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On (2, 3)-generation of Fischer's largest sporadic simple group Fi'_{24} ☆

Sur la (2, 3)*-génération du plus grand groupe simple sporadique de Fischer* Fi'_{24}

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A R T I C L E I N F O

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A B S T R A C T

A group G is said to be (2, 3)-*generated* if it can be generated by an involution x and an element y of order three. For G a sporadic simple group, it was proved by the third author Woldar (1989) [26] that G is (2, 3)-generated if and only if $G \notin \{M_{11}, M_{22}, M_{23}, McL\}$. In this paper, we investigate all possible (2, 3)-generations of Fischer's largest sporadic simple group Fi'_{24} under the assumption that the product xy has prime order.

R É S U M É

Un groupe G est dit (2, 3)-engendré s'il peut être engendré par une involution x et un élément y d'ordre trois. Pour un groupe simple sporadique G, il a été montré par le troisième auteur Woldar (1989) [26] que G est (2, 3)-engendré si et seulement si $G \notin \{M_{11}, M_{22}, M_{23}, McL\}$. Nous étudions ici toutes les (2, 3)-générations du plus grand groupe simple sporadique de Fischer Fi'_{24}, en supposant que le produit xy est d'ordre premier.

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1. Introduction

A group G is said to be (l, m, n)*-generated* if $G = \langle x, y\rangle$ where the elements x, y, xy have respective orders $o(x) = l$, $o(y) = m$, $o(xy) = n$. In such case, G is a quotient group of the Von Dyck group $D(l, m, n)$, and therefore it is also $(\pi(l), \pi(m), \pi(n))$-generated for any $\pi \in S_3$. Thus we may assume throughout that $l \leq m \leq n$.

Initially, the study of (l, m, n)-generations of a group G had deep connections to the topological problem of determining the least genus of an orientable surface on which G admits an effective, orientation-preserving, conformal action. In [24], such investigations were extended well beyond the "minimum genus problem" to all possible (p, q, r)-generations, assuming G to be finite non-abelian simple and p, q, r distinct primes.

In this paper, we restrict attention to $(2, 3, r)$-generations of Fischer's largest sporadic group Fi'_{24} where r is prime. The non-prime case will be treated in a separate article.

Groups that are $(2, 3)$-generated have been of particular interest to combinatorists and group theorists. The quintessential example of an infinite $(2, 3)$-generated group is the modular group $PSL(2, \mathbb{Z})$ which, being the free product of the groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$, acts as a universal cover. This implies that any $(2, 3)$-generated group is a quotient of $PSL(2, \mathbb{Z})$. Connections with Hurwitz groups, regular maps, Beauville surfaces and structures provide additional motivation for the study of these groups, e.g., see [7,8,17]. (Recall that a *Hurwitz group* is one that can be $(2, 3, 7)$-generated.)

The following simple groups are known to be $(2, 3)$-generated: the alternating group A_n, $2 < n \neq 6, 7, 8$ [23]; the projective special linear group $PSL(2, q)$, $q \neq 9$ [22]; all sporadic simple groups with the exception of M_{11}, M_{22}, M_{23}, and McL [26]. Also, a large number of classical linear groups and exceptional Lie groups are known to be $(2, 3)$-generated [12]. Recently, Liebeck & Shalev [19,20,18] showed, using probabilistic methods, that all finite classical groups are $(2, 3)$-generated with the exception of the families $PSp(4, 2^k)$, $PSp(4, 3^k)$ and finitely many other groups. In addition to the references provided above, we direct the reader's attention to [11] for further details related to the generation of finite simple groups by two elements.

In a series of papers, the authors established all possible $(2, 3, r)$-generations of the sporadic groups He, HS, Co_1, Co_2, J_3, J_4, and Fi_{22} (cf. [1–6], [12], [21], [26]) for r a prime. Presently, we focus our attention on Fischer's sporadic group Fi'_{24}.

Groups that act conformally on the sphere (genus 0) and the torus (genus 1) have been classified, see [15], and the only simple group among them is A_5, which acts conformally on the sphere. The implication of this is that if $\mathcal{S}$ is a surface admitting a conformal action of a simple group $G \neq A_5$, then genus $(\mathcal{S}) \geq 2$. Applying the Reimann–Hurwitz formula, we see that G can only be $(2, 3, r)$-generated provided $\frac{1}{2} + \frac{1}{3} + \frac{1}{r} < 1$. Thus Fi'_{24} cannot be $(2, 3, r)$-generated for any $r < 7$, in which case we need only consider the primes $r = 7, 11, 13, 17, 23, 29$ in what follows. A separate section will be devoted to each such value of r.

For convenience, we summarize the main results of our paper as follows.

Theorem. *Fischer's largest sporadic simple group* Fi'_{24} *is* $(2, 3, r)$*-generated for every prime divisor* r *of* $|Fi'_{24}|$ *with* $r \geq 7$. *More explicitly, denoting by* rZ *the* Fi'_{24}*-class containing the element* xy, *we have that* $Fi'_{24} = \langle x, y\rangle$ *if and only if*

(1) $x \in 2A$, $y \in 3E$ and $rZ \in \{17A, 23A/B, 29A/B\}$.
(2) $x \in 2B$, $y \in 3C$ and $rZ \in \{17A, 23A/B, 29A/B\}$.
(3) $x \in 2B$, $y \in 3D$ and $rZ \in \{11A, 13A, 17A, 23A/B, 29A/B\}$.
(4) $x \in 2B$, $y \in 3E$ and $rZ \in \{7B, 11A, 13A, 17A, 23A/B, 29A/B\}$.

2. Preliminaries

Throughout this article, we use the same notation and terminology as can be found in [1,2,6,14,25]. In particular, for a finite group G with conjugacy classes C_1, C_2, C_3, we denote the corresponding structure constant of G by $\Delta(G) = \Delta_G(C_1, C_2, C_3)$, which, by definition, is the cardinality of the set $\Omega = \{(x, y) | xy = z\}$ where $x \in C_1$, $y \in C_2$ and z is a fixed representative in the conjugacy class C_3. It is well known that the value of $\Delta(G)$ can be computed from the character table of G (e.g., see [16, p.45]) via the formula

$$\Delta_G(C_1, C_2, C_3) = \frac{|C_1||C_2|}{|G|} \sum_{i=1}^{m} \frac{\chi_i(x)\chi_i(y)\overline{\chi_i(z)}}{\chi_i(1)}$$

where $\chi_1, \chi_2, \ldots, \chi_m$ are the irreducible complex characters of G, and the bar denotes complex conjugation. We denote by $\Delta^*(G) = \Delta^*_G(C_1, C_2, C_3)$ the number of distinct ordered pairs $(x, y) \in \Omega$ such that $G = \langle x, y\rangle$. Clearly, if $\Delta^*(G) > 0$ then G is (l, m, n)-generated where l, m, n are the respective orders of elements from C_1, C_2, C_3. In this instance we shall also say that G is (C_1, C_2, C_3)*-generated* and we shall refer to (C_1, C_2, C_3) as a *generating triple* for G.

Further, if H is a subgroup of G containing the fixed element $z \in C_3$ above, we denote by $\Sigma(H) = \Sigma_H(C_1, C_2, C_3)$ the total number of distinct ordered pairs $(x, y) \in \Omega$ such that $\langle x, y\rangle \leq H$. The value of $\Sigma_H(C_1, C_2, C_3)$ is obtained as the sum of all structure constants $\Delta_H(c_1, c_2, c_3)$ where the c_i are conjugacy classes of H that fuse to C_i in G, i.e., $c_i \subseteq H \cap C_i$. The

Table 1
The maximal subgroups of Fi'_{24}.

