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Complex analysis

Coefficient estimates and integral mean estimates for certain classes of analytic functions

Estimations de coefficients et de valeurs moyennes pour certaines classes de fonctions analytiques

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ABSTRACT

In this article, we consider a class of functions that are subordinate to certain convex functions in one direction and determine the closed convex hull and its extreme points for functions in this class. Using these results, we solve two extremal problems, namely, coefficient estimates and L^p mean estimates for functions in this class.

RÉSUMÉ

Nous considérons dans cette Note une classe de fonctions subordonnées à certaines fonctions convexes dans une direction, dont nous déterminons l'enveloppe convexe fermée et les points extrêmes. À l'aide de ces résultats, nous résolvons deux problèmes extrémaux, à savoir des estimations de coefficients et des estimations de moyenne L^p pour les fonctions de cette classe.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Then $\mathcal{H}$ is a locally convex topological vector space endowed with the topology of uniform convergence over compact subsets of $\mathbb{D}$. Let $\mathcal{S}$ denote the class of functions $f \in \mathcal{H}$ that are univalent (i.e. one-to-one) in the unit disk $\mathbb{D}$ with the normalization $f(0) = 0 = f'(0) - 1$. If $f \in \mathcal{S}$, then $f(z)$ has the following representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

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Recently, Aleman and Constantin [2] provided a nice connection between univalent function theory and fluid dynamics. They seek explicit solutions to the incompressible two-dimensional Euler equations by means of a univalent harmonic map. More precisely, the problem of finding all solutions describing the particle paths of the flow in Lagrangian variables is reduced to find harmonic functions satisfying an explicit nonlinear differential system in $\mathbb{C}^n$ with $n = 3$ or $n = 4$ (see also [6]).

A function $f \in \mathcal{S}$ is said to belong to the class $\mathcal{S}^*(\alpha)$ where $0 \le \alpha < 1$, called starlike functions of order α, if $\operatorname{Re}(zf'(z)/f(z)) > \alpha$ for $z \in \mathbb{D}$, and is said to belong to the class $\mathcal{C}(\alpha)$ where $0 \le \alpha < 1$, called convex functions of order α, if $\operatorname{Re}(1 + zf''(z)/f'(z)) > \alpha$ for $z \in \mathbb{D}$. The classes $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{C} := \mathcal{C}(0)$ are the familiar classes of starlike and convex functions, respectively. From the above, it is easy to see that $f \in \mathcal{C}(\alpha)$ if, and only if, $zf' \in \mathcal{S}^*(\alpha)$. A function $f \in \mathcal{A}$ is said to be close to convex if there exists a starlike function $g \in \mathcal{S}^*$ and a real number $\alpha \in (-\pi/2, \pi/2)$, such that

$$\operatorname{Re}\left(e^{i\alpha}\frac{zf'(z)}{g(z)}\right) > 0, \quad z \in \mathbb{D}.$$

Let $\mathcal{K}$ denote the class of all close-to-convex functions. It is well known that every close-to-convex function is univalent in $\mathbb{D}$ (see [7]). Geometrically, $f \in \mathcal{K}$ means that the complement of the image domain $f(\mathbb{D})$ is the union of rays that are disjoint (except that the origin of one ray may lie on another one of the rays). These standard classes are related by the proper inclusions $\mathcal{C} \subsetneq \mathcal{S}^* \subsetneq \mathcal{K} \subsetneq \mathcal{S}$.

Let $\mathcal{F}$ denote the class of locally univalent analytic functions f of the form (1.1) that satisfy the condition

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2} \quad \text{for } z \in \mathbb{D}.$$

Functions in the class $\mathcal{F}$ are known to be close to convex, but are not necessarily starlike in $\mathbb{D}$. Indeed, functions in $\mathcal{F}$ are convex in one direction (see [17]). The importance of the class $\mathcal{F}$ in the case of certain univalent harmonic mappings has been discussed in [5]. The region of variability for functions in the class $\mathcal{F}$ has been studied by Ponnusamy and Vasudevarao [13].

Suppose that X is a linear topological space and $U \subseteq X$. The closed convex hull of U, denoted by $\overline{co}\,U$, is defined as the intersection of all closed convex sets containing U. For $U \subseteq V \subseteq X$, we say that U is an extremal subset of V if $u = tx + (1-t)y$, where $u \in U$, $x, y \in V$ and $0 < t < 1$ then x and y both belong to U. An extremal subset of U consisting of just one point is called an extreme point of U. We denote the set of all extreme points of U by $E(U)$. For a general reference and for many important results on this topic, we refer the reader to [9]. Extreme points of the classes $\mathcal{S}^*(\alpha)$, $\mathcal{C}(\alpha)$ and $\mathcal{K}$ are well known in the literature (see [3,4]). Recently, Abu Muhana et al. [1] obtained the set of extreme points for the class $\mathcal{F}$. As a first step for application of the knowledge of extreme points of these classes, Brickman et al. [3] pointed out the following interesting general results.

