id}_H+DK(x)\in GL(H)$ for each $x\in U$.*

Then, μ and $\mu\circ T$ are mutually absolutely continuous measures on U.

2.1. Basic reduction

We decompose the solution map $\Phi(t)$ to (1.1) as

$$\Phi(t) = \mathcal{G}_t^{-1}\circ S(t)\circ\Psi_0(t)$$

for $s>1$ and

[6] In view of Lemma 2.2 (iii), quasi-invariance would suffice.

[7] Even in a situation where we have an invariant measure, it is not at all trivial to know how random solutions at different times are correlated. In fact, it is an important open question to study the space-time correlation of a random solution distributed by an invariant (Gibbs) measure for a dispersive PDE.

$$\Phi(t) = \mathcal{G}_t^{-1} \circ \mathcal{J}_t^{-1} \circ S(t) \circ \Psi_1(t)$$

for $\frac{3}{4} < s \le 1$. The following proposition shows that, in order to prove the quasi-invariance of the Gaussian measure μ_s under $\Phi(t)$, it suffices to establish its quasi-invariance under $\Psi_0(t)$ or $\Psi_1(t)$.

Lemma 2.2. (i) *Given a complex-valued mean-zero Gaussian random variable g with variance σ, i.e. $g \in \mathcal{N}_{\mathbb{C}}(0,\sigma)$, let $Tg = \mathrm{e}^{-\mathrm{i}t|g|^2} g$ for some $t \in \mathbb{R}$. Then, $Tg \in \mathcal{N}_{\mathbb{C}}(0,\sigma)$.*

(ii) *Let $t \in \mathbb{R}$. Then, the Gaussian measure μ_s defined in (1.3) is invariant under the linear map $S(t) = \mathrm{e}^{t(\partial_x^3 - \mathrm{i}\beta\,\partial_x^2)}$, the map $\mathcal{G}_t$ in (1.6), and the map $\mathcal{J}_t$ in (1.13).*

(iii) *Let (X,μ) be a measure space. Suppose that T_1 and T_2 are maps on X into itself such that μ is quasi-invariant under T_j for each $j = 1, 2$. Then, μ is quasi-invariant under $T = T_1 \circ T_2$.*

Proof. In view of Lemmas 4.1, 4.2, 4.4, and 4.5 in [30], it remains to prove the invariance of μ_s under $\mathcal{J}_t$. Note that μ_s can be written as an infinite product of Gaussian measures:

$$\mu_s = \bigotimes_{n\in\mathbb{Z}} \rho_n,$$

where ρ_n is the probability distribution for $\widehat{u}_n = \frac{g_n}{\langle n\rangle^s}$ defined in (1.4). In particular, ρ_n is a mean-zero Gaussian probability measure on $\mathbb{C}$ with variance $2\langle n\rangle^{-2s}$. Note that the action of $\mathcal{J}_t$ on $\widehat{u}_n$ is given by T in Part (i), which leaves the Gaussian measure ρ_n invariant. Hence, we conclude that μ_s is invariant under $\mathcal{J}_t$. $\square$

Fix $s > \frac{3}{4}$ and $\sigma < s - \frac{1}{2}$ sufficiently close to $s - \frac{1}{2}$. First, recall that μ_s is a probability measure on $H^\sigma(\mathbb{T})$. Given $R > 0$, let B_R be the open ball of radius R centered at the origin in $H^\sigma(\mathbb{T})$. We also recall from (1.20), (1.11), and (1.18) that

$$\Psi_j(t) = \mathrm{Id} + K_j(t), \quad j = 0, 1,$$

where $K_j(t)$ is given by

$$K_0(t)(u_0) = \int_0^t \mathcal{N}_0(v)(t')\,\mathrm{d}t' + \int_0^t \mathcal{R}_0(v)(t')\,\mathrm{d}t' =: \mathfrak{N}_0(v)(t) + \mathfrak{R}_0(v)(t),$$

$$K_1(t)(u_0) = \int_0^t \mathcal{N}_1(w)(t')\,\mathrm{d}t' + \int_0^t \mathcal{N}_2(w)(t')\,\mathrm{d}t' =: \mathfrak{N}_1(w)(t) + \mathfrak{N}_2(w)(t),$$

where v and w are the solutions to (1.11) and (1.18) with initial data u_0. Then, the proof of Theorem 1.2 is reduced to proving the following proposition, guaranteeing the hypotheses of Ramer's result (Proposition 2.1). See the proof of Theorem 1.2 for $s > 1$ in [30, Subsection 5.2].

Proposition 2.3. *Given $s > \frac{3}{4}$, let $j = 0$ if $s > 1$ and $j = 1$ if $\frac{3}{4} < s \le 1$. Given $R > 0$, there exists $\tau = \tau(R) > 0$ such that, for each $t \in (0, \tau(R)]$, the following statements hold:*

(i) *$\Psi_j(t)$ is a homeomorphism of B_R onto an open subset of $H^\sigma(\mathbb{T})$;*
(ii) *we have $K_j(t)(B_R) \subset H^s(\mathbb{T})$ and $K_j(t) : B_R \to H^s(\mathbb{T})$ is continuous;*
(iii) *for each $u_0 \in B_R$, the map $DK_j(t)|_{u_0}$ is a Hilbert–Schmidt operator on $H^s(\mathbb{T})$; moreover, $DK_j(t) : u_0 \in B_R \mapsto DK_j(t)|_{u_0} \in HS(H^s(\mathbb{T}))$ is continuous;*
(iv) *$\mathrm{Id}_{H^s} + DK_j(t)|_{u_0} \in GL(H^s(\mathbb{T}))$ for each $u_0 \in B_R$.*

Furthermore, arguing as in the proof of Proposition 5.3 in [30], we see that Proposition 2.3 follows once we prove the following nonlinear estimates (Lemmas 2.4 and 2.5), exhibiting $(1+\varepsilon)$-smoothing. See also Remark 2.7. The first lemma shows nonlinear smoothing for the v-equation (1.11). In particular, when $\sigma > \frac{1}{2}$, this lemma exhibits nonlinear smoothing of order $1+\varepsilon$, yielding Proposition 2.3 and hence Theorem 1.2 for $s > 1$.