Group	Order	Group	Order
Fi_{23}	$2^{18}.3^{13}.5^2.7.11.13.17.23$	$2.Fi_{22}{:}2$	$2^{19}.3^9.5^2.7.11.13$
$(3\times O_8^+(3){:}3){:}2$	$2^{13}.3^{14}.5^2.7.13$	$O_{10}^-(2)$	$2^{20}.3^6.5^2.7.11.17$
$3^7{\cdot}O_7(3)$	$2^9.3^{16}.5.7.13$	$3^{1+10}{:}U_5(2){:}2$	$2^{11}.3^{16}.5.11$
$2^{11}{\cdot}M_{24}$	$2^{21}.3^3.5.7.11.23$	$2^2{\cdot}U_6(2){:}S_3$	$2^{18}.3^7.5.7.11$
$2^{1+12}.3U_4(3).2_2$	$2^{21}.3^7.5.7$	$3^3.[3^{10}].GL_3(3)$	$2^5.3^{16}.13$
$3^{2+4+8}.(A_5\times 2A_4).2$	$2^6.3^{16}.5$	$(A_4\times O_8^+(2).3).2$	$2^{15}.3^7.5^2.7$
$He{:}2$ (2 classes)	$2^{11}.3^3.5^2.7^3.17$	$2^{3+12}.(L_3(2)\times A_6)$	$2^{21}.3^3.5.7$
$2^{6+8}.(S_3\times A_8)$	$2^{21}.3^3.5.7$	$(3^2{:}2\times G_2(3)).2$	$2^8.3^8.7.13$
$(A_5\times A_9){:}2$	$2^9.3^5.5^2.7$	$A_6\times L_2(8){:}3$	$2^6.3^3.5.7^2$
$7{:}6\times A_7$	$2^4.3^3.5.7^2$	$U_3(3).2$ (2 classes)	$2^6.3^3.7$
$L_2(13){:}2$ (2 classes)	$2^3.3.7.13$	$29{:}14$	$2.7.29$

number of pairs $(x, y)\in\Omega$ generating a subgroup H of G will be denoted by $\Sigma^*(H)=\Sigma^*_H(C_1, C_2, C_3)$, and the centralizer of a representative of the conjugacy class C by $C_G(C)$. A general conjugacy class of a proper subgroup H of G whose elements are of order n will be denoted by nx, reserving the notation nX for the case where $H=G$.

The number of conjugates of a given subgroup H of G containing a fixed element g is given by $\pi(g)$, where π is the permutation character corresponding to the action of G on the conjugates of H, i.e. π is the induced character $(1_H)^G$ (*cf.* [16, Lemma 5.14]). As the stabilizer of H in this action is clearly $N_G(H)$, in many cases, one can more easily compute the value $\pi(g)$ from the fusion map from $N_G(H)$ into G in conjunction with Lemma 2.1 below. We emphasize that this is an especially useful strategy when the decomposition of π into irreducible characters is not known explicitly.

Lemma 2.1. *[14] Let G be a finite group and let H be a subgroup of G containing a fixed element g such that* $gcd(o(g), |N_G(H):H|)=1$*. Then the number of conjugates of H containing g is given by*

$$\pi(g)=\sum_{i=1}^{m}\frac{|C_G(g)|}{|C_{N_G(H)}(g_i)|}$$

where π *is the permutation character corresponding to the action of G on the cosets of H, and* $g_1, g_2, \ldots, g_m$ *are representatives of the* $N_G(H)$*-conjugacy classes that fuse to the G-class containing g.*

Below we provide some useful techniques for establishing non-generation.

Lemma 2.2. *[27] Let G be a finite group and let* $x, y\in G$*. Suppose that* $\Delta(G)<|C_G(xy)|$*, where* $\Delta(G)=\Delta_G(lX, mY, nZ)$ *with* $x\in lX$*,* $y\in mY$ *and* $xy\in nZ$*. Then* $C_G(\langle x, y\rangle)$ *is non-trivial.*

Lemma 2.3. *[9] Let G be a finite centerless group and suppose lX, mY, nZ are G-conjugacy classes for which*

$$\Delta^*(G):=\Delta^*_G(lX, mY, nZ)<|C_G(nZ)|.$$

Then $\Delta^*(G)=0$*, hence G is not* (lX, mY, nZ)*-generated.*

Note that for all triples we consider in this paper, it is the case that $\Delta(Fi_{24})=\Delta(Fi'_{24})$. Thus, since $C_{Fi_{24}}(Fi'_{24})=1$, we obtain $\Delta^*(Fi'_{24})<|C_{Fi_{24}}(nZ)|$ as a sufficient condition for non-generation of Fi'_{24} via Lemma 2.2 applied to $G=Fi_{24}$. (Compare this result to Lemma 2.3 applied to $G=Fi'_{24}$.) Frequently, we shall invoke Lemma 2.2 in exactly this manner, while at other times Lemma 2.3 will suffice for our purposes.

We list all maximal subgroups of Fi'_{24} in Table 1. In Table 2 we indicate the fusion map from each maximal subgroup M into Fi'_{24}, and we calculate the corresponding value of $\pi(z)$ where π is the permutation character $(1_M)^{Fi'_{24}}$ and $z\in M$ has prime order $o(z)\geq 7$. Many of our computations relied heavily on the use of $\mathbb{GAP}$, as well as certain subroutines provided in [5]. As always, the $\mathbb{ATLAS}$ [10] served as an invaluable source of information, and we adopt its notation for conjugacy classes, maximal subgroups, etc.

3. (2, 3)-Generation of Fi'_{24}

A group G is said to be a 3-*transposition group* if it is generated by a conjugacy class D of involutions in G such that $o(de)\leq 3$ for all $d, e\in D$. In this case, the conjugacy class D is called a 3-*transposition class*. Fischer [13] introduced the notion of a 3-transposition group, and further classified all finite 3-transposition groups with no non-trivial normal soluble subgroups. In the process of his classification, Fischer discovered three new groups, Fi_{22}, Fi_{23} and Fi_{24}, with 3-transposition

Table 2
Partial fusion maps from maximal subgroups into Fi_{24}'.

