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Probability theory

SLE intersecting with random hulls

Intersection de SLE avec une enveloppe aléatoire

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ABSTRACT

Lawler, Schramm, and Werner gave in 2003 an explicit formula of the probability that SLE(8/3) does not intersect a deterministic hull. For general SLE(κ) with $\kappa \neq 8/3$, no such explicit formula has been obtained so far. In this paper, we shall consider a random hull generated by an independent chordal conformal restriction measure and obtain an explicit formula for the probability that SLE(κ) does not intersect this random hull for any $\kappa \in (0, 8)$. As a corollary, we will give a new proof of Werner's result on conformal restriction measures.

RÉSUMÉ

Lawler, Schramm et Werner ont donné en 2003 une formule explicite pour la probabilité que SLE(8/3) ne rencontre pas une enveloppe déterministe. Pour SLE(κ) avec $\kappa \neq 8/3$, aucune formule de ce type ne semble connue. Nous considérons ici une enveloppe aléatoire engendrée par une mesure de restriction conforme indépendante et nous obtenons une formule explicite de la probabilité que SLE(κ) ne la rencontre pas lorsque $\kappa \in (0, 8)$. Comme corollaire, nous donnons une nouvelle démonstration d'un résultat de Werner sur les mesures de restrictions conformes.

1. Statement of the main theorem

Oded Schramm introduced SLE processes in [10] in order to describe the scaling limit of many lattice models in statistical physics. The conformally invariant scaling limits of a series of planar lattice models have been identified as some SLE processes. These models include site percolation on the triangular graph, loop-erased random walk, Ising model, harmonic random walk, discrete Gaussian free field, FK-Ising model, uniform spanning tree, etc.

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Besides the relation to lattice models, the properties of SLE itself as a random curve have their own interest. (For example, the Hausdorff dimension of SLE(κ) is almost surely equal to $\min\{2, 1+\frac{\kappa}{8}\}$. See [1] and [9] for more properties.) In [3], the authors showed that, for a $\mathbb{H}$–hull away from 0, the probability that SLE(8/3) does not intersect A is equal to $\left(\Phi'_A(0)\right)^{5/8}$, which is the *restriction property* of SLE(8/3). But for $\kappa \neq 8/3$, no such nice formula has been obtained so far. In this paper, we will show that, if we change the deterministic hull A into a random hull generated by an independent chordal restriction measure (which is defined in [3] as a generalization of SLE(8/3)), then we have a nice formula for $\mathbb{P}(\gamma \cap A = \emptyset)$ for any $\kappa \in (0, 8)$. We first state our main theorem.

Theorem 1. *Suppose that γ is a SLE(κ) curve from 0 to ∞ in $\mathbb{H}$ with $\kappa \in (0, 8)$. For any $\alpha \geq \frac{5}{8}$, $0 < x < y$, let K be a sample of the chordal conformal restriction measure $\mathbb{P}(\alpha)$ from x to y that is independent from γ. Then*

$$\mathbb{P}(\gamma \cap K = \emptyset) = \frac{1}{C}\left(\frac{x}{y}\right)^{\delta} {}_2F_1\left(2\delta, 1-\frac{4}{\kappa}; 2\delta+\frac{4}{\kappa}; \frac{x}{y}\right), \tag{1}$$

where

$$\delta = \delta(\alpha, \kappa) = \frac{\kappa - 4 + \sqrt{(4-\kappa)^2 + 16\kappa\alpha}}{2\kappa}, \tag{2}$$

and ${}_2F_1$ is the hypergeometric function and $C = {}_2F_1\left(2\delta, 1-\frac{4}{\kappa}; 2\delta+\frac{4}{\kappa}; 1\right)$.