Lemma 2.4. *Let $\sigma > \frac{1}{2}$. Then, we have*

$$\|\mathfrak{N}_0(v)(t)\|_{H^{\sigma+2}} \lesssim \|v(0)\|_{H^\sigma}^3 + \|v(t)\|_{H^\sigma}^3 + t \sup_{t'\in[0,t]} \|v(t')\|_{H^\sigma}^5, \tag{2.1}$$

$$\|\mathfrak{R}_0(v)(t)\|_{H^{3\sigma}} \lesssim t \sup_{t'\in[0,t]} \|v(t')\|_{H^\sigma}^3. \tag{2.2}$$

The proof of Lemma 2.4 follows closely that of Lemma 5.1 in [30]. Namely, we apply a normal form reduction to (1.11) and convert the cubic non-resonant nonlinearity into a quintic nonlinearity (plus cubic boundary terms). See (2.5) below. While we had a gain of two derivatives for 4NLS (1.5) in [30], it is not the case for our problem due to a weaker dispersion. See Lemma 2.6. On the other hand, the resonant part $\mathfrak{R}_0(v)$ is trivially estimated by $\ell_n^2 \subset \ell_n^6$ as in [30]. Note that the amount of smoothing for the resonant part $\mathfrak{R}_0(v)$ is 2σ, imposing the regularity restriction $\sigma > \frac{1}{2}$ in order to have $(1+\varepsilon)$-smoothing.

The next lemma shows nonlinear smoothing in the context of the w-equation (1.18), where the resonant part giving the regularity restriction is now removed. As in the case of Lemma 2.4, we perform a normal form reduction. However, more care is needed due to the lower regularity under consideration.

Lemma 2.5. *Let $\frac{1}{4} < \sigma \leq \frac{1}{2}$. Then, we have*

$$\|\mathfrak{N}_1(w)(t)\|_{H^{\sigma+1+}} \lesssim \sum_{j=0}^{2} t^j \sup_{t' \in [0,t]} \|w(t')\|_{H^\sigma}^{2j+3},$$

$$\|\mathfrak{N}_2(w)(t)\|_{H^{\sigma+1+}} \lesssim \sum_{j=1}^{3} t^j \sup_{t' \in [0,t]} \|w(t')\|_{H^\sigma}^{2j+3}.$$

We present the proofs of Lemmas 2.4 and 2.5 in the next subsections. Before proceeding to the proofs of the nonlinear estimates, we state the following elementary lemma on the phase function $\phi(\bar{n})$ under the non-resonant assumption $\frac{2}{3}\beta \notin \mathbb{Z}$.

Lemma 2.6. *Let $\phi(\bar{n})$ and $\Gamma(n)$ be as in (1.2) and (1.12). Then, one of the following holds on $\Gamma(n)$:*

(i) *With $n_{\max} = \max(|n|, |n_1|, |n_2|, |n_3|)$, we have*

$$|\phi(\bar{n})| \gtrsim n_{\max}^2 \lambda, \tag{2.3}$$

where $\lambda = \min\big(|n - n_1|, |n - n_3|, |n_1 + n_3 - \frac{2}{3}\beta|\big)$.

(ii) $|n| \sim |n_1| \sim |n_2| \sim |n_3|$ *and*

$$|\phi(\bar{n})| \gtrsim n_{\max}\Lambda, \tag{2.4}$$

where $\Lambda = \min\big(|n - n_1||n - n_3|, |n - n_3||n_1 + n_3 - \frac{2}{3}\beta|, |n - n_1||n_1 + n_3 - \frac{2}{3}\beta|\big)$.

The proof of Lemma 2.6 is immediate from the factorization in (1.2). See also [6, (8.21), (8.22)] for a similar property of the phase function for the modified KdV equation.

2.2. Nonlinear estimate: part 1

In this subsection, we present the proof of Lemma 2.4. Fix $\sigma > \frac{1}{2}$. By writing (1.11) in the integral form, we have

$$\begin{aligned}
\widehat{v}_n(t) &= \widehat{v}_n(0) - \mathrm{i} \int_0^t \sum_{\Gamma(n)} e^{\mathrm{i}t'\phi(\bar{n})} \widehat{v}_{n_1} \overline{\widehat{v}_{n_2}} \widehat{v}_{n_3}(t')\, \mathrm{d}t' + \mathrm{i} \int_0^t |\widehat{v}_n|^2 \widehat{v}_n(t')\, \mathrm{d}t' \\
&=: \widehat{v}_n(0) + \widehat{\mathfrak{N}_0(v)}(n,t) + \widehat{\mathfrak{R}_0(v)}(n,t).
\end{aligned}$$

In view of Lemma 2.6, we have a non-trivial oscillation caused by the phase function $\phi(\bar{n})$ in the non-resonant part $\mathfrak{N}_0(v)$. We exploit this fast oscillation by a normal form reduction, i.e. integrating by parts:

$$\begin{aligned}
\widehat{\mathfrak{N}_0(v)}(n,t) &= -\sum_{\Gamma(n)} \frac{e^{\mathrm{i}t'\phi(\bar{n})}}{\phi(\bar{n})} \widehat{v}_{n_1}(t') \overline{\widehat{v}_{n_2}(t')} \widehat{v}_{n_3}(t') \Big|_{t'=0}^{t} + \sum_{\Gamma(n)} \int_0^t \frac{e^{\mathrm{i}t'\phi(\bar{n})}}{\phi(\bar{n})} \partial_t (\widehat{v}_{n_1} \overline{\widehat{v}_{n_2}} \widehat{v}_{n_3})(t')\, \mathrm{d}t' \\
&= -\sum_{\Gamma(n)} \frac{e^{\mathrm{i}t\phi(\bar{n})}}{\phi(\bar{n})} \widehat{v}_{n_1}(t) \overline{\widehat{v}_{n_2}(t)} \widehat{v}_{n_3}(t) + \sum_{\Gamma(n)} \frac{1}{\phi(\bar{n})} \widehat{v}_{n_1}(0) \overline{\widehat{v}_{n_2}(0)} \widehat{v}_{n_3}(0) \\
&\quad + 2 \int_0^t \sum_{\Gamma(n)} \frac{e^{\mathrm{i}t'\phi(\bar{n})}}{\phi(\bar{n})} \big\{ \widehat{\mathcal{N}_0(v)}(n_1) + \widehat{\mathcal{R}_0(v)}(n_1) \big\} \overline{\widehat{v}_{n_2}} \widehat{v}_{n_3}(t')\, \mathrm{d}t'
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \sum_{\Gamma(n)} \frac{e^{it'\phi(\bar n)}}{\phi(\bar n)} \widehat{v}_{n_1} \overline{\{\widehat{\mathcal{N}_0(v)}(n_2) + \widehat{\mathcal{R}_0(v)}(n_2)\}} \widehat{v}_{n_3}(t')\,dt' \\
&=: \widehat{\mathrm{I}}(n,t) - \widehat{\mathrm{I}}(n,0) + \widehat{\mathrm{II}}(n,t) + \widehat{\mathrm{III}}(n,t).
\end{aligned} \tag{2.5}
$$

In view of Lemma 2.6, the phase function $\phi(\bar n)$ appearing in the denominators allows us to exhibit a smoothing for $\mathfrak{N}_0(v)$. In the computation above, we formally switched the order of the time integration and the summation. Moreover, we applied the product rule in time differentiation at the second equality. These steps can be justified, provided $\sigma \geq \frac{1}{6}$. See [30] for details.

We now present the proof of Lemma 2.4.

Proof of Lemma 2.4. We first estimate the non-resonant term $\mathfrak{N}_0(v)$ in (2.1). If $\phi(\bar n)$ satisfies (2.3) in Lemma 2.6, then we can proceed as in the proof of Lemma 5.1 in [30] and establish (2.1) since the proof of Lemma 5.1 in [30] only requires two gains of derivative from the phase function $\phi(\bar n)$ and the algebra property of $H^\sigma(\mathbb{T})$, $\sigma > \frac{1}{2}$. Moreover, the resonant term $\mathfrak{R}_0(v)$ in (2.2) can be estimated exactly as in [30] with $\ell_n^2 \subset \ell_n^6$. Hence, it remains to prove (2.1) under the assumption that $\phi(\bar n)$ satisfies (2.4) in Lemma 2.6. In this case, we have less gain of derivative from $\phi(\bar n)$ in the denominator and hence we need to proceed with more care. Without loss of generality, assume

$$
|\phi(\bar n)| \gtrsim n_{\max}|n-n_1||n-n_3|.
$$

Recall that we have $|n| \sim |n_1| \sim |n_2| \sim |n_3|$ in this case.

We first consider the term I. It follows from the Cauchy–Schwarz inequality that

$$
\begin{aligned}
\|\mathrm{I}(t)\|_{H^{3\sigma+1}} &\lesssim \bigg\| \langle n\rangle^{3\sigma} \sum_{\Gamma(n)} \frac{1}{|n-n_1||n-n_3|} \prod_{j=1}^3 |\widehat{v}_{n_j}(t)| \bigg\|_{\ell_n^2} \\
&\lesssim \sup_{n\in\mathbb{Z}} \bigg(\sum_{\Gamma(n)} \frac{1}{|n-n_1|^2|n-n_3|^2} \bigg)^{\frac{1}{2}} \|v(t)\|_{H^\sigma}^3 \\
&\lesssim \|v(t)\|_{H^\sigma}^3.
\end{aligned} \tag{2.6}
$$

Next, we consider II. The contribution $\mathrm{II}_{\mathrm{res}}$ from $\mathcal{R}_0(v)$ can be estimated as in (2.6). With (1.11), the Cauchy–Schwarz inequality, and $\ell_n^2 \subset \ell_n^6$, we have

$$
\begin{aligned}
\|\mathrm{II}_{\mathrm{res}}(t)\|_{H^{5\sigma+1}} &\lesssim t \sup_{t'\in[0,t]} \bigg\| \langle n\rangle^{5\sigma} \sum_{\Gamma(n)} \frac{1}{|n-n_1||n-n_3|} |\widehat{v}_{n_1}(t')|^3 \prod_{j=2}^3 |\widehat{v}_{n_j}(t')| \bigg\|_{\ell_n^2} \\
&\lesssim t \sup_{n\in\mathbb{Z}} \bigg(\sum_{\Gamma(n)} \frac{1}{|n-n_1|^2|n-n_3|^2} \bigg)^{\frac{1}{2}} \cdot \sup_{t'\in[0,t]} \|v(t')\|_{H^\sigma}^5 \\
&\lesssim t \sup_{t'\in[0,t]} \|v(t')\|_{H^\sigma}^5.
\end{aligned}
$$

We now estimate the contribution $\mathrm{II}_{\mathrm{nr}}$ from $\mathcal{N}_0(v)$ in II. Proceeding as in (2.6) with the algebra property of $H^\sigma(\mathbb{T})$, $\sigma > \frac{1}{2}$, we have

$$
\begin{aligned}
\|\mathrm{II}_{\mathrm{nr}}(t)\|_{H^{3\sigma+1}} &\lesssim t \sup_{t'\in[0,t]} \bigg\| \langle n\rangle^{3\sigma} \sum_{\substack{n=n_1-n_2+n_3\\ n\neq n_1,n_3}} \frac{1}{|n-n_1||n-n_3|} |\widehat{\mathcal{N}_0(v)}(n_1,t')| \prod_{j=2}^3 |\widehat{v}_{n_j}(t')| \bigg\|_{\ell_n^2} \\
&\lesssim t \sup_{t'\in[0,t]} \|\mathcal{N}_0(v)(t')\|_{H^\sigma} \|v(t')\|_{H^\sigma}^2 \\
&\lesssim t \sup_{t'\in[0,t]} \|v(t')\|_{H^\sigma}^5.
\end{aligned}
$$