Fi_{23}-class	2a	2b	2c	3a	3b	3c	3d	7a	11a	13a	13b	17a	23a	23b
$\to Fi_{24}'$	2A	2A	2B	3A	3B	3C	3D	7A	11A	13A	13A	17A	23A	23B
$\pi(z)$								21	3	6	6	1	1	1
$2.Fi_{22}{:}2$-class	2a	2b	2c	2d	2e	2f	2g	3a	3b	3c	3d	7a	11a	13a
$\to Fi_{24}'$	2A	2A	2A	2B	2B	2A	2B	3A	3B	3C	3D	7A	11A	13A
$\pi(z)$												105	3	9
$(3\times O_8^+(3){:}3){:}2$-class	2a/c	2b/d	3a/d/l	3b/f	3m/o	3c/h	3j/k	3e/g/i	3n/p/q	3r/s	3t/u	7a	13a	13b
$\to Fi_{24}'$	2A	2B	3B	3C	3C	3A	3A	3D	3D	3E	3E	7A	13A	13A
$\pi(z)$												35	1	1
$O_{10}^-(2)$-class	2a	2b	2c	2d	3a	3b	3c	3d/e	3f	7a	11a	11b	17a	17b
$\to Fi_{24}'$	2B	2A	2A	2B	3A	2B	3B	3C	3F	7A	11A	11A	17A	17A
$\pi(z)$										42	8	8	2	2
$3^7\cdot O_7(3)$-class	2a	2b	2c	3a	3b	3c	3d	3e	3f	3g	3h	3i	3j	3k
$\to Fi_{24}'$	2B	2A	2B	3B	3C	3A	3D	3C	3D	3D	3A	3B	3D	3C
$\pi(z)$														
$3^7\cdot O_7(3)$-class	3l	3m	3n	3o	3p	3q	3r	3s	7a	13a	13b			
$\to Fi_{24}'$	3B	3C	3D	3D	3A	3E	3E	3E	7B	13A	13A			
$\pi(z)$									49	12	12			
$3^{1+10}{:}U_5(2){:}2$-class	2a	2b	2c	3a/c	3g/j	3b/f	3n	3d/h	3k/s/t	3e/i	3l/m	3o/p/u	3q/r/v	11a
$\to Fi_{24}'$	2B	2A	2B	3B	3B	3A	3A	3C	3C	3D	3D	3D	3E	11A
$\pi(z)$														4
$2^{11}\cdot M_{24}$-class	2a	2b	2c	2d	2e	3a	3b	7a	7b	11a	23a	23b		
$\to Fi_{24}'$	2A	2B	2A	2B	2B	3C	3E	7A	7A	11A	23A	23B		
$\pi(z)$									210	6	1	1		
$2^2\cdot U_6(2){:}S_3$-class	2a	2b	2c/d	2e	2f	2g	2h	3a	3b/d	3c/e	3f	3g	7a	11a
$\to Fi_{24}'$	2A	2B	2A	2B	2B	2A	2B	3A	3B	3C	3D	3E	7A	11A
$\pi(z)$													105	1
$2^{1+12}.3U_4(3).2_2$	2a	2b	2c	2d	2e/f	2g	2h/i	3a/f	3b	3c	3d	3e	3g	7a
$\to Fi_{24}'$	2B	2A	2B	2A	2B	2A	2B	3C	3D	3B	3A	3D	3C	7B
$\pi(z)$														49
$3^3.[3^{10}].GL_3(3)$-class	2a	2b/c	3a/c/g	3h/o	3u/z	3ac/ar	3l/t/x	3d/e/j	3r/v/w	3ac/ag	3ai/al	3aw	3b/f	3i/k
$\to Fi_{24}'$	2A	2B	3B	3A	3A	3A	3B	3C	3C	3C	3C	3C	3D	3D
$\pi(z)$														
$3^3.[3^{10}].GL_3(3)$-class	3m/n	3p/q	3s/y	3aa/ab	3ad/af	3ah/am	3aq/as	3at/au	3av/ax	3aj/ak	3an/ao	ap	13a/b	13c/d
$\to Fi_{24}'$	3D	3D	3D	3D	3D	3D	3D	3D	3D	3E	3E	3E	13A	13A
$\pi(z)$													12	12
$He{:}2$-class	2a	2b	2c	3a	3b	7a	7b	7c	17a					
$\to Fi_{24}'$	2A	2B	2B	3C	3E	7A	7B	7B	17A					
$\pi(z)$						15	22	22	1					
$(3^2{:}2\times G_2(3)).2$-class	2a	2b	2c	3a/b	3e/f	3d	3c/g	3h	3i/j	3k/l	3m/n	3o/p	7a	13a
$\to Fi_{24}'$	2B	2A	2B	3B	3A	3C	3D	3D	3D	3E	3E	3E	7A	13A
$\pi(z)$													1	70
$U_3(3).2$-class	2a	2b	3ac	3b	7a									
$\to Fi_{24}'$	2B	2B	3D	3E	7B									
$\pi(z)$					294									
$L_2(13){:}2$-class	2a	2b	3a	7a	7b	7c	13a							
$\to Fi_{24}'$	2B	2B	3D	7B	7B	7B	13A							
$\pi(z)$				441	441	441	18							

classes of respective sizes 3 510, 31 671, and 306 936. Of these, the first two groups are simple, whereas the third group has simple commutator subgroup Fi_{24}' of index 2 and order

$$1\,255\,205\,709\,190\,661\,721\,292\,800 = 2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29.$$

The group Fi_{24}' has 108 conjugacy classes in total, including two classes of involutions (viz. $2A$, $2B$) and five classes of elements of order 3 (viz. $3A, 3B, 3C, 3D, 3E$). Linton & Wilson [21] investigated the subgroup structure of Fi_{24}' and classified all maximal subgroups of Fi_{24}' as well as those of its automorphism group Fi_{24}.

We now proceed to a case-by-case analysis of all $(2, 3, r)$-generations of Fi_{24}'.

3.1. The case $r = 7$

In all, there are 20 triples of classes in Fi_{24}' to consider, two classes of elements of order 2, five of order 3, and two of order 7.

We begin with the triple $(2B, 3D, 7B)$. By [21] every proper $(2B, 3D, 7B)$-subgroup of Fi_{24}' lies in some conjugate of $H = 3^7{\cdot}O_7(3)$. Thus, to investigate $(2B, 3D, 7B)$-generation of Fi_{24}', we may apply the principle of inclusion–exclusion to the 49 conjugates of H that contain a fixed $z \in 7B$.

Lemma 3.1. *A fixed element $z \in 7B$ lies in 18 conjugates of a fixed $L = L_2(13)$ in $H = 3^7{\cdot}O_7(3)$. As there are two classes of $L_2(13)$ in H, this implies that z lies in a total of 36 copies of L in H (i.e. 18 conjugates from each class).*

Proof. The number of such conjugates is given by $k = |N_H(\langle z\rangle)|/|N_K(\langle z\rangle)|$, where $K = N_H(L)$. We first claim that $K = L$. Indeed suppose that L is a proper subgroup of K. Then K must contain an element w of order 3 that is not in L. As $Aut(L) = L:2$, w must centralize L. In particular, w centralizes z, hence w must be of Fi_{24}'-type 3C. However, elements of Fi_{24}' type 3C do not commute with elements of order 13. Thus $w = 1$ so $K = L$ as claimed. As $|N_L(\langle z\rangle)| = 14$, we therefore have $k = N_H(\langle z\rangle)/14$.

Now as $N_H(\langle z\rangle) < N_{Fi_{24}'}(\langle z\rangle)$, we see that $|N_H(\langle z\rangle)|$ must divide $2^2 \cdot 3^2 \cdot 7$ and be divisible by $2 \cdot 7$. By Sylow's theorem, $|\mathrm{Syl}_7(H)| = |H : N_H(\langle z\rangle)| \equiv 1 (\mod 7)$, which can only occur if $|N_H(\langle z\rangle)| = 2^2 \cdot 3^2 \cdot 7$. Thus $k = (2^2 \cdot 3^2 \cdot 7)/14 = 18$, which proves the claim. □

Lemma 3.2. *Every $L = L_2(13)$ containing z lies in exactly two conjugates of $H = 3^7{\cdot}O_7(3)$.*

Proof. By [21], each L lies in some conjugate of H. However, $L:2$ does not lie in H, in which case we have $L = H \cap H^t$, where t is an outer involution in $L:2$. We claim that H and H^t are the only conjugates of H that contain L. We proceed as follows.

Let L and L' be representatives of the two conjugacy classes of $L_2(13)$ in Fi_{24}'. We count the set

$$\{(L^*, H^*) | L^* \text{ is conjugate to either } L \text{ or } L', H^* \text{ is conjugate to } H, z \in L^* < H^*\}.$$

On one hand, this set has size 49×36 since for each of the 49 choices of H^* there are 36 choices for L^*. On the other hand, this set has size *at least* 882×2. Indeed, by the above each L^* lies in at least two H^*. But $49 \times 36 = 882 \times 2$, from which we conclude that each L^* lies in *exactly* two H^*. □

Lemma 3.3. *Every non-splitting $K = 3^7{\cdot}L_2(13)$ containing z lies in exactly one conjugate of H.*

Proof. Suppose $K < H \cap H^g$ with $H^g \neq H$. Let T denote the normal subgroup of K with $T \cong 3^7$. As $K \leq H^g$ and $T^g \trianglelefteq H^g$, we have that $L = KT^g$ is a subgroup of H^g.

Observe that L must properly contain K. Indeed, if $L = K$ then $T^g \leq K$ in which case $TT^g \trianglelefteq K$. As $L_2(13)$ is simple, this implies $T^g = T$. However, it now follows that $g \in N(T) = H$, contradicting $H^g \neq H$.

We therefore conclude that either $L = H^g \cong 3^7{\cdot}O_7(3)$ or $L \cong 3^7{\cdot}G_2(3)$. But this implies that the quotient group L/T^g is isomorphic to either $O_7(3)$ or $G_2(3)$. However, we also have that $L/T^g = KT^g/T^g \cong K/(K \cap T^g)$, which is clearly impossible as neither $O_7(3)$ nor $G_2(3)$ can be a homomorphic image of $K \cong 3^7{\cdot}L_2(13)$. The result follows. □

Lemma 3.4. *Let I denote the intersection of any three distinct conjugates of $H = 3^7{\cdot}O_7(3)$. Then $\Sigma(I) = 0$.*

Proof. The only proper $(2B, 3D, 7B)$-subgroups of Fi_{24}' are $L_2(13)$ and $3^7{\cdot}L_2(13)$ (non-splitting), see [21]. But by Lemmas 3.2 and 3.3, neither group can occur in a triple intersection of such conjugates. □

Proposition 3.5. *Fi_{24}' is not $(2B, 3D, 7B)$-generated.*

Table 3
Contributions toward $\Delta_G(2B, 3D, 7A)$ where $G = Fi_{24}$.

