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On the structure of diffuse measures for parabolic capacities ☆

Sur la structure des mesures diffuses des capacités paraboliques

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ABSTRACT

Let $Q = (0, T) \times \Omega$, where Ω is a bounded open subset of $\mathbb{R}^d$. We consider the parabolic p-capacity on Q naturally associated with the usual p-Laplacian. Droniou, Porretta, and Prignet have shown that if a bounded Radon measure μ on Q is diffuse, i.e. charges no set of zero p-capacity, $p > 1$, then it is of the form $\mu = f + \mathrm{div}(G) + g_t$ for some $f \in L^1(Q)$, $G \in (L^{p'}(Q))^d$ and $g \in L^p(0, T; W_0^{1,p}(\Omega) \cap L^2(\Omega))$. We show the converse of this result: if $p > 1$, then each bounded Radon measure μ on Q admitting such a decomposition is diffuse.

RÉSUMÉ

Soit $Q = (0, T) \times \Omega$, où Ω est un ouvert borné dans $\mathbb{R}^d$. On considère la p-capacité parabolique dans Q naturellement associée au p-laplacien. Droniou, Porretta et Prignet ont démontré que, si une mesure de Radon bornée μ dans Q est diffuse, c'est-à-dire si μ ne charge pas les ensembles de p-capacité nulle, elle est alors de la forme $\mu = f + \mathrm{div}(G) + g_t$, où $f \in L^1(Q)$, $G \in (L^{p'}(Q))^d$ et $g \in L^p(0, T; W_0^{1,p}(\Omega) \cap L^2(\Omega))$. Nous montrons l'inverse de ce résultat : si $p > 1$, alors toute mesure Radon bornée qui admet une telle décomposition est diffuse.

1. Introduction

Let Ω be a bounded open set in $\mathbb{R}^d$ and $Q = (0, T) \times \Omega$ for some $T > 0$. For $p > 1$, the parabolic p-capacity of an open subset U of Q is defined by (see [5,13])

$$\mathrm{cap}_p(U) = \inf\{\|u\|_W : u \in W,\ u \geq \mathbf{1}_U \text{ a.e. in } Q\},$$

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where $W=\{u\in L^p(0,T;V): u_t\in L^{p'}(0,T;V')\}$, $V=W_0^{1,p}(\Omega)\cap L^2(\Omega)$ and V' is the dual of V; we endow V with the norm $\|u\|_V=\|u\|_{W_0^{1,p}(\Omega)}+\|u\|_{L^2(\Omega)}$, and W with the norm $\|u\|_W=\|u_t\|_{L^{p'}(0,T;V')}+\|u\|_{L^p(0,T;V)}$. The capacity cap_p is then extended to an arbitrary Borel subset of Q in the usual way.

Let $\mathcal{M}_b(Q)$ denote the space of all (signed) bounded Radon measures on Q equipped with the norm $\|\mu\|_{TV}=|\mu|(Q)$, where $|\mu|$ stands for the variation of μ. We say that $\mu\in\mathcal{M}_b(Q)$ is diffuse if it charges no set of zero parabolic p-capacity, i.e. if $\mu(B)=0$ for any Borel $B\subset Q$ such that $\mathrm{cap}_p(B)=0$. We denote by $\mathcal{M}_{0,b}(Q)$ the subset of $\mathcal{M}_b(Q)$ consisting of all diffuse measures. Droniou, Poretta, and Prignet [5] have shown that for every $\mu\in\mathcal{M}_{0,b}(Q)$, there exists $f\in L^1(Q)$, $G=(G^1,\dots,G^d)$ with $G^i\in L^{p'}(Q)$, $i=1,\dots,d$, and $g\in L^p(0,T;V)$ such that

$$\mu=f+\mathrm{div}(G)+g_t. \tag{1.1}$$

The decomposition (1.1) plays a crucial role in the study of evolution problems with measure data whose model example is

$$\begin{cases} u_t-\Delta_p u+h(u)=\mu & \text{in } Q, \\ u=u_0 & \text{on } \{0\}\times\Omega, \\ u=0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{1.2}$$

where $\Delta_p u=\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the usual p-Laplace operator, $p>1$, $u_0\in L^1(\Omega)$ and $h:\mathbb{R}\to\mathbb{R}$ (see [5,9,11]).