The fourth term III in (2.5) can be estimated in an analogous manner. Finally, by noting that $3\sigma+1 > \sigma+2$ for $\sigma > \frac{1}{2}$, we conclude that the estimate (2.1) holds for $\sigma > \frac{1}{2}$. $\square$

2.3. Nonlinear estimate: part 2

In this subsection, we present the proof of Lemma 2.5. Fix $\sigma > \frac{1}{4}$. By writing (1.18) in the integral form, we have

$$\begin{aligned}\widehat{w}_n(t) &= \widehat{w}_n(0) + \int_0^t \widehat{\mathcal{N}_1(w)}(n,t')\,\mathrm{d}t' + \int_0^t \widehat{\mathcal{N}_2(w)}(n,t')\,\mathrm{d}t' \\ &=: \widehat{w}_n(0) + \widehat{\mathfrak{N}_1(w)}(n,t) + \widehat{\mathfrak{N}_2(w)}(n,t).\end{aligned} \tag{2.7}$$

As in the previous subsection, our basic strategy is to apply a normal form reduction. The main difference comes from the time-dependent nature of the phase function $\psi(\bar{n})$ in (1.19).

For a short-hand notation, we set

$$\theta(\bar{n}) = \phi(\bar{n}) + \psi(\bar{n})$$

and

$$\Xi(n,t) = 2\,\mathrm{Im}\Bigg(\sum_{(m_1,m_2,m_3)\in\Gamma(n)} \mathrm{e}^{\mathrm{i}t\theta(n,\bar{m})}\widehat{w}_{m_1}\overline{\widehat{w}_{m_2}}\widehat{w}_{m_3}\overline{\widehat{w}_n}\Bigg), \tag{2.8}$$

where $(n,\bar{m}) = (n,m_1,m_2,m_3)$. Then, from (1.18), we have

$$\begin{aligned}\widehat{\mathfrak{N}_1(w)}(n,t) &= -\mathrm{i}\int_0^t \sum_{\Gamma(n)} \partial_t\bigg(\frac{\mathrm{e}^{\mathrm{i}t'\phi(\bar{n})}}{i\phi(\bar{n})}\bigg)\mathrm{e}^{\mathrm{i}t'\psi(\bar{n})}\widehat{w}_{n_1}\overline{\widehat{w}_{n_2}}\widehat{w}_{n_3}(t')\,\mathrm{d}t' \\ &= -\sum_{\Gamma(n)}\frac{\mathrm{e}^{\mathrm{i}t\theta(\bar{n})}}{\phi(\bar{n})}\widehat{w}_{n_1}(t)\,\overline{\widehat{w}_{n_2}(t)}\widehat{w}_{n_3}(t) + \sum_{\Gamma(n)}\frac{1}{\phi(\bar{n})}\widehat{w}_{n_1}(0)\,\overline{\widehat{w}_{n_2}(0)}\widehat{w}_{n_3}(0) \\ &\quad + \mathrm{i}\int_0^t \sum_{\Gamma(n)}\frac{\mathrm{e}^{\mathrm{i}t\theta(\bar{n})}}{\phi(\bar{n})}\big\{\psi(\bar{n}) + t'\,\partial_t\psi(\bar{n})\big\}\widehat{w}_{n_1}\overline{\widehat{w}_{n_2}}\widehat{w}_{n_3}(t')\,\mathrm{d}t' \\ &\quad - 2i\int_0^t \sum_{\substack{(n_1,n_2,n_3)\in\Gamma(n)\\(m_1,m_2,m_3)\in\Gamma(n_1)}} \frac{\mathrm{e}^{\mathrm{i}t'(\theta(\bar{n})+\theta(n_1,\bar{m}))}}{\phi(\bar{n})}(\widehat{w}_{m_1}\overline{\widehat{w}_{m_2}}\widehat{w}_{m_3})\overline{\widehat{w}_{n_2}}\widehat{w}_{n_3}(t')\,\mathrm{d}t' \\ &\quad + i\int_0^t \sum_{\substack{(n_1,n_2,n_3)\in\Gamma(n)\\(m_1,m_2,m_3)\in\Gamma(n_2)}} \frac{\mathrm{e}^{\mathrm{i}t'(\theta(\bar{n})-\theta(n_2,\bar{m}))}}{\phi(\bar{n})}\widehat{w}_{n_1}(\overline{\widehat{w}_{m_1}}\widehat{w}_{m_2}\overline{\widehat{w}_{m_3}})\widehat{w}_{n_3}(t')\,\mathrm{d}t' \\ &\quad - 2i\int_0^t t'\sum_{\Gamma(n)}\frac{\mathrm{e}^{\mathrm{i}t'\theta(\bar{n})}}{\phi(\bar{n})}\Xi(n_1,t')\widehat{w}_{n_1}\overline{\widehat{w}_{n_2}}\widehat{w}_{n_3}(t')\,\mathrm{d}t' \\ &\quad + i\int_0^t t'\sum_{\Gamma(n)}\frac{\mathrm{e}^{\mathrm{i}t'\theta(\bar{n})}}{\phi(\bar{n})}\Xi(n_2,t')\widehat{w}_{n_1}\overline{\widehat{w}_{n_2}}\widehat{w}_{n_3}(t')\,\mathrm{d}t' \\ &=: \widehat{\mathrm{I}}(n,t) - \widehat{\mathrm{I}}(n,0) + \widehat{\mathrm{II}}(n,t) + \widehat{\mathrm{III}}_1(n,t) \\ &\quad + \widehat{\mathrm{III}}_2(n,t) + \widehat{\mathrm{IV}}_1(n,t) + \widehat{\mathrm{IV}}_2(n,t).\end{aligned} \tag{2.9}$$