$\langle S\rangle$	S_7	S_6	S_5	$S_5\times S_2$	S_4	$S_4\times S_2$	$S_4\times S_3$
$C_G(S)$	7:6	$L_2(8):3$	S_9	$2\times S_7$	$O_8^+(2):S_3$	$2\times S_6(2)$	$U_3(3):2$
$\Sigma(C_G(S))$	0	0	0	0	0	0	0
$\langle S\rangle$	S_3	$S_3\times S_2$	$S_3\times S_2^2$	S_3^2	S_2	S_2^2	S_2^3
$C_G(S)$	$O_8^+(3):S_3$	$2\times O_7(3)$	$2\times 2U_4(3)$	$G_2(3)$	$2\times Fi_{23}$	$2\times 2Fi_{22}$	$2\times 2^2U_6(2)$
$\Sigma(C_G(S))$	756	182	0	42	15680	1848	0

Proof. We compute the structure constant $\Delta_{Fi'_{24}}(2B, 3D, 7B) = 61\,740$. As alluded to earlier, we will apply the principle of inclusion–exclusion to the 49 conjugates of H that contain a fixed $z \in 7B$. By Lemma 3.2, those subgroups isomorphic to $L_2(13)$ are counted exactly twice, while Lemma 3.3 asserts that those isomorphic to $3^7{\cdot}L_2(13)$ (non-splitting) are counted only once. By Lemma 3.4, there is no contribution coming from intersections of three or more distinct conjugates of H. Thus the number of triples that generate a proper $(2B, 3D, 7B)$-subgroup of Fi'_{24} is given by

$$49 \cdot \Sigma(3^7{\cdot}O_7(3)) - 882 \cdot \Sigma(L_2(13)) = 49(1512) - 882(14) = 61740.$$

However, this is precisely the structure constant of Fi'_{24} from which the result follows. □

The case $(2B, 3D, 7A)$ is quite more delicate, and its treatment will require some additional lemmas.

Lemma 3.6. *Let $G = Fi_{24}$ (so that $G' = Fi'_{24}$) and fix $z \in 7A$. Then $C_G(z) = \langle z\rangle \times H$ with $H \cong S_7$. Identifying H with S_7, we have that the 21 transpositions $(a, b) \in H$ are of G-class $2C$, the 105 involutions of type $(a, b)(c, d) \in H$ are of G-class $2A$, and the 105 involutions of type $(a, b)(c, d)(e, f) \in H$ are of G-class $2D$. Finally, the 70 elements of type $(a, b, c) \in H$ are of G-class $3A$.*

Proof. As $C_{G'}(z) \cong 7 \times A_7$, we see at once that elements of type $(a, b) \in H$ and those of type $(a, b)(c, d)(e, f) \in H$ must lie outside G'. Thus elements of each type are either of G-class $2C$ or $2D$. Since $(1, 2) \in H$ centralizes $z(3, 4, 5, 6, 7) \in C_G(z)$, we see that $w = z(1, 2)(3, 4, 5, 6, 7)$ is of order 70 and $w^{35} = (1, 2)$. But this can only occur if $(1, 2) \in 2C$.

Note that despite the fact that $(a, b) \in H$ and $(a, b)(c, d)(e, f) \in H$ are obviously not conjugate in H, they could be conjugate in G. However, by character computation, we know that z lies in exactly 21 G'-conjugates of $F \cong Fi_{23}$ (see Table 2). Thus z lies in exactly 21 G-conjugates of $C_G(t) = \langle t\rangle \times F$ where $t \in 2C$. These 21 conjugates correspond to the 21 elements $(a, b) \in 2C$, hence there can be no other elements in $C_G(z)$ of type $2C$. Thus elements of type $(a, b)(c, d)(e, f) \in H$ must be in the G-class $2D$.

As elements of the form $(a, b)(c, d)$ are in $H' \cong A_7$, they must be in G', and thus they are either of class $2A$ or $2B$. However, nothing in class $2B$ commutes with $z \in 7A$, hence $(a, b)(c, d) \in 2A$.

Finally, elements of type (a, b, c) are of G-class $3A$ because $v = z(1, 2, 3)(4, 5, 6, 7)$ is of order 84 with $v^{28} = (1, 2, 3)$. But this can only occur if $(1, 2, 3) \in 3A$. (Alternatively, one could argue that the product of two non-commuting involutions of Fi_{24}-type $2C$ must be of Fi_{24}-type $3A$, a known property of the 3-transposition class $2C$.) □

Lemma 3.7. *With notation as in Lemma 3.6, let S be any set of transpositions in H. Then the subgroup $\langle S\rangle < H$ generated by S is isomorphic to one of the following groups: S_7, S_6, S_5, $S_5 \times S_2$, S_4, $S_4 \times S_2$, $S_4 \times S_3$, S_3, $S_3 \times S_2$, $S_3 \times S_2^2$, S_3^2, S_2, S_2^2, S_2^3. Moreover, the numbers of such subgroups of H that so arise are indicated as follows:*

S_7	S_6	S_5	$S_5\times S_2$	S_4	$S_4\times S_2$	$S_4\times S_3$	S_3	$S_3\times S_2$	$S_3\times S_2^2$	S_3^2	S_2	S_2^2	S_2^3
1	7	21	21	35	105	35	35	210	105	70	21	105	105

Proof. This is an easy counting exercise which we leave to the reader. □

Again set $G = Fi_{24}$. Our next goal is to determine the centralizer $C_G(S)$ of each subgroup $\langle S\rangle \le H \cong S_7$, as well as its contribution $\Sigma(C_G(S))$ to the structure constant $\Delta(G) = 147\,000$ of type $(2B, 3D, 7A)$. All results, which depend on our Lemma 3.6 and are corroborated by [21, Table 10.2, p. 153], are indicated in Table 3. Note that if S and T are sets of transpositions in $H \cong S_7$ with $\langle S\rangle \cong \langle T\rangle$, then $\langle S\rangle$ and $\langle T\rangle$ are conjugate in H, hence they are conjugate in G.

In Table 4, we indicate the number of ways a fixed subgroup $\langle S\rangle$ of $H \cong S_7$ of given isomorphism type can be generated by exactly n transpositions. This information is crucial to us, since n is also the precise number of transposition-centralizers whose intersection gives the corresponding group $C_G(S)$. (Indeed, $C_G(S) = \bigcap_{s\in S} C_G(s)$.) We shall use this information later on, when we invoke the principle of inclusion–exclusion to prove that Fi'_{24} is not $(2B, 3D, 7A)$-generated.

The reader will observe that the table stops at $n = 11$. This is because larger values of n fail to contribute to $\Delta(Fi_{24})$. We also remark that if one multiplies each entry in Table 4 by the number of subgroups $\langle S\rangle \le H$ of indicated isomorphism type, then the resulting column sums will be $\binom{21}{n}$, $1 \le n \le 11$. This provides a valuable check on the accuracy of Table 4.

Table 4
Number of ways a specified subgroup of H can be generated by a set S of n transpositions ($1 \le n \le 11$).