The decomposition (1.1) is a counterpart to the decomposition of diffuse measures proved in the stationary case by Boccardo, Gallouët, and Orsina [2] (see also [7] for an extension to the Dirichlet forms setting). In the stationary case, each finite Borel measure μ on Ω that charges no set of zero p-capacity admits a decomposition of the form

$$\mu=f+\mathrm{div}(G), \tag{1.3}$$

where $f\in L^1(\Omega)$, $G=(G^1,\dots G^d)$ with $G^i\in L^{p'}(\Omega)$, $i=1,\dots,d$. The decomposition (1.3) proved to be important and useful in the study of elliptic equations with measure data (see, e.g., [3–5,8]).

In the stationary case, it is also known that if μ is a bounded Borel measure on Ω admitting decomposition (1.3), then it is diffuse (see [2] and also [7] for a related result concerning the capacity associated with a general Dirichlet operator). In the parabolic setting, only a partial result in this direction is known. The difficulty is caused by the term g_t appearing in (1.1). Petitta, Ponce, and Porretta [11] (see also [10]) have shown that, if $\mu\in\mathcal{M}_b(Q)$ admits decomposition (1.1) with g having the additional property that $g\in L^\infty(Q)$, then μ is indeed diffuse. The problem whether one can dispense with this additional assumption was left open. It is worth noting here that not every diffuse measure can be written in the form (1.1) with bounded g (see [10,11]).

In this note, we show that if $p>1$, then in the parabolic case the situation is the same as in the stationary case, i.e. if $\mu\in\mathcal{M}_b(Q)$ satisfies (1.1), then it is diffuse.

2. Main result

Define V, V', W as in Section 1. We denote by $\langle\cdot,\cdot\rangle$ the duality pairing between V' and V, and by $\langle\langle\cdot,\cdot\rangle\rangle$ the duality pairing between the dual space W' of W and W.

We start with recalling the decompositions of $\Phi\in W'$ and $\mu\in\mathcal{M}_{0,b}(Q)$ proved in [5].

Lemma 2.1. *For every $\Phi\in W'$ there exist $h\in L^{p'}(0,T;L^2(\Omega))$, $g\in L^p(0,T;V)$ $G=(G^1,\dots,G^d)$ with $G^i\in L^{p'}(Q)$, $i=1,\dots,d$ such that, for every $u\in W$,*

$$\langle\langle\Phi,u\rangle\rangle=\int_Q hu\,\mathrm{d}t\,\mathrm{d}x-\int_Q G\nabla u\,\mathrm{d}t\,\mathrm{d}x-\int_0^T\langle u_t,g\rangle\,\mathrm{d}t. \tag{2.1}$$

Proof. See [5, Lemma 2.24]. □

If $\Phi\in W'$ satisfies (2.1), then we write

$$\Phi=h+\mathrm{div}\,G+g_t.$$

Theorem 2.2. *If $\mu\in\mathcal{M}_{0,b}(Q)$, then there exists $f\in L^1(Q)$, $g\in L^p(0,T;V)$ and $G=(G^1,\dots,G^d)$ with $G^i\in L^{p'}(Q)$, $i=1,\dots,d$, such that, for every $\eta\in C_c^\infty([0,T]\times\Omega)$,*

$$\int_Q\eta\,\mathrm{d}\mu=\int_Q f\eta\,\mathrm{d}t\,\mathrm{d}x-\int_Q G\cdot\nabla\eta\,\mathrm{d}t\,\mathrm{d}x-\int_0^T\langle\eta_t,g\rangle\,\mathrm{d}t. \tag{2.2}$$