As in (2.5), switching the order of the time integration and the summation in the computation above can be justified for $\sigma \geq \frac{1}{6}$. See [30]. Similarly, we have

$$\begin{aligned}\widehat{\mathfrak{N}_2(w)}(n,t) &= -2\mathrm{i}\int_0^t t'\widehat{w}_n\,\mathrm{Im}\Bigg(\sum_{\Gamma(n)}\partial_t\bigg(\frac{\mathrm{e}^{\mathrm{i}t'\phi(\bar{n})}}{i\phi(\bar{n})}\bigg)\mathrm{e}^{\mathrm{i}t'\psi(\bar{n})}\widehat{w}_{n_1}\overline{\widehat{w}_{n_2}}\widehat{w}_{n_3}\overline{\widehat{w}_n}\Bigg)(t')\,\mathrm{d}t' \\ &= 2\mathrm{i}t\,\widehat{w}_n\,\mathrm{Re}\Bigg(\sum_{\Gamma(n)}\frac{\mathrm{e}^{\mathrm{i}t\theta(\bar{n})}}{\phi(\bar{n})}\widehat{w}_{n_1}\overline{\widehat{w}_{n_2}}\widehat{w}_{n_3}\overline{\widehat{w}_n}\Bigg)(t)\end{aligned}$$

$$
\begin{aligned}
& - 2i \int_0^t \widehat{w}_n \operatorname{Re}\bigg(\sum_{\Gamma(n)} \frac{e^{it'\theta(\bar{n})}}{\phi(\bar{n})} e^{it'\psi(\bar{n})} \widehat{w}_{n_1} \overline{\widehat{w}_{n_2}} \widehat{w}_{n_3} \overline{\widehat{w}_n} \bigg)(t')\, dt' \\
& + 2i \int_0^t t' \widehat{w}_n \operatorname{Im}\bigg(\sum_{\Gamma(n)} \frac{e^{it'\theta(\bar{n})}}{\phi(\bar{n})} \big\{ \psi(\bar{n}) + t'\,\partial_t \psi(\bar{n}) \big\} \widehat{w}_{n_1} \overline{\widehat{w}_{n_2}} \widehat{w}_{n_3} \overline{\widehat{w}_n} \bigg)(t')\, dt' \\
& - 2i \int_0^t t' (\partial_t \widehat{w}_n) \operatorname{Re}\bigg(\sum_{\Gamma(n)} \frac{e^{it'\theta(\bar{n})}}{\phi(\bar{n})} e^{it'\psi(\bar{n})} \widehat{w}_{n_1} \overline{\widehat{w}_{n_2}} \widehat{w}_{n_3} \overline{\widehat{w}_n} \bigg)(t')\, dt' \\
& - 4i \int_0^t t' \widehat{w}_n \operatorname{Re}\bigg(\sum_{\Gamma(n)} \frac{e^{it'\theta(\bar{n})}}{\phi(\bar{n})} e^{it'\psi(\bar{n})} (\partial_t \widehat{w}_{n_1}) \overline{\widehat{w}_{n_2}} \widehat{w}_{n_3} \overline{\widehat{w}_n} \bigg)(t')\, dt' \\
& - 2i \int_0^t t' \widehat{w}_n \operatorname{Re}\bigg(\sum_{\Gamma(n)} \frac{e^{it'\theta(\bar{n})}}{\phi(\bar{n})} e^{it'\psi(\bar{n})} \widehat{w}_{n_1} (\overline{\partial_t \widehat{w}_{n_2}}) \widehat{w}_{n_3} \overline{\widehat{w}_n} \bigg)(t')\, dt' \\
& - 2i \int_0^t t' \widehat{w}_n \operatorname{Re}\bigg(\sum_{\Gamma(n)} \frac{e^{it'\theta(\bar{n})}}{\phi(\bar{n})} e^{it'\psi(\bar{n})} \widehat{w}_{n_1} \overline{\widehat{w}_{n_2}} \widehat{w}_{n_3} (\overline{\partial_t \widehat{w}_n}) \bigg)(t')\, dt' \\
& =: \widehat{\widetilde{\mathrm{I}}}(n,t) + \widehat{\widetilde{\mathrm{II}}}(n,t) + \widehat{\widetilde{\mathrm{III}}}(n,t) + \widehat{\widetilde{\mathrm{IV}}}_0(n,t) \\
& + \widehat{\widetilde{\mathrm{IV}}}_1(n,t) + \widehat{\widetilde{\mathrm{IV}}}_2(n,t) + \widehat{\widetilde{\mathrm{IV}}}_3(n,t),
\end{aligned}
\tag{2.10}
$$

Modulo the extra phase factor $\psi(\bar{n})$, the terms I, III_1, and III_2 in (2.9) already appear in (2.5). While the other terms in (2.9) and (2.10) are new, it turns out that they can be estimated in a similar manner to I, III_1, and III_2 with small modifications.

We now present the proof of Lemma 2.5.

Proof of Lemma 2.5. In the following, we first estimate the terms I, III_1, and III_2 in (2.9). We then show how the estimates for the other terms in (2.9) and (2.10) follow from those for I, III_1, and III_2.

Main argument: We first consider I in (2.9). If $\phi(\bar{n})$ satisfies (2.4), then we estimate I as in (2.6). Next, suppose that $\phi(\bar{n})$ satisfies (2.3). Without loss of generality, assume that

$$
|\phi(\bar{n})| \gtrsim n_{\max}^2 |n - n_1|. \tag{2.11}
$$

Then, by the Cauchy–Schwarz inequality with $\langle n\rangle^\sigma \lesssim \max_{j=1,2,3} \langle n_j\rangle^\sigma$ for $\sigma \geq 0$, we have

$$
\begin{aligned}
\|\mathrm{I}(t)\|_{H^{\sigma+1+}} & \lesssim \bigg\| \langle n\rangle^\sigma \sum_{\substack{n=n_1-n_2+n_3 \\ n\neq n_1,n_3}} \frac{1}{|n-n_1| n_{\max}^{1-}} \prod_{j=1}^3 |\widehat{w}_{n_j}(t)| \bigg\|_{\ell^2_n} \\
& \lesssim \sup_{n\in\mathbb{Z}} \bigg(\sum_{\substack{n=n_1-n_2+n_3 \\ n\neq n_1,n_3}} \frac{1}{|n-n_1|^2 n_{\max}^{2-}} \bigg)^{\frac{1}{2}} \|w(t)\|_{H^\sigma}^3 \\
& \lesssim \|w(t)\|_{H^\sigma}^3.
\end{aligned}
\tag{2.12}
$$

Next, we consider the fourth term III_1 in (2.9). The fifth term III_2 in (2.9) can be estimated in an analogous manner. In the following, we only estimate the integrand of III_1. With abuse of notation, we also denote the integrand as III_1.