$\langle S\rangle$	1	2	3	4	5	6	7	8	9	10	11
S_7	0	0	0	0	0	17227	68295	156555	258125	331506	343140
S_6	0	0	0	0	1296	3660	5700	6165	4945	2997	1365
S_5	0	0	0	125	222	205	120	45	10	1	0
$S_5 \times S_2$	0	0	0	0	125	222	205	120	45	10	1
S_4	0	0	16	15	6	1	0	0	0	0	0
$S_4 \times S_2$	0	0	0	16	15	6	1	0	0	0	0
$S_4 \times S_3$	0	0	0	0	48	61	33	9	1	0	0
S_3	0	3	1	0	0	0	0	0	0	0	0
$S_3 \times S_2$	0	0	3	1	0	0	0	0	0	0	0
$S_3 \times S_2^2$	0	0	0	3	1	0	0	0	0	0	0
S_3^2	0	0	0	9	6	1	0	0	0	0	0
S_2	1	0	0	0	0	0	0	0	0	0	0
S_2^2	0	1	0	0	0	0	0	0	0	0	0
S_2^3	0	0	1	0	0	0	0	0	0	0	0

Proposition 3.8. *Fi_{24}' is not $(2B, 3D, 7A)$-generated.*

Proof. As discussed above, we proceed by inclusion–exclusion, but first we must lay some additional groundwork. Note that all computations will proceed in Fi_{24} rather than in Fi_{24}'.

We start by determining the precise number of $C_G(z)$-conjugates of the centralizer $C_G(t) \cong 2 \times Fi_{23}$ where $t \in H \le C_G(z)$, as well as the contribution to $\Delta(Fi_{24})$ coming from all $(2B, 3D, 7A)$-subgroups contained in these conjugates. In subsequent steps, we do this for intersections of two $C_G(z)$-conjugates of $C_G(t)$, and then for intersections of three such conjugates and so on, ultimately reaching 11.

All data used in the following computations may be gleaned from Tables 3 and 4 as well as from Lemma 3.7. Observe that there are only five rows of Table 4 that are relevant to our proof, viz. those headed by S_3, $S_3 \times S_2$, S_3^2, S_2, S_2^2, S_2^3, as these are the only subgroups $\langle S\rangle < H \cong S_7$ for which $C_G(S)$ contributes a positive value to $\Delta_{Fi_{24}}(2B, 3D, 7A)$.

One conjugate. There are 21 $C_G(z)$-conjugates of $C_G(t) \cong 2 \times Fi_{23}$ and each contributes 15680 to $\Delta(Fi_{24})$. This gives a total contribution of $21 \times 15\,680 = 329\,280$.

Two conjugates. There are 210 intersections of two $C_G(z)$-conjugates of $C_G(t)$ and these are of two types: (1) $3 \times 35 = 105$ intersections isomorphic to $O_8^+(3):S_3$, giving a contribution of $105 \times 756 = 79\,380$, and (2) $1 \times 105 = 105$ intersections isomorphic to $2 \times 2Fi_{22}$, giving a contribution of $105 \times 1\,848 = 19\,404$. Thus the total contribution to $\Delta(Fi_{24})$ coming from intersections of two conjugates is $79\,380 + 194\,040 = 273\,420$.

Three conjugates. Relevant intersections here are of two types: (1) $1 \times 35 = 35$ intersections isomorphic to $O_8^+(3):S_3$, giving a contribution of $35 \times 756 = 26\,460$ and (2) $3 \times 210 = 630$ intersections isomorphic to $2 \times O_7(3)$, giving a contribution of $630 \times 182 = 114\,660$. Thus the total contribution to $\Delta(Fi_{24})$ coming from intersections of three conjugates is $26\,460 + 114\,660 = 141\,120$.

Four conjugates. Relevant intersections here are of two types: (1) $1 \times 210 = 210$ intersections isomorphic to $2 \times O_7(3)$, giving a contribution of $210 \times 182 = 38\,220$ and (2) $9 \times 70 = 630$ intersections isomorphic to $G_2(3)$, giving a contribution of $630 \times 42 = 26\,460$. Thus the total contribution to $\Delta(Fi_{24})$ coming from intersections of four conjugates is $38\,220 + 26\,460 = 64\,680$.

Five conjugates. The only relevant intersections of five conjugates are the $6 \times 70 = 420$ ones isomorphic to $G_2(3)$. They contribute $420 \times 42 = 17\,640$ to $\Delta(Fi_{24})$.

Six conjugates. The only relevant intersections of six conjugates are the $1 \times 70 = 70$ ones isomorphic to $G_2(3)$. They contribute $70 \times 42 = 2\,940$ to $\Delta(Fi_{24})$.

Finally observe from Tables 3 and 4 that the intersection of seven or more conjugates contribute zero to $\Delta(Fi_{24})$.

We are now prepared to invoke the principle of inclusion–exclusion to account for all $(2B, 3D, 7A)$-triples in $G = Fi_{24}$ that lie in at least one conjugate of $C_G(t)$, $t \in C_G(z)$. This involves adding all contributions coming from intersections of an odd number of conjugates, from which we subtract all contributions coming from intersections of an even number of conjugates. Specifically, this gives

$$329\,280 - 273\,420 + 141\,120 - 64\,680 + 17\,640 - 2\,940 = 147\,000$$

which is the precise value of $\Delta(G)$. We conclude that every $(2B, 3D, 7A)$-subgroup of $G = Fi_{24}$ must lie in at least one $C_G(z)$-conjugate of $C_G(t) \cong Fi_{23} \times 2$, whence Fi_{24}' is not $(2B, 3D, 7A)$-generated. □

Theorem 3.9. *The only* $(2,3,7)$*-triple that generates* Fi_{24}' *is of type* $(2B,3E,7B)$.

Proof. We first establish that Fi_{24}' is $(2B,3E,7B)$-generated. The only maximal subgroups of Fi_{24}' with non-empty intersection with all three conjugacy classes are $3^7{\cdot}O_7(3)$, $2^{1+12}.3U_4(3).2_2$ and $U_3(3).2$ (two classes). We compute $\Delta_{Fi_{24}'}(2B,3E,7B)=224322$, $\Sigma(3^7{\cdot}O_7(3))=2079$, $\Sigma(2^{1+12}.3U_4(3).2_2)=322$ and $\Sigma(U_3(3).2)=7$. For a fixed subgroup of Fi_{24}' isomorphic to each of $3^7{\cdot}O_7(3)$, $2^{1+12}.3U_4(3).2_2$ and $U_3(3).2$, a fixed element $z\in 7B$ is contained in precisely 49, 49, and 294 conjugates thereof, respectively. In particular, z lies in 588 copies of $U_3(3).2$ in all, i.e. 294 from each class. Consulting Table 2, we therefore have

$$\begin{aligned}\Delta^*_{Fi_{24}'}(2B,3E,7B) &\geq \Delta(Fi_{24}')-49\,\Sigma(3^7{\cdot}O_7(3))-49\,\Sigma(2^{1+12}.3U_4(3).2_2)\\ &\quad -588\,\Sigma(U_3(3).2)\\ &=224\,322-49(2\,079)-49(322)-588(7)>0,\end{aligned}$$

which proves that Fi_{24}' is $(2B,3E,7B)$-generated.

We now establish non-generation in all remaining cases.

Non-generation by triples of type $(2B,3D,7B)$ and $(2B,3D,7A)$ follows from Propositions 3.5 and 3.8, respectively. To prove non-generation of type $(2B,3E,7A)$, we compute $\Delta_{Fi_{24}'}(2B,3E,7A)=70\,560$ and $\Sigma(2^3{\cdot}L_2(7))=14$. As the elementary abelian subgroup 2^3 of $2^3{\cdot}L_2(7)$ is an irreducible $L_2(7)$-module, we further have $\Sigma^*(2^3{\cdot}L_2(7))=14$. (Indeed, by [21, p.156] all $(2,3E,7)$-subgroups of Fi_{24}' have trivial centralizer and $L_2(7)$ does not meet the class $3E$.) As there are 5040 conjugates of $2^3.L_2(7)$ containing a fixed element $z\in 7A$, we thereby obtain

$$\Delta^*(Fi_{24}')=\Delta(Fi_{24}')-5\,040\,\Sigma(2^3.L_2(7))=70\,560-5\,040\times 14=0,$$

which establishes that Fi_{24}' is not $(2B,3E,7A)$.