Proof. See [5, Theorem 2.28]. □

Definition. Let $\Phi \in W'$. We say that $w \in L^p(0,T;V)$ is a weak solution to the Cauchy–Dirichlet problem

$$w_t - \Delta_p w = \Phi, \qquad w(0,\cdot) = 0, \qquad w = 0 \text{ on } (0,T)\times\partial\Omega \tag{2.3}$$

if

$$-\int_0^T \langle \eta_t, w\rangle \,\mathrm{d}t + \int_Q |\nabla w|^{p-2}\nabla w\,\nabla\eta\,\mathrm{d}t\,\mathrm{d}x = \langle\langle \Phi, \eta\rangle\rangle$$

for all $\eta \in W$ with $\eta(T,\cdot) = 0$.

In what follows, $\{j_n\}$ is a family of symmetric mollifiers defined on $\mathbb{R}\times\mathbb{R}^d$. For a given $\Phi \in W'$ and a given decomposition (2.1) with h, G, g having compact supports in Q, we define (for sufficiently large $n \ge 1$) $\Phi_n \in W'$ by

$$\langle\langle \Phi_n, u\rangle\rangle = \int_Q h_n u\,\mathrm{d}t\,\mathrm{d}x - \int_Q G_n \nabla u\,\mathrm{d}t\,\mathrm{d}x - \int_0^T \langle g_n, u_t\rangle\,\mathrm{d}t, \quad u \in W, \tag{2.4}$$

where $h_n = h * j_n$, $G_n = G * j_n$ and $g_n = g * j_n$.

Proposition 2.3. *Let $\Phi \in W'$.*

(i) *There exists a unique weak solution w to (2.3).*
(ii) *Assume that Φ admits decomposition (2.1) with some h, G, g having compact supports in Q. Let Φ_n be given by (2.4) and let w_n be a weak solution to the problem*

$$(w_n)_t - \Delta_p w_n = \Phi_n, \qquad w_n(0,\cdot) = 0, \qquad w_n = 0 \textit{ on } (0,T)\times\partial\Omega.$$

Then $w_n \to w$ in $L^p(0,T;V)$.

Proof. Part (i) is proved in [5, Theorem 3.1]. To prove (ii), we modify slightly the proof of [5, Theorem 3.1]. By the definition of a weak solution and (2.4), for sufficiently large $n \ge 1$,

$$-\int_0^T \langle \eta_t, w_n - g_n\rangle\,\mathrm{d}t + \int_Q |\nabla w_n|^{p-2}\nabla w_n\,\nabla\eta\,\mathrm{d}t\,\mathrm{d}x = \int_Q h_n\eta\,\mathrm{d}t\,\mathrm{d}x + \int_0^T \langle \chi_n, \eta\rangle\,\mathrm{d}t,$$

for every $\eta \in C_c^\infty([0,T]\times D)$ such that $\eta(T) = 0$. From the above equality, it follows that $w_n - g_n \in W$ and, by a standard approximation argument, that

$$-\int_0^t \langle \eta_s, w_n - g_n\rangle\,\mathrm{d}s + (\eta(t), (w_n - g_n)(t))_{L^2(\Omega)} + \int_0^t\int_\Omega |\nabla w_n|^{p-2}\nabla w_n\,\nabla\eta\,\mathrm{d}s\,\mathrm{d}x$$
$$= \int_0^t\int_\Omega h_n\eta\,\mathrm{d}s\,\mathrm{d}x + \int_0^t \langle \chi_n, \eta\rangle\,\mathrm{d}s, \quad t \in (0,T],$$

for every $\eta \in W$. Therefore, from the proof of [5, Theorem 3.1] (see the last two equations in [5, page 131]) and [1, Lemma 5], it follows that $\nabla w_n \to \nabla w$ in $L^p(Q)$ and $w_n \to w$ in $L^p(0,T;L^2(\Omega))$. By this and [5, (3.6)] (see also the comment following it), the sequence $\{w_n - g_n\}$ is bounded in W. Therefore, by [14, Corollary 4] and uniqueness of the solution to (2.3), $w_n - g_n \to u - g$ in $L^p(Q)$. Since $g_n \to g$ in $L^p(Q)$, it follows that $w_n \to w$ in $L^p(Q)$. By what has been proved, $w_n \to w$ in $L^p(0,T;V)$. □

Lemma 2.5 below is the key to proving our main result. To state and prove it, we need some more notation.