- **Case (i):** $\phi(\bar{n})$ satisfies (2.3).

  We first consider the case that (2.11) holds. In this case, for $\frac{1}{6} < \sigma \leq \frac{1}{2}$, we have

$$
\frac{\langle n\rangle^{\sigma+1+}}{|\phi(\bar{n})|} \frac{1}{\langle n_1\rangle^{\sigma-\frac{1}{6}} \langle n_2\rangle^\sigma \langle n_3\rangle^\sigma} \lesssim \frac{1}{\langle n\rangle^{\frac{1}{2}+} |n-n_1|^{1-} \langle n_2\rangle^\sigma \langle n_3\rangle^{\frac{1}{2}+}}. \tag{2.13}
$$

By the triangle inequality: $\langle n_1\rangle^{\sigma-\frac{1}{6}} \lesssim \max_{j=1,2,3} \langle m_j\rangle^{\sigma-\frac{1}{6}}$ and Sobolev's inequality, we have

$$
\bigg\| \langle n_1\rangle^{\sigma-\frac{1}{6}} \sum_{(m_1,m_2,m_3)\in\Gamma(n_1)} \widehat{w}_{m_1} \overline{\widehat{w}_{m_2}} \widehat{w}_{m_3} \bigg\|_{\ell^\infty_{n_1}} \lesssim \|w\|^2_{H^{\frac{1}{6}}} \|w\|_{H^\sigma} \leq \|w\|^3_{H^\sigma} \tag{2.14}
$$

for $\sigma \geq \frac{1}{6}$. Then, it follows from Cauchy–Schwarz inequality with (2.13) and (2.14) that

$$\begin{aligned}\|\mathrm{III}_1\|_{H^{\sigma+1+}} &\lesssim \|w\|_{H^\sigma}^3 \Bigg\| \sum_{(n_1,n_2,n_3)\in\Gamma(n)} \frac{1}{\langle n\rangle^{\frac{1}{2}+}|n-n_1|^{1-}\langle n_2\rangle^\sigma\langle n_3\rangle^{\frac{1}{2}+}} \prod_{j=2}^{3}\langle n_j\rangle^\sigma|\widehat{w}_{n_j}|\Bigg\|_{\ell_n^2} \\ &\lesssim \|w\|_{H^\sigma}^5 \Bigg(\sum_{n,n_3\in\mathbb{Z}}\sum_{\substack{n_2\in\mathbb{Z}\\ n_2\neq n_3}} \frac{1}{\langle n\rangle^{1+}|n_2-n_3|^{2-}\langle n_2\rangle^{2\sigma}\langle n_3\rangle^{1+}}\Bigg)^{\frac{1}{2}}\end{aligned}$$

By first summing in n_2, then in n_3 and n,

$$\lesssim \|w\|_{H^\sigma}^5 \tag{2.15}$$

for $\frac{1}{6} < \sigma \leq \frac{1}{2}$. The upper bound $\sigma \leq \frac{1}{2}$ is by no means sharp, but it suffices for our purpose.

Next, suppose that

$$|\phi(\bar{n})| \gtrsim n_{\max}^2|n_1+n_3-\tfrac{2}{3}\beta| = n_{\max}^2|n+n_2-\tfrac{2}{3}\beta|.$$

In this case, we have

$$\frac{\langle n\rangle^{\sigma+1+}}{|\phi(\bar{n})|}\frac{1}{\langle n_2\rangle^\sigma\langle n_3\rangle^\sigma} \lesssim \frac{1}{\langle n\rangle^{\frac{1}{2}+}\langle n+n_2-\frac{2}{3}\beta\rangle^{1-}\langle n_2\rangle^{\sigma-}\langle n_3\rangle^{\frac{1}{2}+}}.$$

Then, we can repeat a computation analogous to (2.15) once we notice

$$\Bigg(\sum_{n,n_3\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}} \frac{1}{\langle n\rangle^{1+}\langle n+n_2-\frac{2}{3}\beta\rangle^{2-}\langle n_2\rangle^{2\sigma-}\langle n_3\rangle^{1+}}\Bigg)^{\frac{1}{2}} < \infty.$$

Similarly, we can handle the case

$$|\phi(\bar{n})| \gtrsim n_{\max}^2|n-n_3|$$

by noting

$$\frac{\langle n\rangle^{\sigma+1+}}{|\phi(\bar{n})|}\frac{1}{\langle n_2\rangle^\sigma\langle n_3\rangle^\sigma} \lesssim \frac{1}{\langle n\rangle^{\frac{1}{2}+}\langle n-n_3\rangle^{1-}\langle n_2\rangle^{\frac{1}{2}+}\langle n_3\rangle^{\sigma-}}$$

and

$$\Bigg(\sum_{n,n_2\in\mathbb{Z}}\sum_{n_3\in\mathbb{Z}} \frac{1}{\langle n\rangle^{1+}\langle n-n_3\rangle^{2-}\langle n_2\rangle^{1+}\langle n_3\rangle^{2\sigma-}}\Bigg)^{\frac{1}{2}} < \infty.$$

- **Case (ii):** $\phi(\bar{n})$ satisfies (2.4).