Remark 2. The method used to prove Theorem 1 has been used in SLE literature (see, for example [5] and [6]). By scaling invariance, the probability involved is a function φ of the ratio x/y. Therefore, by Markov's property of SLE and the scaling covariance of restriction measures, the martingale $M_t := \mathbb{P}(\gamma \cap K = \emptyset | \gamma[0,t])$ may be written as $\varphi((g_t(x) - W_t)/(g_t(y) - W_t))$ multiplied by a deterministic function of $(g_t(x) - W_t,\ g_t(y) - W_t)$. Then Ito's formula applied to M_t gives a differential equation of φ with appropriate boundary conditions. Solving the differential equation leads to the determination of φ.

Remark 3. In a first version of this paper, the theorem was stated with the range $\kappa \in (0, 4]$, but the proof is valid without change for the full range, if we interpret $\gamma \cap K = \emptyset$ as "γ swallows all the points in the hull generated by K at the same time" in the range $\kappa \in (4, 8)$.

Remark 4. As the referee pointed out to us, the formula (1) shows some similarity with Theorem 3.2 in [5], as well as with the remark at the end of paragraph therein and Lemma 3.2 of [6]. In particular, the exponent of x/y is the same in both cases, but the parameters inside the hypergeometric functions involved are different. Let us compare the two results.

We first recall the setups in [5]. Given $\kappa \in (4, 8)$ and $x \in (0, 1)$. Suppose that $\tilde{\gamma}$ is the SLE(κ) curve from $1-x$ to ∞ in $\mathbb{H}$ and g_t is the corresponding Loewner flow. Then 0 and 1 are swallowed by the SLE(κ) hull in finite time. Let T_0 and T_1 be the swallowing time of 0 and 1, respectively. Define $f_t(z) = \frac{g_t(z)-g_t(0)}{g_t(1)-g_t(0)}$ for $t < \min\{T_0, T_1\}$. Define $\Lambda(x, \alpha) := \mathbf{E}\left[1_{T_0<T_1}\left(f'_{T_0}(1)\right)^{\alpha}\right]$. By sampling an independent chordal restriction measure $\mathbb{P}(\alpha)$ (denoted by $\tilde{K}$) from 1 to ∞ in $\mathbb{H}$, one observes that

$$\Lambda(x, \alpha) = \mathbf{E}\left[\frac{1_{T_0<T_1} 1_{\tilde{K}\cap\tilde{\gamma}[0,T_0]=\emptyset}}{\left(g_{T_0}(1) - g_{T_0}(0)\right)^{\alpha}}\right]. \tag{3}$$

In [5], the authors gives an explicit formula of $\Lambda(x, \alpha)$ as follows:

$$\Lambda(x, \alpha) = \frac{1}{c_{\kappa,\alpha}} x^{\delta} {}_2F_1\left(\delta, \delta + \frac{8-\kappa}{\kappa}; 2\delta + \frac{4}{\kappa}; x\right) \tag{4}$$

where δ is the same as in (2).

Now coming to (1), performing a conformal mapping of the upper half plane onto itself, we may assume that our situation in Theorem 1 is a sample (still denoted by K) of chordal restriction measure $\mathbb{P}(\alpha)$ from 1 to ∞ and a sample of SLE(κ) (still denoted by γ) from $1-x$ to 0. Then Theorem 1 reads

$$F(x, \alpha) := \mathbf{E}[1_{\gamma\cap K=\emptyset}] = \frac{1}{C_{\kappa,\alpha}} x^{\delta} {}_2F_1\left(2\delta, 1-\frac{4}{\kappa}; 2\delta+\frac{4}{\kappa}; x\right) \tag{5}$$

Comparing (3), (4) with (5), we see that our result gives the probability that the whole part of SLE(κ) does not intersect with the chordal restriction measure and applies to $\kappa \in (0, 8)$. And the result in [5] may be explained as giving the "probability" (in fact, (4) is the probability of this event factored by an extra term) that a partial path of the SLE curve does not intersect the chordal restriction measure for $\kappa \in (4, 8)$. In this setting, it is thus clear that the two results have a close connection that could explain the fact that the exponents δ are the same, but also compute different probabilities, explaining the difference in the parameters inside the hypergeometric functions.