Theorem A. *Let $\mathcal{G}$ be a compact subset of $\mathcal{H}$ and J be a complex-valued continuous linear functional on $\mathcal{H}$. Then* $\max\{\operatorname{Re} J(f) : f \in \overline{co}\,\mathcal{G}\} = \max\{\operatorname{Re} J(f) : f \in \mathcal{G}\} = \max\{\operatorname{Re} J(f) : f \in E(\overline{co}\,\mathcal{G})\}$.

Theorem B. *Let $\mathcal{G}$ be a compact subset of $\mathcal{H}$ and J be a real-valued, continuous and convex functional on $\overline{co}\,\mathcal{G}$. Then* $\max\{J(f) : f \in \overline{co}\,\mathcal{G}\} = \max\{J(f) : f \in \mathcal{G}\} = \max\{J(f) : f \in E(\overline{co}\,\mathcal{G})\}$.

The proof of Theorem A and Theorem B can be found in [9, Theorem 4.5, Theorem 4.6]. In order to solve such linear extremal problems over $\mathcal{G}$, it suffices to solve them over the smaller class $E(\overline{co}\,\mathcal{G})$. This reduction thereby becomes an effective technique for solving various linear extremal problems. Using this technique, we solve two extremal problems, namely, coefficient estimates and integral mean estimates for functions associated with the class $\mathcal{F}$ through subordination or majorization.

Let f and g be two analytic functions in $\mathbb{D}$. We say that f is subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function $\omega : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$ such that $f(z) = g(\omega(z))$ for $z \in \mathbb{D}$. Furthermore, if g is univalent in $\mathbb{D}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. We say that f is majorized by g in $\mathbb{D}$ if $|f(z)| \le |g(z)|$ for each $z \in \mathbb{D}$. In other words, f is majorized by g in $\mathbb{D}$ if there exists an analytic function $\omega : \mathbb{D} \to \overline{\mathbb{D}}$ such that $f = \omega g$. If $\mathcal{G} \subseteq \mathcal{H}$, we use the notation $s(\mathcal{G}) = \{f : f \prec g \text{ for some } g \in \mathcal{G}\}$ and $m(\mathcal{G}) = \{f : f \text{ is majorized by } g \text{ for some } g \in \mathcal{G}\}$. If $\mathcal{G}$ is a compact subset of $\mathcal{H}$, then it is not very difficult to show that both $s(\mathcal{G})$ and $m(\mathcal{G})$ are compact subsets of $\mathcal{H}$ (for instance, see [9, Lemma 5.19]).

The coefficient bounds for the families $s(\mathcal{C})$ and $s(\mathcal{K})$ were first obtained by Rogosinski [15] and Robertson [14], respectively. For the investigation of coefficient bounds for the families $s(\mathcal{S}^*)$, $m(\mathcal{S}^*)$, $s(\mathcal{C})$, $m(\mathcal{C})$, $s(\mathcal{K})$, and $m(\mathcal{K})$ with arguments using extreme point methods, we refer the reader to [8,9,11,12]. Another problem that has an independent interest in the theory of univalent functions is the estimation of the L^p mean for certain classes of analytic functions. Corresponding to each analytic function f in $\mathbb{D}$, we let

$$J(f)=\frac{1}{2\pi}\int_0^{2\pi}|f^{(n)}(re^{i\theta})|^p\,d\theta, \tag{1.2}$$

where $0<r<1$, $p>0$ and $n=0,1,2,\ldots$. We shall be interested in maximizing $J(f)$ over certain families of functions. It is generally more convenient to consider the functional $\|f\|=[J(f)]^{1/p}$. In particular, if $p\geq 1$ then $\|tf+(1-t)g\|\leq t\|f\|+(1-t)\|g\|$ because of Minkowski's inequality. In other words, if $p\geq 1$, then $\|f\|$ is a convex functional. Consequently, because of Theorem B, in order to maximize $\|f\|$ over a compact family $\mathscr{G}$, it suffices to maximize $\|f\|$ over $E(\overline{co}\,\mathscr{G})$. Thus, if $p\geq 1$ then $\max\{J(f):f\in\mathscr{G}\}=\max\{J(f):f\in E(\overline{co}\,\mathscr{G})\}$. The above argument is due to MacGregor [12]. For the study of L^p mean bounds for many interesting families such as $\mathcal{C}$, $\mathcal{K}$, $s(\mathcal{C})$, $s(\mathcal{K})$ with arguments using extreme point methods, we refer to [8,12].