  In this case, we have $|n| \sim |n_1| \sim |n_2| \sim |n_3|$. We first consider the case

$$|\phi(\bar{n})| \gtrsim n_{\max}|n-n_1||n-n_3|.$$

By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\|\mathrm{III}_1\|_{H^{\sigma+1+}} &\lesssim \Bigg\| \sum_{(n_1,n_2,n_3)\in\Gamma(n)} \frac{1}{\langle n_1\rangle^{\sigma-}|n-n_1||n-n_3|} \\ &\qquad \times \sum_{(m_1,m_2,m_3)\in\Gamma(n_1)} \prod_{i=1}^{3}|\widehat{w}_{m_i}| \prod_{j=2}^{3}\langle n_j\rangle^\sigma|\widehat{w}_{n_j}|\Bigg\|_{\ell_n^2} \\ &\lesssim \|w\|_{H^\sigma}^2 \Bigg\| \frac{1}{\langle n_1\rangle^{\sigma-}\langle n-n_1\rangle\langle n-n_3\rangle} \sum_{(m_1,m_2,m_3)\in\Gamma(n_1)} \prod_{i=1}^{3}|\widehat{w}_{m_i}|\Bigg\|_{\ell_{n,n_1,n_3}^2}\end{aligned} \tag{2.16}$$

By summing in n_3 and then in n and applying Hölder inequality (in n_1) and Young's inequality (with $\frac{1-2\sigma+}{2}+2 = \frac{1}{q}+\frac{1}{q}+\frac{1}{q}$),

$$\lesssim \|w\|_{H^\sigma}^2 \Big\| \sum_{(m_1,m_2,m_3)\in\Gamma(n_1)} \prod_{i=1}^{3} |\widehat{w}_{m_i}| \Big\|_{\ell_{n_1}^{\frac{2}{1-2\sigma+}}} \lesssim \|w\|_{H^\sigma}^2 \|\widehat{w}_n\|_{\ell_n^q}^3$$

By Hölder's inequality,

$$\lesssim \|w\|_{H^\sigma}^2 \Big(\|\langle n\rangle^{-\frac{2-q}{2q}-}\|_{\ell_n^{\frac{2q}{2-q}}} \|\langle n\rangle^{\frac{2-q}{2q}+}\widehat{w}_n\|_{\ell_n^2}\Big)^3$$
$$\lesssim \|w\|_{H^\sigma}^2 \|w\|_{H^{\frac{1-\sigma+}{3}}}^3 \le \|w\|_{H^\sigma}^5,$$

provided that $\frac{1-\sigma+}{3} \le \sigma$. This gives the regularity restriction $\sigma > \frac{1}{4}$ stated in the hypothesis.

When $|\phi(\bar{n})| \gtrsim n_{\max}|n-n_1||n_1+n_3-\frac{2}{3}\beta|$ or $|\phi(\bar{n})| \gtrsim n_{\max}|n-n_3||n_1+n_3-\frac{2}{3}\beta|$, we can proceed as in the computation above but we need to first sum in n and then in n_3 the corresponding factors at (2.16).

Remaining terms: We now estimate the remaining terms. The main idea is to reduce the estimates to the main argument presented above (for $\mathrm{I}, \mathrm{III}_1$, and III_2) by noticing that the extra factors appearing in the remaining terms are all bounded in the ℓ_n^∞-norm. From (1.19), we have

$$\|\psi(\bar{n})\|_{\ell_{\bar{n}}^\infty} \lesssim \|\widehat{w}_n\|_{\ell_n^\infty}^2 \le \|w\|_{L^2}^2, \tag{2.17}$$

where $\ell_{\bar{n}}^\infty = \ell_{n,n_1,n_2,n_3}^\infty$. On the one hand, it follows from (1.15), (1.19), and (2.8) that

$$\partial_t \psi(\bar{n}) = -\Xi(n) + \Xi(n_1) - \Xi(n_2) + \Xi(n_3). \tag{2.18}$$

On the other hand, from (2.14) with $\sigma = \frac{1}{6}$, we have

$$\|\Xi(n)\|_{\ell_n^\infty} \lesssim \|\widehat{w}_n\|_{\ell_n^\infty} \|w\|_{H^{\frac{1}{6}}}^3 \le \|w\|_{H^{\frac{1}{6}}}^4. \tag{2.19}$$

Combining (2.18) and (2.19), we obtain

$$\|\partial_t \psi(\bar{n})\|_{\ell_{\bar{n}}^\infty} \lesssim \|w\|_{H^{\frac{1}{6}}}^4. \tag{2.20}$$

Hence, from the estimate (2.6) and (2.12) on I with (2.17) and (2.20), we can estimate II in (2.9) by

$$\|\mathrm{II}(t)\|_{H^{\sigma+1+}} \lesssim t \sup_{t'\in[0,t]} \|w(t')\|_{H^\sigma}^5 + t^2 \sup_{t'\in[0,t]} \|w(t')\|_{H^\sigma}^7$$

for $\sigma \ge \frac{1}{6}$. Noting that the terms IV_j, $j=1,2$, in (2.9) have the same structure as I with an extra factors of $\Xi(n_j)$, it follows from the estimates on I and (2.19) that

$$\|\mathrm{IV}_j(t)\|_{H^{\sigma+1+}} \lesssim t^2 \sup_{t'\in[0,t]} \|w(t')\|_{H^\sigma}^7$$

for $j=1,2$. Similarly, by noting that $\widetilde{\mathrm{I}}$, $\widetilde{\mathrm{II}}$, and $\widetilde{\mathrm{III}}$ in (2.10) basically have the same structure as I in (2.9) with two extra factors of $\widehat{w}_n$ (and $\psi(\bar{n}) + t'\,\partial_t\psi(\bar{n})$ for $\widetilde{\mathrm{III}}$), we have

$$\|\widetilde{\mathrm{I}}(t)\|_{H^{\sigma+1+}} \lesssim t\|w(t)\|_{H^\sigma}^5,$$
$$\|\widetilde{\mathrm{II}}(t)\|_{H^{\sigma+1+}} \lesssim t \sup_{t'\in[0,t]} \|w(t')\|_{H^\sigma}^5,$$
$$\|\widetilde{\mathrm{III}}(t)\|_{H^{\sigma+1+}} \lesssim t^2 \sup_{t'\in[0,t]} \|w(t')\|_{H^\sigma}^7 + t^3 \sup_{t'\in[0,t]} \|w(t')\|_{H^\sigma}^9.$$