Since cap_p is subadditive (see [5, Proposition 2.8]), each $\mu \in \mathcal{M}_b(Q)$ has a unique decomposition (see [6]) of the form

$$\mu = \mu_{\mathrm{d}} + \mu_{\mathrm{c}}, \tag{2.5}$$

where $\mu_{\rm d} \in \mathcal{M}_{0,b}(Q)$ (the diffuse part of μ) and $\mu_{\rm c} \in \mathcal{M}_b(Q)$ is concentrated on a set of zero p-capacity (the so-called concentrated part of μ). For $\mu \in \mathcal{M}_b(Q)$ with decomposition (2.5), we set

$$\mu_n = \mu * j_n, \qquad \mu_{\rm d}^n = \mu_{\rm d} * j_n, \qquad \mu_{\rm c}^n = \mu_{\rm c} * j_n.$$

We denote by $\omega(n,m)$ (resp. $\omega(n,\delta)$) any quantity such that

$$\lim_{m\to\infty} \limsup_{n\to\infty} |\omega(n,m)| = 0 \quad (\text{resp.} \quad \lim_{\delta\downarrow 0} \limsup_{n\to\infty} |\omega(n,\delta)| = 0).$$

For $m > 0$, we set $T_m(t) = ((-m)\wedge t) \vee m$, $t \in \mathbb{R}$.

Let D be an open subset of Q. We denote by $\mathcal{M}_b(D) \cap W'$ the set of elements $\Phi \in W'$ for which there exists $c > 0$ such that $|\langle\langle \Phi, \eta\rangle\rangle| \le c\|\eta\|_\infty$, $\eta \in C_c^\infty(D)$. For given $\Phi \in \mathcal{M}_b(D) \cap W'$, we denote by $\Phi^{\rm meas, D} \in \mathcal{M}_b(Q)$ the unique measure such that

$$\langle\langle \Phi, \eta\rangle\rangle = \int_D \eta \, {\rm d}\Phi^{\rm meas, D}, \quad \eta \in C_c^\infty(D)$$

(see the comments following [5, Definition 2.22]).

Remark 2.4. In the proof of Lemma 2.5, we will use [9, Lemma 5], which was proved in [9] under the assumption that $p > (2d+1)/(d+1)$. A close inspection of the proof of [9, Lemma 5] reveals that this additional assumption on p is unnecessary. The reason is that this assumption on p is needed in [9] to apply [9, Lemma 4]. However, from [5, Remark 2.3], it follows that the assertion of [9, Lemma 4] holds true for any $p > 1$.

Lemma 2.5. *Let D be an open subset of Q and $\Phi \in \mathcal{M}_b(D) \cap W'$. Assume that Φ admits decomposition (2.1) with some h, G, g having compact supports in Q and by $u_n \in L^p(0,T;V)$ denote a weak solution to the problem*

$$(u_n)_t - \Delta_p u_n = \Phi_n, \qquad u_n(0,\cdot) = 0, \qquad u_n = 0 \text{ on } (0,T)\times\partial\Omega \tag{2.6}$$

with Φ_n defined by (2.4). Then for every $\eta \in C_c^\infty(D)$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} I(n,m) = \int_D \eta \, {\rm d}(\Phi^{\rm meas, D})_c, \tag{2.7}$$

where

$$I(n,m) = \frac{1}{m} \int_{\{m \le u_n \le 2m\}} |\nabla u_n|^p \eta \, {\rm d}t \, {\rm d}x - \frac{1}{m} \int_{\{-2m \le u_n \le -m\}} |\nabla u_n|^p \eta \, {\rm d}t \, {\rm d}x.$$