In this article, we characterize the sets $\overline{co}\,s(\mathcal{F})$ and $E(\overline{co}\,s(\mathcal{F}))$ and hence, we determine the coefficient bounds and L^p mean bounds for functions in the class $s(\mathcal{F})$. Further, we determine the coefficient bounds for functions in the class $m(\mathcal{F})$.

2. Coefficient estimates and L^p mean estimates

Recently, Abu Muhana et al. [1, Lemma 3.1] proved that each function $f\in\mathcal{F}$ of the form (1.1) has the following representation

$$f(z)=\int_{|x|=1}\frac{z-(x/2)z^2}{(1-xz)^2}\,d\mu(x) \tag{2.1}$$

where μ is a probability measure on $|x|=1$. Moreover, $\overline{co}\,\mathcal{F}$ consists of the functions represented by (2.1), where μ varies over the set of probability measures on $|x|=1$ and $E(\overline{co}\,\mathcal{F})$ consists of the functions defined by

$$F(z)=\frac{z-(x/2)z^2}{(1-xz)^2},\quad |x|=1. \tag{2.2}$$

Let $\mathscr{G}$ be a compact subset of $\mathcal{H}$ of functions f such that $f(0)=0$. If $f\in E(\overline{co}\,s(\mathscr{G}))$, then the argument given in [12, page 366] (see also [9, page 65]) implies that either $f=0$ or $f\prec g$, where $g\in E(\overline{co}\,\mathscr{G})$. Similarly, if $f\in E(\overline{co}\,m(\mathscr{G}))$, then either $f=0$ or f is majorized by g for some $g\in E(\overline{co}\,\mathscr{G})$.

Theorem 2.1. *The set $\overline{co}\,s(\mathcal{F})$ consists of the functions represented by*

$$f(z)=\int_R x\frac{z-(y/2)z^2}{(1-yz)^2}\,d\mu(x,y)$$

where μ varies over the set of probability measure on $R=\partial\mathbb{D}\times\partial\mathbb{D}$ and

$$E(\overline{co}\,s(\mathcal{F}))=\left\{f:f(z)=x\frac{z-(y/2)z^2}{(1-yz)^2},|x|=1,|y|=1\right\}.$$

Proof. Since z and $-z$ are in $s(\mathcal{F})$, it is clear that $0\notin E(\overline{co}\,s(\mathcal{F}))$. If $f\in E(\overline{co}\,s(\mathcal{F}))$ then by using the arguments at the beginning of this section, we conclude that $f\prec F$ for some x with $|x|=1$, where F is defined by (2.2). For a given x with $|x|=1$, let

$$L(x)=\left\{g:g(z)\prec F(z)=\frac{z-(x/2)z^2}{(1-xz)^2}\right\}.$$

Let $l(z):=1/(1-z)^2$ and $s(\{l\})=\{h:h(z)\prec l(z)\}$. Since

$$F(z)=\frac{z-(x/2)z^2}{(1-xz)^2}=\frac{1}{2x}\left[\frac{1}{(1-xz)^2}-1\right],$$

the linear map $g\mapsto(1/2x)(h-1)$ exhibits a one-to-one correspondence between functions in the classes $L(x)$ and $s(\{l\})$. Then, in view of [9, Theorem 5.7], we obtain

$$\begin{aligned}E(\overline{co}\,L(x))&=\frac{1}{2x}[E(\overline{co}\,s(\{l\}))-1]\\&=\left\{g:g(z)=\frac{1}{2x}\left[\frac{1}{(1-yz)^2}-1\right],|y|=1\right\}.\end{aligned}$$

We note that $\overline{co}\, L(x) \subseteq \overline{co}\, s(\mathcal{F})$ for every x with $|x| = 1$. Since $f \in E(\overline{co}\, s(\mathcal{F}))$ and $f \in L(x) \subseteq \overline{co}\, L(x)$ for some x with $|x| = 1$, it follows that $f \in E(\overline{co}\, L(x))$. Thus

$$f(z) = \frac{1}{2x}\left[\frac{1}{(1-yz)^2} - 1\right] = \bar{x}y\frac{z-(y/2)z^2}{(1-yz)^2} = u\frac{z-(y/2)z^2}{(1-yz)^2}$$

where $|u| = |y| = 1$ and hence

$$f \in E := \left\{ g : g(z) = x\frac{z-(y/2)z^2}{(1-yz)^2}, |x| = 1, |y| = 1 \right\}.$$