As a corollary, let $\kappa = \frac{8}{3}$ and $\alpha = \frac{5}{8}$ (by [3], it means that K is the SLE(8/3) curve from x to y), we get the pure partition function of two global SLE(8/3) with the patterns $\big\{\{0,\infty\},\{x,y\}\big\}$ (see definitions in [8]).

Corollary 5 *(see [8]). The pure partition function of two global* SLE(8/3) *with the patterns* $\big\{\{0,\infty\},\{x,y\}\big\}$ *is equal to*

$$(y-x)^{-\frac{5}{4}}\left(\frac{x}{y}\right)^{\frac{3}{4}}{}_2F_1\left(\frac{3}{2},-\frac{1}{2};3;\frac{x}{y}\right).$$

If we only fix $\kappa = \frac{8}{3}$, we get the following.

Corollary 6. *Suppose that γ is a* SLE(8/3) *curve from* 0 *to* ∞ *in* $\mathbb{H}$*. For any $\alpha \geq \frac{5}{8}$ and $0 < x < y$, let K be a sample of the chordal conformal restriction measure $\mathbb{P}(\alpha)$ that is independent from γ. Then*

$$\mathbb{P}(\gamma \cap K = \emptyset) = \frac{1}{C}\left(\frac{x}{y}\right)^{\delta}{}_2F_1\left(2\delta,-\frac{1}{2};2\delta+\frac{3}{2};\frac{x}{y}\right), \tag{6}$$

where

$$\delta = \delta(\alpha) = \frac{1}{4}(-1+\sqrt{1+24\alpha}), \tag{7}$$

and ${}_2F_1$ is the hypergeometric function and $C = {}_2F_1\left(2\delta,-\frac{1}{2};2\delta+\frac{3}{2};1\right)$.

Taking $x \to 0$, $y \to \infty$ in (6), we provide a new proof of the following result on conformal restriction measures due to Werner (see [11] or [7]).

Theorem 7. *Suppose that γ is a* SLE(8/3) *curve from* 0 *to* ∞ *in* $\mathbb{H}$*. Given γ, take K_2 to be a chordal conformal restriction measure $\mathbb{P}(\alpha)$ in the right connected component of $\mathbb{H}\backslash\gamma$. Let K be the union of γ, K_2 and the area between them. Then K follows the law of a chordal conformal restriction measure $\mathbb{P}(\beta(\alpha))$, where*

$$\beta(\alpha) := \alpha + \frac{3}{8} + \frac{1}{4}\sqrt{1+24\alpha}. \tag{8}$$

Remark 8. It should be noticed that we only get here a part of Werner's results, which state that one can more generally get $\mathbb{P}(\tilde{\xi}(\beta_1,\beta_2,...,\beta_p))$ from $\mathbb{P}(\beta_1),...,\mathbb{P}(\beta_p)$, $\tilde{\xi}$ being the half-plane Brownian intersection exponents (see [11]). But we use the explicit formula instead of the estimates that are used in [11].

The paper is organized as follows: background on SLE and chordal restriction measure will be given in the following section, the last one being devoted to the proof of the main theorem.

2. SLE and chordal restriction measure

In this section, basic knowledge about SLE processes and chordal conformal restriction measures will be given. In this paper, we denote by $\mathbb{H} := \{x + \mathrm{i}y \in \mathbb{C} : y > 0\}$ the upper half plane.