Proof. Set $\nu = \Phi^{\rm meas, D}$, $\nu_n = (\Phi_n)^{\rm meas, D}$ and $\theta_m(s) = \frac{1}{m}(T_{2m}(s) - T_m(s))$, $\theta = |\theta_m|$, $\psi(s) = \theta(s) - 1$, $\Psi(t) = \int_0^t \psi(s)\,{\rm d}s$, $\Theta(t) = \int_0^t \theta(s)\,{\rm d}s$. We extend ν, ν_n to measures on Q by putting $\nu(Q\setminus D) = \nu_n(Q \setminus D) = 0$. Observe that $|\nu_n| \ll {\rm d}t \otimes {\rm d}x$, so, by a standard approximation argument, for all $w \in W$ with compact support in D,

$$\langle\langle \Phi_n, w\rangle\rangle = \int_Q w \, {\rm d}\nu_n.$$

Moreover, for every fixed $w \in W$ with compact support in D, there exists $N \ge 1$ such that

$$\int_Q w \, {\rm d}\nu_n = \int_Q w \, {\rm d}(\nu * j_n), \quad n \ge N. \tag{2.8}$$

Indeed, for sufficiently large $n \ge 1$,

$$\begin{aligned}
\int_Q w \, {\rm d}(\nu * j_n) &= \int_Q (w * j_n)\,{\rm d}\nu = \langle\langle \Phi, w * j_n\rangle\rangle \\
&= \int_Q h(w * j_n)\,{\rm d}t\,{\rm d}x - \int_Q G(\nabla w_n * j_n)\,{\rm d}t\,{\rm d}x - \int_Q (w * j_n)_t g \,{\rm d}t\,{\rm d}x \\
&= \langle\langle \Phi_n, w\rangle\rangle = \int_Q w \, {\rm d}\nu_n.
\end{aligned}$$

Let $E \subset Q$ be a Borel set such that $\mathrm{cap}_p(E) = 0$ and ν_c is concentrated on E. By regularity of the measure ν and [9, Lemma 5], for every $\delta > 0$ there exists a compact set $K_\delta \subset E$, an open set $U_\delta \subset D$ such that $K_\delta \subset U_\delta$, and $\psi_\delta \in C_c^1(U_\delta)$ with $0 \le \psi_\delta \le 1$ such that

$$|\nu|(U_\delta \setminus K_\delta) \le \delta, \qquad \int_Q (1-\psi_\delta)\,\mathrm{d}|\nu_c| \le \delta, \tag{2.9}$$

$$\|(\psi_\delta)_t\|_{L^1(Q)+L^{p'}(0,T;W^{-1,p'}(\Omega))} + \|\psi_\delta\|_{L^p(0,T;V)} \le \delta, \tag{2.10}$$

$$\psi_\delta \to 0 \quad \text{weakly}^* \text{ in } L^\infty(Q) \text{ as } \delta \downarrow 0. \tag{2.11}$$

Let $\eta \in C_c^\infty(D)$. Taking $\psi(u_n)\psi_\delta\eta$ as a test function in (2.6), we obtain

$$\int_Q \psi(u_n)\psi_\delta\eta\,\mathrm{d}\nu_n = \int_Q (u_n)_t\psi(u_n)\psi_\delta\eta\,\mathrm{d}t\,\mathrm{d}x + \int_Q |\nabla u_n|^{p-2}\nabla u_n \nabla(\psi(u_n)\psi_\delta\eta)\,\mathrm{d}t\,\mathrm{d}x =: I_1 + I_2.$$