Of the remaining triples, the only ones that contribute a positive value to $\Delta(Fi_{24}')$ are the following:

$$(2A,3C,7A),\ (2A,3E,7B),\ (2B,3B,7A),\ (2B,3C,7A).$$

However, we easily compute

$$\begin{aligned}\Delta_{Fi_{24}'}(2A,3C,7A)&=294<17\,640=|C_{Fi_{24}'}(7A)|\\ \Delta_{Fi_{24}'}(2A,3E,7B)&=196<2\,058=|C_{Fi_{24}'}(7B)|\\ \Delta_{Fi_{24}'}(2B,3B,7A)&=49<17\,640=|C_{Fi_{24}'}(7A)|\\ \Delta_{Fi_{24}'}(2B,3C,7A)&=5\,439<17\,640=|C_{Fi_{24}'}(7A)|\end{aligned}$$

from which the result follows from Lemma 2.3. □

3.2. The case $r=11$

As Fi_{24}' has a unique class of elements of order 11, we have 10 triples of classes to consider in this case.

Proposition 3.10. Fi_{24}' *is not* $(2A,3E,11A)$*-generated.*

Proof. The only maximal subgroups of Fi_{24}' with nonzero structure constant are conjugates of $M\cong 3^{1+10}{:}U_5(2){:}2$ and $K\cong 2^{11}{\cdot}M_{24}$ for which we have $\Sigma_M(2A,3E,11A)=\Sigma_K(2A,3E,11A)=44$. We further observe that a fixed element $z\in 11A$ is contained in four conjugates of M in Fi_{24}'. Specifically, these are M, M^u, M^v, M^{uv}, where $\langle u,v\rangle$ is a Klein-four subgroup in the centralizer $C_{Fi_{24}'}(z)\cong 11\times 2\times S_3$. We claim that the intersection of any two such conjugates contributes zero to the structure constant $\Delta_{Fi_{24}'}(2A,3E,11A)=440$. By symmetry (and conjugation by elements of $\langle u,v\rangle$), it suffices to prove the claim for the intersection $H=M\cap M^u$.

First observe that a central element $t\in O_3(M)\cong 3^{1+10}$ commutes with z, i.e. $t\in C_{Fi_{24}'}(z)$. From the structure of $C_{Fi_{24}'}(z)$, we see at once that $u\in C_{Fi_{24}'}(z)$ normalizes $\langle t\rangle$ (indeed, $\langle t,u\rangle$ is isomorphic to either $\mathbb{Z}_6$ or S_3). Hence $t\in H$. Furthermore, u normalizes H since, being an involution, it merely interchanges M and M^u.

Now set $N=N_{Fi_{24}'}(H)$. As $|C_M(z)|=33$, we have $u\notin M$ whence $N\not\leq M$. Similarly, since $|C_K(z)|=44$, we have $t\notin K$ so $N\not\leq K$. Naturally, $N\neq Fi_{24}'$, so N must be contained in a maximal subgroup of Fi_{24}', say X. But this implies that $\Sigma(X)=0$, since X is neither isomorphic to M nor K. It follows that $\Sigma(H)=0$, in which case we are getting a contribution of $4\times 44=176$ coming from the four conjugates M, M^u, M^v, M^{uv}.

We next claim that the intersection of $K=2^{11}{\cdot}M_{24}$ with any conjugate of M cannot contain any $(2A,3E,11A)$-subgroup. Indeed, set $U:=K\cap M$ (the cases of other conjugates of M are similar) and suppose that $L\leq U$ is a potential

$(2A, 3E, 11A)$-subgroup. Then, since $L_2(11)$ is the only $(2, 3, 11)$-subgroup of $U_5(2)$, we have $L/O_3(L) \cong L_2(11)$. However, since $L < K$ it now follows that $O_3(L) = 1$. Thus $L \cong L_2(11)$, in which case $\Sigma(L) = 0$, since all $L_2(11)$-subgroups of $U_5(2)$ fail to meet the Fi'_{24}-class $3E$.

This means that K contributes an additional 44 to the structure constant $\Delta(Fi'_{24}) = 440$. The contribution thus far is $176 + 44 = 220$ and this is sufficient to complete the proof. Indeed, applying Lemma 2.2, we obtain

$$\Delta^*(Fi'_{24}) \leq \Delta(Fi'_{24}) - 220 = 440 - 220 = 220 < 264 = |C_{Fi_{24}}(z)|$$

whence Fi'_{24} is not $(2A, 3E, 11A)$-generated. □

Theorem 3.11. *Let $X \in \{A, B\}$, $Y \in \{A, B, C, D, E\}$. Then Fischer's group Fi'_{24} is $(2X, 3Y, 11A)$-generated if and only if $(X, Y) \in \{(B, D), (B, E)\}$*

Proof. Let $(X, Y) \in \{(A, A), (B, A), (A, B), (B, B), (A, C), (A, D)\}$. Then because $\Delta_{Fi'_{24}}(2X, 3Y, 11A) < |C_{Fi_{24}}(11A)| = 264$ for all of these choices, it follows from Lemma 2.2 that Fi'_{24} is not $(2X, 3Y, 11A)$-generated for such (X, Y). Moreover, non-generation of Fi'_{24} for $(X, Y) = (A, E)$ was established in Proposition 3.10.

Next consider $(X, Y) = (B, C)$. Here $\Delta_{Fi'_{24}}(2B, 3C, 11A) = 726$, while $\Sigma(Fi_{23}) = 374$ and $\Sigma(2Fi_{22} : 2) = 154$. We need only consider two of the three conjugates of Fi_{23} that contain $z \in 11A$. Letting t be the outer automorphism of $2Fi_{22}$ in $2Fi_{22} : 2$, we have that $2Fi_{22} = Fi_{23} \cap (Fi_{23})^t$. This already accounts for a contribution of $2 \times 374 - 154 = 594$ coming from just Fi_{23} and $(Fi_{23})^t$. Non-generation of type $(2B, 3C, 11A)$ now follows from Lemma 2.2, since

$$\Delta^*(Fi'_{24}) \leq 726 - 594 = 132 < 264 = |C_{Fi_{24}}(z)|.$$

Finally, we consider the remaining two cases $(2B, 3D, 11A)$ and $(2B, 3E, 11A)$. We calculate the relevant structure constants to be $\Delta_{Fi'_{24}}(2B, 3D, 11A) = 55\,044$ and $\Delta_{Fi'_{24}}(2B, 3E, 11A) = 165\,792$. The only maximal subgroups of Fi'_{24} that meet these classes are isomorphic to Fi_{23}, $2.Fi_{22}{:}2$, $O^-_{10}(2)$, $3^{1+10}{:}U_5(2){:}2$ and $2^2{\cdot}U_6(2){:}S_3$. However, $\Sigma_{3^{1+10}:U_5(2):2}(2B, 3D, 11A) = \Sigma_{2^2\cdot U_6(2):S_3}(2B, 3D, 11A) = 0$. Further observe that any proper $(2B, 3E, 11A)$-generated subgroup of Fi'_{24} must be contained in one of $3^{1+10}{:}U_5(2){:}2$, $2^{11}{\cdot}M_{24}$, $2^2{\cdot}U_6(2){:}S_3$. However, $\Sigma_{3^{1+10}:U_5(2):2}(2B, 3E, 11A) = \Sigma_{2^2\cdot U_6(2):S_3}(2B, 3E, 11A) = 0$. Hence, consulting the fusion maps of Fi_{23}, $2.Fi_{22}{:}2$, $O^-_{10}(2)$, $2^{11}{\cdot}M_{24}$ in Table 2, we obtain

$$\begin{aligned}
\Delta^*_{Fi'_{24}}(2B, 3D, 11A) &\geq \Delta_{Fi'_{24}}(2B, 3D, 11A) - 3\Sigma_{Fi_{23}}(2B, 3D, 11A) \\
&\quad - 3\Sigma_{2.Fi_{22}:2}(2B, 3D, 11A) - 8\Sigma_{O^-_{10}(2)}(2B, 3D, 11A) \\
&= 55\,044 - 3(11\,616) - 3(1\,980) - 8(594) > 0, \\
\Delta^*_{Fi'_{24}}(2B, 3E, 11A) &\geq \Delta_{Fi'_{24}}(2B, 3E, 11A) - 6\Sigma_{2^{11}\cdot M_{24}}(2B, 3E, 11A) \\
&= 165\,792 - 6(1\,012) > 0.
\end{aligned}$$

Thus generation by both triples is established and the proof is complete. □

3.3. The case $r = 13$

As in the previous case, Fi'_{24} has a unique class of elements of order 13. Hence we have 10 triples of classes to consider.