We next show that $E \subseteq E(\overline{co}\, s(\mathcal{F}))$. Since $E \subseteq s(\mathcal{F}) \subseteq \overline{co}\, s(\mathcal{F})$, for any real-valued continuous linear functional J,

$$\max_{f\in E} J(f) \le \max_{f\in \overline{co}\, s(\mathcal{F})} J(f) = \max_{f\in E(\overline{co}\, s(\mathcal{F}))} J(f) \le \max_{f\in E} J(f).$$

Thus all the above quantities are equal. Hence, it is sufficient to show that each function in E uniquely maximizes a real-valued continuous linear functional over E. Let $J(f) = \alpha f'(0) + \beta f''(0)/2$, whenever $f \in \mathcal{H}$ and $|\alpha| = |\beta| = 1$. As

$$g(z) = x\frac{z-(y/2)z^2}{(1-yz)^2} = xz + \frac{3}{2}xyz^2 + \cdots,$$

we obtain

$$\operatorname{Re} J(g) = \operatorname{Re}\left(\alpha x + \frac{3}{2}\beta xy\right) \le \left|\alpha x + \frac{3}{2}\beta xy\right| \le |\alpha| + \frac{3}{2}|\beta| = \frac{5}{2}. \tag{2.3}$$

The equality holds in (2.3) when $x = 1/\alpha$ and $y = \alpha/\beta$. Thus, a unique function in E maximizes $\operatorname{Re} J(f)$. By varying α and β, we obtain all possible pairs (x, y) with $|x| = 1$, $|y| = 1$ such that $\operatorname{Re} J(g) = 5/2$. Hence $E(\overline{co}\, s(\mathcal{F})) = E$ and the statement about $\overline{co}\, s(\mathcal{F})$ follows from the Krein–Milman theorem [16, Theorem 3.23] and the weak-star compactness of the probability measures. □

Theorem 2.2. *If $f \in s(\mathcal{F})$ and $f(z) = \sum_{n=1}^{\infty} a_n z^n$ then $|a_n| \le \frac{n+1}{2}$, $n \ge 1$. The estimate is sharp.*

Proof. Since the functional $J(f) = |a_n|$ is a real-valued continuous and convex functional, in view of Theorem B, it is sufficient to verify the inequality for functions of the form

$$f(z) = x\frac{z-(y/2)z^2}{(1-yz)^2}, \quad |x| = |y| = 1.$$

For these functions

$$|a_n| = \left|\frac{n+1}{2}xy^{n-1}\right| = \frac{n+1}{2}, \quad n \ge 1.$$

This completes the proof. □

To prove our next result, which deals with the coefficient bounds for functions in the class $m(\mathcal{F})$, we need the following lemma due to Kakeya [10].

Lemma 2.1. *[10] If $c_0 > c_1 > \ldots > c_n > 0$ then all the roots of $P(z) = c_0 + c_1 z^2 + \cdots + c_n z^n$ satisfy $|z| > 1$, that is $P(z) \ne 0$ when $|z| \le 1$.*

Theorem 2.3. *If $f \in m(\mathcal{F})$ and $f(z) = \sum_{n=1}^{\infty} a_n z^n$ then $|a_n| \le \sum_{k=0}^{n-1} A_k^2$ $(n \ge 1)$, where*

$$A_k = \sum_{j=0}^{k} \binom{1/2}{j}\left(-\frac{1}{2}\right)^j$$

The estimate is sharp.

Proof. By the arguments at the beginning of this section, it is sufficient to verify the inequality for functions of the form

$$f(z)=\omega(z)\frac{z-(x/2)z^2}{(1-xz)^2},$$

where $\omega:\mathbb{D}\to\overline{\mathbb{D}}$ and $|x|=1$. If $0<r<1$ then by Cauchy's theorem

$$\begin{aligned}a_n&=\frac{1}{2\pi i}\int\limits_{|z|=r}\frac{f(z)}{z^{n+1}}\,dz=\frac{1}{2\pi i}\int\limits_{|z|=r}\frac{\omega(z)}{z^n}(\phi(z))^2\,dz\\&=\frac{1}{2\pi i}\int\limits_{|z|=r}\frac{\omega(z)}{z^n}\left(\sum_{k=0}^{n-1}A_k(x)z^k\right)^2dz,\end{aligned}$$

where

$$\phi(z)=\frac{(1-\frac{x}{2}z)^{1/2}}{1-xz}=\sum_{k=0}^{\infty}A_k(x)z^k.$$

Here

$$A_k(x)=\sum_{j=0}^{k}x^k\binom{1/2}{j}\left(-\frac{1}{2}\right)^j=x^kA_k(1).$$

We notice that

$$\begin{aligned}A_k:=A_k(1)&=\sum_{j=0}^{k}\binom{1/2}{j}\left(-\frac{1}{2}\right)^j\\&>\sum_{j=0}^{\infty}\binom{1/2}{j}\left(-\frac{1}{2}\right)^j\\&=\left(1-\frac{1}{2}\right)^{1/2}>0.\end{aligned}$$