2.1. SLE process

The chordal SLE process from 0 to ∞ in $\mathbb{H}$ is a random family of conformal maps $(g_t : t \geq 0)$ that satisfies

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}, \quad 0 \leq t < \tau(z), \quad g_0(z) = z, \tag{9}$$

where B is a standard Brownian motion and $\tau(z)$ is the blow-up time for the differential equation (9). The SLE(κ) process is generated by a continuous curve γ, (see [9] and [2]). That is, the following limits

$$\gamma(t) = \lim_{y\downarrow 0} g_t^{-1}(\mathrm{i}y + \sqrt{\kappa}B_t),$$

exist for all $t \geq 0$ and define a continuous parametrized curve γ. The curve γ is a simple curve if and only if $\kappa \in (0,4]$, (see [9]). The SLE(κ) curve satisfies the scaling invariance property (see [9]), i.e. for any $r > 0$, $r\gamma$ has the same distribution as γ (with a time rescaling). So, given any triple set (D,a,b), where D is a simply connected domain with two given boundary points a and b, we can define the SLE process on D from a to b as the conformal map from $\mathbb{H}$ to D that sends 0 to a and ∞ to b.

2.2. Chordal conformal restriction measure

Let Ω be the collection of subsets K of $\mathbb{H}$ that satisfy the following conditions:

(1) K a connected closed set, $K \cap \mathbb{R} = \{0\}$;
(2) $\mathbb{H} \backslash K$ has two connected unbounded components.

We call a bounded subset $A \subset \overline{\mathbb{H}}$ a compact $\mathbb{H}$-*hull* if $A = \overline{A \cap \mathbb{H}}$ and $H \backslash A$ is simply connected. Let $\mathcal{A}_h$ be the collection of $\mathbb{H}$-hulls and $\mathcal{A}_h^* := \{A \in \mathcal{A}_h : 0 \notin A\}$. Suppose that $\mathcal{F}_h$ is the σ -algebra on Ω generated by the class below:

$$\left\{\{K : K \cap A = \emptyset\} : A \in \mathcal{A}_h^*\right\}.$$

For any $A \in \mathcal{A}_h^*$, by the Riemann mapping theorem, there is a unique conformal map $\Phi_A : \mathbb{H} \backslash A \mapsto \mathbb{H}$ that fixes 0, ∞ and $\lim_{z \to \infty} \Phi_A(z)/z = 1$.

Definition 9. If a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}_h)$ satisfies: for any $A \in \mathcal{A}_h^*$, conditioned on $K \cap A = \emptyset$, $\Phi_A(K)$ has the same law as K (here K is a sample of $\mathbb{P}$), we call $\mathbb{P}$ a **chordal conformal restriction measure** on $\mathbb{H}$ from 0 to ∞.

Notice that if two probability measures $\mathbb{P}$, $\mathbb{P}'$ on $(\Omega, \mathcal{F}_h)$ satisfy, for any $A \in \mathcal{A}_h^*$, $\mathbb{P}[K \cap A = \emptyset] = \mathbb{P}'[K \cap A = \emptyset]$, then $\mathbb{P} = \mathbb{P}'$.

By definition and by the analogous property of SLE(κ), the chordal conformal restriction measure satisfies the scaling invariance property, i.e. for any $r > 0$, rK has the same distribution as K. So, given any triple set (D, a, b), where D is a simply connected domain with two given boundary points a and b, we can also define the chordal conformal restriction measure on D from a to b by the conformal map from $\mathbb{H}$ to D that sends 0 to a and ∞ to b.

Theorem 10 *(see [3]). The conformal restriction measure on $\mathbb{H}$ can be characterized as follows.*

(1) *If $\mathbb{P}$ is a conformal restriction measure on $\mathbb{H}$, there exists a unique $\alpha \in \mathbb{R}$ such that for any $A \in \mathcal{A}_h^*$,*

$$\mathbb{P}[K \cap A = \emptyset] = \Phi_A'(0)^\alpha.$$

So the conformal restriction can be characterized by one parameter α, which is denoted by $\mathbb{P}(\alpha)$.
(2) *$\mathbb{P}(\alpha)$ exists if and only if $\alpha \geq \frac{5}{8}$.*

We can obtain directly from the above theorem the Radon–Nikydom derivative of the chordal conformal restriction measure of a subdomain with respect to the original domain.