Clearly

$$I_1 = \int_Q (\Psi(u_n))_t\psi_\delta\eta\,\mathrm{d}t\,\mathrm{d}x = -\int_Q \Psi(u_n)(\psi_\delta\eta)_t\,\mathrm{d}t\,\mathrm{d}x = -\int_Q \Psi(u_n)(\psi_\delta)_t\eta\,\mathrm{d}t\,\mathrm{d}x - \int_Q \Psi(u_n)\psi_\delta\eta_t\,\mathrm{d}t\,\mathrm{d}x.$$

Since Ψ is continuous and bounded, it follows from Proposition 2.3 and (2.10) that $I_1 = \omega(n,\delta)$. We have

$$I_2 = \int_Q |\nabla u_n|^p\psi'(u_n)\psi_\delta\eta\,\mathrm{d}t\,\mathrm{d}x + \int_Q |\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\delta\psi(u_n)\eta\,\mathrm{d}t\,\mathrm{d}x + \int_Q |\nabla u_n|^{p-2}\nabla u_n\psi(u_n)\nabla\eta\psi_\delta\,\mathrm{d}t\,\mathrm{d}x. \tag{2.12}$$

Using Proposition 2.3 and (2.11) shows that $\int_Q |\nabla u_n|^p\psi'(u_n)\psi_\delta\eta\,\mathrm{d}t\,\mathrm{d}x = \omega(n,\delta)$. Applying Hölder's inequality, Proposition 2.3 and (2.10) also shows that the last two integrals on the right-hand side of (2.12) are quantities of the form $\omega(n,\delta)$. Hence, $I_2 = \omega(n,\delta)$, and consequently

$$\int_Q \psi(u_n)\psi_\delta\eta\,\mathrm{d}\nu_n = \omega(n,\delta). \tag{2.13}$$

Since $K_\delta \subset E$, $\mathrm{cap}_p(K_\delta) = 0$. Therefore, by (2.9), $|\nu_d|(U_\delta) = |\nu_d|(U_\delta \setminus K_\delta) \le \delta$. We also have $|\int_Q \psi(u_n)\psi_\delta\eta\,\mathrm{d}\nu_d^n| \le \|\eta\|_\infty \int_Q \psi_\delta\,\mathrm{d}|\nu_d|^n$ with $|\nu_d|^n = |\nu_d| * j_n$, which converges to $\|\eta\|_\infty \int_Q \psi_\delta d|\nu_d|$ as $n \to \infty$ since $|\nu_d|^n \to |\nu_d|$ locally weakly *. Thus $\int_Q \psi(u_n)\psi_\delta\eta\,\mathrm{d}\nu_d^n = \omega(n,\delta)$. By this, (2.8) and (2.13),

$$\int_Q \psi(u_n)\psi_\delta\eta\,\mathrm{d}\nu_c^n = \omega(n,\delta). \tag{2.14}$$

Taking $\theta(u_n)\eta$ as a test function in (2.6), we obtain:

$$\begin{aligned}\int_Q \theta(u_n)\eta\,\mathrm{d}\nu_n &= \int_Q (u_n)_t\theta(u_n)\eta\,\mathrm{d}t\,\mathrm{d}x + \int_Q |\nabla u_n|^{p-2}\nabla u_n\nabla(\theta(u_n)\eta)\,\mathrm{d}t\,\mathrm{d}x \\ &= \int_Q (\Theta(u_n))_t\eta\,\mathrm{d}t\,\mathrm{d}x + \int_Q |\nabla u_n|^p\theta'(u_n)\eta\,\mathrm{d}t\,\mathrm{d}x \\ &\quad + \int_Q |\nabla u_n|^{p-2}\nabla u_n\theta(u_n)\nabla\eta\,\mathrm{d}t\,\mathrm{d}x.\end{aligned} \tag{2.15}$$