Since $|\omega(z)|\le 1$, we see that if $z=r\,e^{i\theta}$, then

$$\begin{aligned}|a_n|&\le\frac{1}{2\pi}\int\limits_0^{2\pi}\frac{1}{r^{n-1}}\left|\sum_{k=0}^{n-1}A_k(x)z^k\right|^2d\theta\\&=\frac{1}{r^{n-1}}\sum_{k=0}^{n-1}|A_k(x)|^2r^{2k}\\&=\frac{1}{r^{n-1}}\sum_{k=0}^{n-1}|A_k|^2r^{2k}.\end{aligned}$$

By letting $r\to 1$, we obtain the desired result.

To see the sharpness of the result, we let

$$\psi(z)=\frac{(1-z/2)^{1/2}}{1-z}=\sum_{k=0}^{\infty}A_kz^k$$

and $P(z)=\sum_{k=0}^{n-1}A_kz^k$. Since $A_k<A_{k-1}$, by Lemma 2.1, $P(z)\neq 0$ for $z\in\overline{\mathbb{D}}$. If $\omega(z)=z^{n-1}P(1/z)/P(z)$, then $\omega(z)$ is analytic in $\overline{\mathbb{D}}$. For $|z|=1$, we have

$$|\omega(z)|=\frac{|P(1/z)|}{|P(z)|}=\frac{|P(e^{-i\theta})|}{|P(e^{i\theta})|}=1.$$

Then, by the maximum modulus theorem, $|\omega(z)|\le 1$ in $\overline{\mathbb{D}}$. If $g(z)=z\omega(z)\psi^2(z)$, then $g\in m(\mathcal{F})$. If $g(z)=\sum_{n=1}^{\infty}b_nz^n$, then, by using the earlier arguments, we see that

$$
\begin{aligned}
b_n &= \frac{1}{2\pi i} \int\limits_{|z|=r} \frac{\omega(z)}{z^n} \psi^2(z)\, dz \\
&= \frac{1}{2\pi i} \int\limits_{|z|=r} \frac{\omega(z)}{z^n} (P(z))^2\, dz \\
&= \frac{1}{2\pi i} \int\limits_{|z|=1} \frac{P(1/z)P(z)}{z}\, dz \\
&= \frac{1}{2\pi} \int\limits_0^{2\pi} P(e^{i\theta})P(e^{-i\theta})\, d\theta \\
&= \frac{1}{2\pi} \int\limits_0^{2\pi} |P(e^{i\theta})|^2\, d\theta \\
&= \sum_{k=0}^{n-1} A_k^2.
\end{aligned}
$$

This completes the proof of the sharpness of the result. □

Theorem 2.4. *If $f \in s(\mathcal{F})$ and $g(z) = \dfrac{z - z^2/2}{(1-z)^2}$ then*

$$
\frac{1}{2\pi} \int\limits_0^{2\pi} |f^{(n)}(r e^{i\theta})|^p\, d\theta \le \frac{1}{2\pi} \int\limits_0^{2\pi} |g^{(n)}(r e^{i\theta})|^p\, d\theta \tag{2.4}
$$

whenever $0 < r < 1$, $p \ge 1$ *and* $n = 0, 1, \ldots$.

Proof. For $p \ge 1$, as mentioned in Section 1, it is sufficient to consider functions of the form

$$
f(z) = x\frac{z - (y/2)z^2}{(1-yz)^2}, \quad |x| = |y| = 1.
$$

For these functions $f(z) = xy\, g(yz)$ and so $f^{(n)}(z) = xy^{n+1} g^{(n)}(yz)$. Thus

$$
\begin{aligned}
\frac{1}{2\pi} \int\limits_0^{2\pi} |f^{(n)}(r e^{i\theta})|^p\, d\theta &= \frac{1}{2\pi} \int\limits_0^{2\pi} |g^{(n)}(yr e^{i\theta})|^p\, d\theta \\
&= \frac{1}{2\pi} \int\limits_0^{2\pi} |g^{(n)}(r e^{i\theta})|^p\, d\theta.
\end{aligned}
$$

Hence (2.4) holds for any $f \in s(\mathcal{F})$. □

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