First we recall briefly some facts about boundary Poisson kernel (for more details, see chapter five of [4]). For a simply connected domain Ω and two boundary points x, y lying on analytic boundary segments, the *boundary Poisson kernel* $H_\Omega(x, y)$ is characterized by the following conditions:

(1) it is conformally covariant, i.e., for any conformal map $\psi : \Omega \mapsto \psi(\Omega)$, we have $\psi'(x)\psi'(y)H_{\psi(\Omega)}(\psi(x), \psi(y)) = H_\Omega(x, y)$.
(2) For the upper half plane, if $x \neq y \in \mathbb{R}$, we have $H_{\mathbb{H}}(x, y) = \frac{1}{(x-y)^2}$.

Corollary 11. *Suppose that $U \subset \Omega$ are two simply connected domains and $x, y \in \partial U \cap \partial \Omega$. Let $\mathbb{P}(U; x, y)$ (resp. $\mathbb{P}(\Omega; x, y)$) denote the chordal conformal restriction measure in U (resp. Ω) with parameter α. Then we have*

$$\frac{\mathrm{d}\mathbb{P}(U; x, y)}{\mathrm{d}\mathbb{P}(\Omega; x, y)}(K) = 1_{K \subset U}\left(\frac{H_\Omega(x, y)}{H_U(x, y)}\right)^\alpha, \tag{10}$$

where $H_U(x, y)(H_\Omega(x, y))$ are the Poisson kernel in $U(\Omega)$.

This corollary is a consequence of Theorem 10 and of the covariance property of the boundary Poisson kernel.

3. Proof of the main theorem

In this section, we complete the proof of the theorems.

Proof of Theorem 1. Denote by $F(x, y)$ the probability that $\gamma \cap K = \emptyset$. Then by Corollary 11 and the Markov property of SLE(κ), we have

$$\tilde{M}_t := \mathbb{E}[1_{\gamma\cap K=\emptyset}|\gamma[0,t]] = \left(\frac{H_{H_t}(x,y)}{H_{\mathbb{H}}(x,y)}\right)^{\alpha} F(X_t, Y_t),$$

where $X_t := g_t(x) - \sqrt{\kappa}B_t$, $Y_t := g_t(y) - \sqrt{\kappa}B_t$. Notice that by definition $\tilde{M}_t$ is a martingale. By the conformal covariance property of the boundary Poisson kernel,

$$\tilde{M}_t = (x-y)^{2\alpha}(g_t'(x)g_t'(y))^{\alpha}(g_t(x)-g_t(y))^{-2\alpha}F(X_t, Y_t).$$

Since $(x-y)^{2\alpha}$ is a constant, we have

$$M_t := (g_t'(x)g_t'(y))^{\alpha}(g_t(x)-g_t(y))^{-2\alpha}F(X_t, Y_t),$$

is a martingale. Using Ito's formula,[1] we can get that $F(x,y)$ satisfies the following differential equation:

$$\frac{2}{x}\frac{\partial F}{\partial x} + \frac{2}{y}\frac{\partial F}{\partial y} + \frac{\kappa}{2}\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial x \partial y}\right) - 2\alpha\frac{(x-y)^2}{x^2y^2}F = 0. \tag{11}$$

Notice that $F(x,y)$ satisfies the following boundary condition:

$$\lim_{x\to 0, y\to\infty} F(x,y) = 0. \tag{12}$$

Since SLE(κ) and the chordal conformal restriction measure are scaling invariant, we have, for any $\lambda > 0$, $F(\lambda x, \lambda y) = F(x,y)$. Therefore, by defining $G(z) := F(z,1)$, we get $F(x,y) = G(\frac{x}{y})$. Then, from (11), $G(z)$ satisfies

$$\frac{\kappa}{2}z^2(z-1)G''(z) + zG'(z)\big((\kappa-2)z-2\big) - 2\alpha(z-1)G(z) = 0, \tag{13}$$

with the boundary conditions $G(0)=0$, $G(1)=1$. Suppose that $G(z)$ has the form $G(z) = z^{\delta}H(z)$ with δ taken the same as in (2). Then we have

$$z(1-z)H''(z) + \left[2\delta + \frac{4}{\kappa} - 2\left(\delta + 1 - \frac{2}{\kappa}\right)z\right]H'(z) + 2\delta\frac{\kappa-4}{\kappa}H(z) = 0. \tag{14}$$