Theorem 3.12. *Let $X \in \{A, B\}$, $Y \in \{A, B, C, D, E\}$. Then Fischer's group Fi'_{24} is $(2X, 3Y, 13A)$-generated if and only if $(X, Y) \in \{(B, D), (B, E)\}$.*

Proof. For $(X, Y) \in \{(A, A), (B, A), (A, B), (A, C), (A, E)\}$, non-generation by the corresponding triples $(2X, 3Y, 13A)$ follows from $\Delta_{Fi'_{24}}(2X, 3Y, 13A) = 0$. Likewise, non-generation of type $(X, Y) \in \{(B, B), (A, E)\}$ follows from Lemma 2.3 since

$$\Delta_{Fi'_{24}}(2B, 3B, 13A) = 13 < 234 = |C_{Fi'_{24}}(13A)|,$$

$$\Delta_{Fi'_{24}}(2A, 3E, 13A) = 156 < 234 = |C_{Fi'_{24}}(13A)|.$$

We next treat the case $(X, Y) = (B, C)$. Here we obtain $\Delta_{Fi'_{24}}(2B, 3C, 13A) = 975$, $\Sigma_{Fi_{23}}(2B, 3C, 13A) = 429$ and $\Sigma_{2Fi_{22}:2}(2B, 3C, 13A) = 195$. To prove non-generation, it suffices to consider just two of the six conjugates of Fi_{23} that contain $z \in 13A$. Letting t be the outer automorphism of $2Fi_{22}$ in $2Fi_{22} : 2$, we have that $2Fi_{22} = Fi_{23} \cap (Fi_{23})^t$. This already accounts for a contribution of $2 \times 429 - 195 = 663$ coming from just Fi_{23} and $(Fi_{23})^t$. Non-generation of type $(2B, 3C, 13A)$ now follows from Lemma 2.2, since

$$\Delta^*_{Fi'_{24}}(2B, 3C, 13A) \leq \Delta(Fi'_{24}) - 663 = 975 - 663 = 312 < 468 = |C_{Fi_{24}}(z)|.$$

We now turn our attention to triples of type $(2B, 3D, 13A)$ and $(2B, 3E, 13A)$.

Generation by $(2B, 3D, 13A)$ triples was established by Linton & Wilson [21, pp. 149–151] using their "fingerprint" computational approach, e.g., see [21, p. 151]. In fact, 28 non-conjugate classes of such generating triples were identified via this method.

For the final remaining triple $(2B, 3E, 13A)$, we compute the structure constant $\Delta_{Fi'_{24}}(2B, 3E, 13A) = 185\,328$. The maximal subgroups of Fi'_{24} with order divisible by 13 are Fi_{23}, $2.Fi_{22}{:}2$, $(3\times O_8^+(3){:}3){:}2$, $3^7{\cdot}O_7(3)$, $3^3.[3^{10}].GL_3(3)$, $(3^2{:}2\times G_2(3)).2$ and $L_2(13){:}2$. But none of Fi_{23}, $2.Fi_{22}{:}2$, $L_2(13){:}2$ meets all classes in the triple $(2B, 3E, 13A)$. Further we calculate

$$\Sigma((3\times O_8^+(3){:}3){:}2) = \Sigma(3^3.[3^{10}].GL_3(3)) = \Sigma((3^2{:}2\times G_2(3)).2) = 0.$$

Thus the only maximal subgroup that can admit $(2B, 3E, 13A)$-generated subgroups is isomorphic to $3^7{\cdot}O_7(3)$. We now compute

$$\begin{aligned}\Delta^*_{Fi'_{24}}(2B, 3E, 13A) &\geq \Delta_{Fi'_{24}}(2B, 3E, 13A) - 12\Sigma_{3^7\cdot O_7(3)}(2B, 3E, 13A)\\ &= 185\,328 - 12(1\,404) > 0,\end{aligned}$$

which establishes that Fi'_{24} is $(2B, 3E, 13A)$-generated, completing the proof. □

3.4. The case $r = 17$

Once again there are 10 cases to consider, since Fi'_{24} has a unique class of elements of order 17.

Proposition 3.13. *Fi'_{24} is $(2B, 3C, 17A)$-generated.*

Proof. Our mode of proof is to construct a partial subgroup lattice for Fi'_{24}, consisting only of those groups that could potentially contribute a positive value to the structure constant $\Delta_{Fi'_{24}}(2B, 3C, 17A) = 408$, i.e. those that meet all three Fi'_{24}-classes.

We begin by analyzing certain subchains of the lattice, starting at the bottom and gradually working our way up.

(1) $L_2(16) < S_4(4) < S_8(2)$. We compute $\Sigma^*(L_2(16)) = 17$, $\Sigma(S_4(4)) = 34$ and $\Sigma(S_8(2)) = 68$. As a fixed $z \in 17A$ lies in two $S_4(4)$-conjugates of $L_2(16)$, we obtain $\Sigma^*(S_4(4)) = 0$. Also observe that the only contribution toward $\Sigma(S_8(2))$ so far is coming from these two conjugates of $L_2(16)$.

(2) $L_2(16) < O_8^-(2) < S_8(2)$. We compute $\Sigma(O_8^-(2)) = 51$. A fixed $z \in 17A$ lies in two $O_8^-(2)$-conjugates of $L_2(16)$, hence we obtain $\Sigma^*(O_8^-(2)) = 17$. In addition to the two conjugates of $L_2(16)$ mentioned in **(1)**, z lies in two $S_8(2)$-conjugates of $O_8^-(2)$. Thus we obtain

$$\Sigma^*(S_8(2)) = \Sigma(S_8(2)) - 2\,\Sigma^*(O_8^-(2)) - 2\,\Sigma^*(L_2(16)) = 68 - 2\cdot 17 - 2\cdot 17 = 0.$$

(3) $L_2(16) < S_4(4) < He$. For each of the two classes of He in Fi'_{24}, we have $\Sigma^*(He) = 0$, since $\Sigma(He) = 34$ and $z \in 17A$ lies in two He-conjugates of $L_2(16)$.

(4) $S_8(2) < O_{10}^-(2)$. We compute $\Sigma(O_{10}^-(2)) = 136$. From **(1)** and **(2)** above, the only $(2B, 3C, 17A)$-generated subgroups of $O_{10}^-(2)$ are $L_2(16)$ and $O_8^-(2)$. As a fixed element $z \in 17A$ lies in two $O_{10}^-(2)$-conjugates of each of $L_2(16)$ and $O_8^-(2)$, we obtain $\Sigma^*(O_{10}^-(2)) = \Sigma(O_{10}^-(2)) - 2\,\Sigma^*(L_2(16)) - 2\,\Sigma^*(O_8^-(2)) = 136 - 2\cdot 17 - 2\cdot 17 = 68$.