By the definition of θ,

$$\int_Q |\nabla u_n|^p \theta'(u_n)\eta \,\mathrm{d}t\,\mathrm{d}x = I(n,m).$$

We have

$$\Big|\int_Q (\Theta(u_n))_t \eta \,\mathrm{d}t\,\mathrm{d}x\Big| = \Big|\int_Q \Theta(u_n)\eta_t \,\mathrm{d}t\,\mathrm{d}x\Big| \le \int_{\{|u_n|\ge m\}} |u_n||\eta_t|\,\mathrm{d}t\,\mathrm{d}x$$

and

$$\Big|\int_Q |\nabla u_n|^{p-2}\nabla u_n \theta(u_n)\nabla\eta \,\mathrm{d}t\,\mathrm{d}x\Big| \le \int_{\{|u_n|\ge m\}} |\nabla u_n|^{p-1}|\nabla\eta|\,\mathrm{d}t\,\mathrm{d}x,$$

so by Proposition 2.3,

$$\int_Q (\Theta(u_n))_t \eta \,\mathrm{d}t\,\mathrm{d}x + \int_Q |\nabla u_n|^{p-2}\nabla u_n \theta(u_n)\nabla\eta \,\mathrm{d}t\,\mathrm{d}x = \omega(n,m).$$

By the above and (2.15),

$$I(n,m) = \int_Q \theta(u_n)\eta \,\mathrm{d}\nu_n + \omega(n,m). \tag{2.16}$$

By [11, Theorem 1.2, Proposition 3.3],

$$\Big|\int_Q \theta(u_n)\eta \,\mathrm{d}\nu_d^n\Big| \le \|\eta\|_\infty \int_{\{|u_n|\ge m\}} \mathrm{d}|\nu_d|^n = \omega(n,m). \tag{2.17}$$

Furthermore, by the definition of ψ,

$$\int_Q \theta(u_n)\eta \,\mathrm{d}\nu_c^n = \int_Q \eta \,\mathrm{d}\nu_c^n + \int_Q \psi(u_n)\eta \,\mathrm{d}\nu_c^n, \tag{2.18}$$

and by (2.9) and (2.14),

$$\int_Q \psi(u_n)\eta \,\mathrm{d}\nu_c^n = \int_Q \psi(u_n)\eta(1-\psi_\delta) \,\mathrm{d}\nu_c^n + \int_Q \psi(u_n)\eta\psi_\delta \,\mathrm{d}\nu_c^n = \omega(n,\delta). \tag{2.19}$$

Since $\int_Q \theta(u_n)\eta \,\mathrm{d}\nu_n$ does not depend on δ, from (2.8) and (2.17)–(2.19), we conclude that

$$\int_Q \theta(u_n)\eta \,\mathrm{d}\nu_n = \int_Q \eta \,\mathrm{d}\nu_c^n + \omega(n,m).$$

Combining this with (2.16) we see that

$$I(n,m) = \int_Q \eta \,\mathrm{d}\nu_c^n + \omega(n,m),$$

which implies (2.7). $\square$

In case Φ is positive, Lemma 2.5 is essentially [12, Proposition 5]. Note that [12, Proposition 5] is proved for any positive $\Phi \in \mathcal{M}_b(Q)$. In Lemma 2.5, we drop the assumption that Φ is positive, but we additionally assume that $\Phi \in W'$.

Theorem 2.6. *Let $\mu \in \mathcal{M}_b(Q)$. If (2.2) is satisfied for all $\eta \in C_c^\infty(Q)$, then $\mu \in \mathcal{M}_{0,b}(Q)$.*

Proof. Let $\nu = \mu - f\,\mathrm{d}t\,\mathrm{d}x$ and $\Phi = \mathrm{div}(G) + g_t$, i.e.

$$\langle\langle \Phi, \eta\rangle\rangle = -\int_Q G\nabla\eta\,\mathrm{d}t\,\mathrm{d}x - \int_0^T \langle \eta_t, g\rangle\,\mathrm{d}t, \quad \eta \in W.$$