By the appendix about hypergeometric functions in [4] and the boundary conditions $G(0)=0$, $G(1)=1$, one sees that $H(z)$ has the form

$$\frac{1}{C(\alpha)}\,{}_2F_1\left(2\delta, 1-\frac{4}{\kappa}; 2\delta + \frac{4}{\kappa}; z\right),$$

where $C = {}_2F_1(2\delta, 1-\frac{4}{\kappa}; 2\delta+\frac{4}{\kappa}; 1)$. We finish the proof. □

Proof of Theorem 7. We are ready to give the proof of Werner's theorem. For any $A \in \mathcal{A}_h^*$, let γ_1 be the SLE(8/3) in $\mathbb{H}$ from 0 to ∞ and K_2 the sample of $\mathbb{P}(\alpha)$ in $\mathbb{H}$ from x to y with $0<x<y$. Let K be the union of γ_1, K_2, and the area between them. Then combining above two lemmas, we get

$$\begin{aligned}
&\mathbb{P}(K\cap A=\emptyset|\gamma_1\cap K_2=\emptyset) = \frac{\mathbb{P}(K\cap A=\emptyset, \gamma_1\cap K_2=\emptyset)}{\mathbb{P}(\gamma_1\cap K_2=\emptyset)}\\
=&\frac{\mathbb{P}(\gamma_1\cap A=\emptyset, K_2\cap A=\emptyset)\mathbb{P}(\gamma_1\cap K_2=\emptyset|\gamma_1\cap A=\emptyset, K_2\cap A=\emptyset)}{\mathbb{P}(\gamma_1\cap K_2=\emptyset)}\\
=&\frac{\mathbb{P}(\gamma_1\cap A=\emptyset)\mathbb{P}(K_2\cap A=\emptyset)F(\phi_A(x),\phi_A(y))}{F(x,y)}\\
=&\frac{\big(\phi_A'(0)\big)^{5/8}\,\mathbb{P}(K_2\cap A=\emptyset)\left(\frac{\phi_A(x)}{\phi_A(y)}\right)^{\delta}{}_2F_1\left(2\delta,-\frac{1}{2};2\delta+\frac{3}{2};\frac{\phi_A(x)}{\phi_A(y)}\right)}{\left(\frac{x}{y}\right)^{\delta}{}_2F_1\left(2\delta,-\frac{1}{2};2\delta+\frac{3}{2};\frac{x}{y}\right)}.
\end{aligned}$$

The third equation is by the independence between γ_1 and K_2 and the conformal restriction property of γ_1 and K_2. Notice that when $x\to 0$, $y\to\infty$, $\mathbb{P}(K_2\cap A=\emptyset)$ converges to $\big(\phi_A'(0)\big)^{\alpha}$, we get

$$\lim_{x\to 0, y\to\infty}\mathbb{P}(K\cap A=\emptyset|\gamma_1\cap K_2=\emptyset) = \big(\phi_A'(0)\big)^{5/8+\alpha+\delta}.$$

[1] We assume that F is smooth to find an exact form of F and then using the optimal stopping theorem to show that this is the right form.

We can see that the limit measure follows the law of $\mathbb{P}(\beta)$, where

$$\beta = 5/8 + \alpha + \delta = \alpha + \frac{3}{8} + \frac{1}{4}\sqrt{1 + 24\alpha}.$$

This finishes the proof of Theorem 7. $\square$

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