(5) $S_8(2) < Fi_{23}$. The analysis is similar to **(4)** above, that is, we only have to account for contributions coming from the subgroups $L_2(16)$ and $O_8^-(2)$ of Fi_{23}. As $\Sigma(Fi_{23}) = 238$, and as $z \in 17A$ lies in two Fi_{23}-conjugates of $L_2(16)$ and four conjugates of $O_8^-(2)$, we compute

$$\Sigma^*(Fi_{23}) = \Sigma(Fi_{23}) - 2\,\Sigma^*(L_2(16)) - 4\,\Sigma^*(O_8^-(2)) = 238 - 2\cdot 17 - 4\cdot 17 = 136.$$

To summarize, the only proper $(2B, 3C, 17A)$-subgroups of Fi'_{24} are $L_2(16)$, $O_8^-(2)$, $O_{10}^-(2)$, and Fi_{23}. As the respective numbers of Fi'_{24}-conjugates of these subgroups containing a fixed element $z \in 17A$ are two, two, two and one, we obtain

$$\begin{aligned}\Delta^*(Fi'_{24}) &= \Delta(Fi'_{24}) - 2\,\Sigma^*(L_2(16)) - 2\,\Sigma^*(O_8^-(2)) - 2\,\Sigma^*(O_{10}^-(2)) - \Sigma^*(Fi_{23})\\ &= 408 - (2\cdot 17) - (2\cdot 17) - (2\cdot 34) - 170 = 68.\end{aligned}$$

This establishes that Fi'_{24} is $(2B, 3C, 17A)$-generated as claimed. □

Theorem 3.14. *Let $X \in \{A, B\}$, $Y \in \{A, B, C, D, E\}$. Then Fischer's group Fi'_{24} is $(2X, 3Y, 17A)$-generated if and only if $(X, Y) \in \{(A, E), (B, C), (B, D), (B, E)\}$.*

Proof. For $(X,Y) \in \{(A,A),(A,B),(A,C),(B,A),(B,B)\}$ non-generation of Fi_{24}' follows at once since all corresponding structure constants are zero. For the case $(X,Y)=(A,D)$ we compute $\Delta(Fi_{24}')=34$ and $\Sigma(O_{10}^{-}(2))=17$. Thus non-generation of this type follows from Lemma 2.2, since $\Delta^{*}(Fi_{24}') \le 34-17 < 34 = |C_{Fi_{24}}(17A)|$.

Thus it remains to establish generation for each of $(2B,3C,17A)$, $(2A,3E,17A)$, $(2B,3E,17A)$, and $(2B,3D,17A)$. The first of these cases was proved in Proposition 3.13. For the remaining cases, we observe that the only maximal subgroups of Fi_{24}' with order divisible by 17 are Fi_{23}, $O_{10}^{-}(2)$ and $He{:}2$. However, neither Fi_{23} nor $O_{10}^{-}(2)$ meet all classes in the triples $(2A,3E,17A)$ and $(2B,3E,17A)$, and $He{:}2$ does not meet all classes in the triple $(2B,3D,17A)$. Further, a fixed element of order 17 in Fi_{24}' is contained in a unique conjugate of Fi_{23}, in two conjugates of $O_{10}^{-}(2)$, and in a unique conjugate of each of $He{:}2$. From Table 2, it now follows that

$$\begin{aligned}
\Delta^{*}_{Fi_{24}'}(2B,3D,17A) &\ge \Delta_{Fi_{24}'}(2B,3D,17A) - \Sigma_{Fi_{23}}(2B,3D,17A) \\
&\quad -2\Sigma_{O_{10}^{-}(2)}(2B,3D,17A) \\
&= 49\,844 - 11\,322 - 2(816) > 0, \\
\Delta^{*}_{Fi_{24}'}(2A,3E,17A) &\ge \Delta_{Fi_{24}'}(2A,3E,17A) - 2\Sigma_{He:2}(2A,3E,17A) \\
&= 204 - 2(51) > 0, \\
\Delta^{*}_{Fi_{24}'}(2B,3E,17A) &\ge \Delta_{Fi_{24}'}(2B,3E,17A) - 2\Sigma_{He:2}(2B,3E,17A) \\
&= 191\,114 - 2(374) > 0,
\end{aligned}$$

which establishes generation for each indicated triple, as claimed. □

3.5. *The case $r=23$*

Strictly speaking, there are 20 cases to consider, since Fi_{24}' has two classes of elements of order 23. However, as the classes $23A$ and $23B$ are algebraically conjugate, it clearly suffices to restrict our attention to the 10 cases corresponding to the class $23A$. (Indeed, it is immediate that Fi_{24}' is $(2X,3Y,23B)$-generated if and only if it is $(2X,3Y,23A)$-generated.)

Theorem 3.15. *Let $X \in \{A,B\}$, $Y \in \{A,B,C,D,E\}$. Then Fischer's group Fi_{24}' is $(2X,3Y,23A)$-generated if and only if $(X,Y) \in \{(A,E),(B,C),(B,D),(B,E)\}$.*

Proof. For $(X,Y) \in \{(A,A),(A,B),(A,C),(B,A),(B,B)\}$, all structure constants are zero, thus establishing non-generation. Non-generation of type $(2A,3D,23A)$ also follows easily since $\Delta_{Fi_{24}'}(2A,3D,23A) = \Sigma_{Fi_{23}}(2A,3D,23A) = 23$.

It remains to show that the four remaining cases all lead to generations. From Table 1, we see that the only maximal subgroups containing elements of order 23 are Fi_{23} and $2^{11}{\cdot}M_{24}$. However, the class $3E$ does not meet Fi_{23}, and the class $3D$ does not meet $2^{11}{\cdot}M_{24}$. Furthermore, a fixed element $z \in 23A$ is contained in a unique conjugate class of each of Fi_{23} and $2^{11}{\cdot}M_{24}$. We now calculate $\Sigma_{2^{11}\cdot M_{24}}(2A,3E,23Z)=46$, $\Sigma_{2^{11}\cdot M_{24}}(2B,3C,23Z)=46$, $\Sigma_{2^{11}\cdot M_{24}}(2B,3E,23Z)=506$, $\Sigma_{Fi_{23}}(2B,3C,23Z)=161$, $\Sigma_{Fi_{23}}(2B,3D,23Z)=11592$ and $\Sigma_{2^{11}\cdot M_{24}}(2B,3E,23Z)=506$. Together, this yields

$$\begin{aligned}
\Delta^{*}_{Fi_{24}'}(2A,3E,23Z) &\ge \Delta_{Fi_{24}'}(2A,3E,23Z) - \Sigma_{2^{11}\cdot M_{24}}(2A,3E,23Z) \\
&= 138 - 1(46) > 0, \\
\Delta^{*}_{Fi_{24}'}(2B,3C,23Z) &\ge \Delta_{Fi_{24}'}(2B,3C,23Z) - \Sigma_{Fi_{23}}(2B,3C,23Z) \\
&\quad -\Sigma_{2^{11}\cdot M_{24}}(2B,3C,23Z) \\
&= 345 - 161 - 161 - 46 > 0, \\
\Delta^{*}_{Fi_{24}'}(2B,3D,23Z) &\ge \Delta_{Fi_{24}'}(2B,3D,23Z) - \Sigma_{Fi_{23}}(2B,3D,23Z) \\
&= 52\,302 - 11\,592 > 0, \\
\Delta^{*}_{Fi_{24}'}(2B,3E,23Z) &\ge \Delta_{Fi_{24}'}(2B,3E,23Z) - \Sigma_{2^{11}\cdot M_{24}}(2B,3E,23Z) \\
&= 199\,962 - 506 > 0.
\end{aligned}$$

Hence we have established $(2X,3Y,23A)$-generation for each of the four indicated triples, and the result follows. □

3.6. *The case* $r = 29$

Although Fi'_{24} has two classes of elements of order 29, they are algebraically conjugate. Thus, just as in the previous case, we may restrict our attention to the class $29A$. This means that we have once again 10 cases to consider.

Theorem 3.16. *Let* $X \in \{A, B\}$*,* $Y \in \{A, B, C, D, E\}$*. Then Fischer's group* Fi'_{24} *is* $(2X, 3Y, 29A)$*-generated if and only if* $(X, Y) \in \{(A, E), (B, C), (B, D), (B, E)\}$*.*

Proof. From Table 1, we see that the only maximal subgroup of Fi'_{24} with order divisible by 29 is $N_G(z) \cong 29{:}14$, where $z \in 29A$. As $|N_G(z)|$ is not divisible by 3, Fi'_{24} cannot have any proper $(2X, 3Y, 29A)$-subgroups. This means that Fi'_{24} is $(2X, 3Y, 29A)$-generated precisely when $\Delta_{Fi'_{24}}(2X, 3Y, 29A) > 0$, and this occurs if and only if $(X, Y) \in \{(A, E), (B, C), (B, D), (B, E)\}$. □

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