Clearly $\Phi \in W'$. By (2.2), $\langle\langle \Phi, \eta\rangle\rangle = \int_Q \eta\,\mathrm{d}\mu - \int_Q \eta f\,\mathrm{d}t\,\mathrm{d}x$ for $\eta \in C_c^\infty(Q)$. From this and the assumption that $\mu \in \mathcal{M}_b(Q)$ it follows that $\Phi \in \mathcal{M}_b(Q) \cap W'$ and $\Phi^{\mathrm{meas},Q} = \nu$. Fix an open subset of Q such that $\bar{D} \subset Q$ and choose a nonnegative function $\theta \in C_c^\infty(Q)$ such that $\theta = 1$ on D. Set $G^\theta = G\theta$, $g^\theta = g\theta$, and then $\Phi^\theta = \mathrm{div}(G^\theta) + (g^\theta)_t$, i.e.

$$\langle\langle \Phi^\theta, \eta\rangle\rangle = -\int_Q G^\theta\nabla\eta\,\mathrm{d}t\,\mathrm{d}x - \int_0^T \langle \eta_t, g^\theta\rangle\,\mathrm{d}t, \quad \eta \in W.$$

Next, set $G_n^\theta = G^\theta * j_n$, $g_n^\theta = g^\theta * j_n$, and then $\Phi_n^\theta = \mathrm{div}(G_n^\theta) + (g_n^\theta)_t$, i.e.

$$\langle\langle \Phi_n^\theta, \eta\rangle\rangle = -\int_Q G_n^\theta\nabla\eta\,\mathrm{d}t\,\mathrm{d}x - \int_0^T \langle \eta_t, g_n^\theta\rangle\,\mathrm{d}t, \quad \eta \in W. \tag{2.20}$$

Clearly, $\Phi^\theta, \Phi_n^\theta \in W'$. Since $\theta = 1$ on D, we have $\langle\langle \Phi^\theta, \eta\rangle\rangle = \langle\langle \Phi, \eta\rangle\rangle$ for $\eta \in C_c^\infty(D)$, so $\Phi^\theta \in \mathcal{M}_b(D) \cap W'$. Integrating by parts, we conclude from (2.20) that $\Phi_n^\theta \in \mathcal{M}_b(D) \cap W'$. Moreover,

$$(\Phi^\theta)^{\mathrm{meas},D} = \nu_{|D}, \tag{2.21}$$

where $\nu_{|D}$ denotes the restriction of ν to D. Indeed, for $\eta \in C_c^\infty(D)$, we have $\langle\langle \Phi^\theta, \eta\rangle\rangle = \int_D \eta\,\mathrm{d}(\Phi^\theta)^{\mathrm{meas},D}$, and on the other hand, $\langle\langle \Phi^\theta, \eta\rangle\rangle = \langle\langle \Phi, \eta\rangle\rangle = \int_D \eta\,\mathrm{d}\nu = \int_D \eta\,\mathrm{d}\nu_{|D}$. Let u_n be a solution to (2.6) with Φ_n replaced by Φ_n^θ. From Proposition 2.3 it follows that $\sup_{n\ge 1} \|u_n\|_{L^p(0,T;V)} < \infty$. Hence, for every $\eta \in C_c^\infty(Q)$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{1}{m} \int_{\{m\le |u_n|\le 2m\}} |\nabla u_n|^p \eta\,\mathrm{d}t\,\mathrm{d}x \le \|\eta\|_\infty \lim_{m\to\infty} \limsup_{n\to\infty} \frac{1}{m} \|u_n\|^p_{L^p(0,T;V)} = 0.$$

By Lemma 2.5 and (2.21), this implies that $(\nu_{|D})_c = 0$. Hence $(\mu_c)_{|D} = (\mu_{|D})_c = 0$ since $f\,\mathrm{d}t\,\mathrm{d}x \in \mathcal{M}_{0,b}(Q)$. Since D was an arbitrary open subset of Q with $\bar{D} \subset Q$, we see that $\mu_c = 0$. $\square$

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