As for $\widetilde{\mathrm{IV}}_0$ and $\widetilde{\mathrm{IV}}_3$ in (2.10), we first observe that

$$\|\partial_t \widehat{w}_n\|_{\ell_n^\infty} \lesssim \|w\|_{H^{\frac{1}{6}}}^3 + t\|\widehat{w}_n\|_{\ell_n^\infty}^2 \|w\|_{H^{\frac{1}{6}}}^3 \lesssim \|w\|_{H^{\frac{1}{6}}}^3 + t\|w\|_{H^{\frac{1}{6}}}^5, \tag{2.21}$$

which follows from (1.18) and (2.14). Then, by noting that $\widetilde{\mathrm{IV}}_0$ and $\widetilde{\mathrm{IV}}_3$ are basically I with extra factors of $\partial_t \widehat{w}_n$ and $\widehat{w}_n$, we obtain

$$\|\mathrm{IV}_j(t)\|_{H^{\sigma+1+}} \lesssim t^2 \sup_{t'\in[0,t]} \|w(t')\|_{H^\sigma}^7 + t^3 \sup_{t'\in[0,t]} \|w(t')\|_{H^\sigma}^9$$

for $j = 0, 3$.

It remains to consider $\widetilde{\mathrm{IV}}_j$, $j = 1, 2$, in (2.10). Note that they are basically III_j in (2.9), where we replaced

$$\sum_{(m_1,m_2,m_3)\in\Gamma(n_j)} e^{it'\theta(n_j,\bar{m})}\widehat{w}_{m_1}\overline{\widehat{w}_{m_2}}\widehat{w}_{m_3}$$

by $\partial_t \widehat{w}_{n_j}$ and added two extra factors of $\widehat{w}_n$. By a small modification of (2.21), we have

$$\|\langle n_j\rangle^{\sigma-\frac{1}{6}}\,\partial_t \widehat{w}_{n_j}\|_{\ell^\infty_{n_j}} \lesssim \|w\|^3_{H^\sigma} + t\|w\|^5_{H^\sigma} \tag{2.22}$$

for $\sigma \geq \frac{1}{6}$. Then, by repeating the computation in Case (i) above with (2.22), we obtain

$$\|\mathrm{IV}_j(t)\|_{H^{\sigma+1+}} \lesssim t^2 \sup_{t'\in[0,t]}\|w(t')\|^7_{H^\sigma} + t^3 \sup_{t'\in[0,t]}\|w(t')\|^9_{H^\sigma} \tag{2.23}$$

for $j = 1, 2$ when $\phi(\bar{n})$ satisfies (2.3). Lastly, noting from (1.18) that the contribution from $\widehat{\mathcal{N}_2(w)}(n_j)$ in $\partial_t\widehat{w}_{n_j}$ is basically $\widehat{\mathcal{N}_1(w)}(n_j)$ with two extra factors of $\widehat{w}_{n_j}$. Hence, by repeating the computation in Case (ii), we also obtain (2.23) when $\phi(\bar{n})$ satisfies (2.4).

This completes the proof of Lemma 2.5. □

Remark 2.7. As mentioned above, once we have Lemma 2.5, we can prove Proposition 2.3 with $j = 1$ and $\frac{3}{4} < s \leq 1$ by repeating the proof of Proposition 5.3 in [30]. In this case, we need to interpret the nonlinear part $K_1(t)(u_0) = \mathfrak{N}_1(w)(t) + \mathfrak{N}_2(w)(t)$ of the dynamics (2.7) as those given by the right-hand sides of (2.9) and (2.10). In particular, in computing the derivative $DK_j(t)|_{u_0}$ for $u_0 \in B_R \subset H^\sigma(\mathbb{T})$, we need to take derivatives of the complex exponentials such as $e^{it\psi(\bar{n})}$ since $\psi(\bar{n})$ depends on w. While this introduces extra terms, it does not cause any issue, since such derivatives can be easily bounded. For example, let $F(t) = e^{it\psi(\bar{n})}$. Then, with (1.19), we have

$$\begin{aligned} DF(t)|_{u_0}(\mathbf{w}(0)) &= it\,F(w)(t)D\psi(\bar{n})(t)|_{u_0}(\mathbf{w}(0)) \\ &= 2it\,F(w)(t)\,\mathrm{Re}(-\widehat{w}_n\overline{\widehat{\mathbf{w}}_n} + \widehat{w}_{n_1}\overline{\widehat{\mathbf{w}}_{n_1}} - \widehat{w}_{n_2}\overline{\widehat{\mathbf{w}}_{n_2}} + \widehat{w}_{n_3}\overline{\widehat{\mathbf{w}}_{n_3}}), \end{aligned}$$

where $\mathbf{w}$ is the solution to the linearized equation for (2.7) around the solution w to (2.7) with $w|_{t=0} = u_0$. Hence, we have

$$\big\|DF(t)|_{u_0}(\mathbf{w}(0))\big\|_{\ell^\infty_{\bar{n}}} \lesssim t\|w(t)\|_{L^2}\|\mathbf{w}(t)\|_{L^2}. \tag{2.24}$$

By combining this with (the proof of) Lemma 2.5, we obtain

$$\big\|\langle\partial_x\rangle^{\frac{1}{2}+}DK_1(t)|_{u_0}(\mathbf{w}(0))\big\|_{H^s} \lesssim \sum_{j=0}^{4} t^j \sup_{t'\in[0,t]}\|w(t')\|^{2j+2}_{H^{s-\frac{1}{2}-}}\|\mathbf{w}(t')\|_{H^{s-\frac{1}{2}-}} \tag{2.25}$$

for $\frac{3}{4} < s \leq 1$. Note that when differentiation hits the complex exponentials, (2.24) increases the value of j by 1 in the statement of Lemma 2.5, and hence we needed to include $j = 4$ in (2.25). Once we have (2.25), one can follow the argument in [30] and prove $DK_1(t)|_{u_0} \in HS(H^s(\mathbb{T}))$ for any $u_0 \in B_R \subset H^{s-\frac{1}{2}-}(\mathbb{T})$.

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