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Mathematical problems in mechanics

Thin layer approximations in mechanical structures: The Dirichlet boundary condition case

Approximations de couche mince dans les structures mécaniques : le cas de la condition aux limites de Dirichlet

Frédérique Le Louër

Sorbonne Universités, Université de technologie de Compiègne, LMAC EA2222 Laboratoire de mathématiques appliquées de Compiègne, CS 60 319, 60203 Compiègne cedex, France

A R T I C L E I N F O

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A B S T R A C T

We consider the solution to a transmission problem at a thin layer interface of thickness $\varepsilon > 0$ in a mechanical structure. We build a multi-scale expansion for that solution as $\varepsilon \to 0$, which enables to replace the thin layer with an improved boundary condition and leads to optimal estimates for the remainders. This short note presents new results when a Dirichlet condition is imposed on the internal boundary of the thin layer and is the counterpart of F. Caubet, D. Kateb, F. Le Louër, J. Elast. 136 (1) (2019) 17–53, where the Neumann case was considered.

R É S U M É

Cette note concerne un problème de transmission dans une structure mécanique contenant une couche d'épaisseur mince $\varepsilon > 0$. Nous construisons un développement asymptotique de la solution lorsque $\varepsilon \to 0$ qui permet de remplacer la couche mince par une condition aux limites approchées et nous en déduisons des estimations d'erreurs optimales. Nous présentons de nouveaux résultats lorsqu'une condition de Dirichlet est imposée sur la frontière interne de la couche mince, tandis que le cas d'une condition de Neumann est étudié dans F. Caubet, D. Kateb, F. Le Louër, J. Elast. 136 (1) (2019) 17–53.

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1. Introduction and problem settings

Let Ω be a Lipschitz bounded open set of $\mathbb{R}^d$, where $d \geq 2$ is an integer representing the dimension. We assume that the solid Ω consists of an isotropic material with a linear behavior. The boundary of Ω is such that $\partial\Omega =: \Gamma_D \cup \Gamma_N$, where Γ_D

E-mail address: frederique.le-louer@utc.fr.

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and Γ_N are two non-empty open sets of $\partial\Omega$ and $|\Gamma_D| > 0$. We consider a nonempty inclusion $\omega \subset\subset \Omega$ with analytic boundary $\partial\omega =: \Gamma$. We denote by $\boldsymbol{n}$ the unit normal vector to $\partial\Omega$ and Γ directed outward to $\Omega\backslash\overline{\omega}$.

Let $\varepsilon > 0$ be small enough. We consider that Γ has an interior thin layer with thickness ε bordering ω defined by

$$\omega_i^\varepsilon := \{x + s\boldsymbol{n}(x) \mid x \in \Gamma \text{ and } 0 < s < \varepsilon\}.$$

We recall that the normal vector $\boldsymbol{n}$ is directed inward the inclusion ω. We set $\omega^\varepsilon := \omega\backslash\overline{\omega_i^\varepsilon}$ and we denote its boundary by Γ^ε. In the sequel, we use the lower index e for all quantities related to $\Omega\backslash\overline{\omega}$ and the lower index i for all quantities related to ω_i^ε. These notations are illustrated in [2, Fig. 1].

We denote by A_e the Hooke's law defined, for any symmetric matrix ξ, by

$$A_e\,\xi := 2\,\mu_e\,\xi + \lambda_e\,\mathrm{Tr}(\xi)\,\mathrm{I}_d,$$

where $\mu_e > 0$ and $\lambda_e > 0$ are two positive constants that represent the Lamé coefficients in $\Omega\backslash\overline{\omega}$. The Hooke's law associated with ω_i^ε is denoted by A_i with Lamé coefficients $\mu_i > 0$ and $\lambda_i > 0$. Moreover, the stress vector relative to the material properties A_i on Γ is defined by

$$\boldsymbol{T}_i(\boldsymbol{u}) := A_i e(\boldsymbol{u})\boldsymbol{n}, \quad \text{where} \quad e(\boldsymbol{u}) := \frac{1}{2}\left(\nabla\boldsymbol{u} + {}^{\mathsf{T}}\nabla\boldsymbol{u}\right),$$

and we define similarly $\boldsymbol{T}_e$ the stress vector relative to the material properties A_e on either Γ or Γ_N.

For a smooth bounded open set ω of $\mathbb{R}^d$ ($d \geq 2$) with a boundary Γ, we denote by $H^s(\omega)$ and $H^s(\Gamma)$ the standard complex valued, Hilbert–Sobolev spaces of order $s \in \mathbb{R}$ defined on ω and Γ, respectively (with the convention $H^0 = L^2$). The spaces of vector functions will be denoted by boldface letters, thus $\boldsymbol{H}^s = (H^s)^d$. We introduce the following Sobolev space:

$$\boldsymbol{H}^1_{\Gamma_D}(\Omega\backslash\overline{\omega}) := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega\backslash\overline{\omega})\ ;\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D\}.$$

The dual space of $\boldsymbol{H}^1_{\Gamma_D}(\Omega\backslash\overline{\omega})$ is denoted by $\widetilde{\boldsymbol{H}}^{-1}_{\Gamma_D}(\Omega\backslash\overline{\omega})$. Let $\boldsymbol{f} \in \widetilde{\boldsymbol{H}}^{-1}_{\Gamma_D}(\Omega\backslash\overline{\omega})$ be some exterior forces and a load $\boldsymbol{g} \in \boldsymbol{H}^{-1/2}(\Gamma_N)$. We are concerned with the following transmission problem

$$\begin{cases} -\operatorname{div}\left(A_e e(\boldsymbol{u}_e^\varepsilon)\right) = \boldsymbol{f} & \text{in } \Omega\backslash\overline{\omega} \\ -\operatorname{div}\left(A_i e(\boldsymbol{u}_i^\varepsilon)\right) = \boldsymbol{0} & \text{in } \omega_i^\varepsilon \\ \boldsymbol{u}_e^\varepsilon = \boldsymbol{0} & \text{on } \Gamma_D \\ \boldsymbol{T}_e(\boldsymbol{u}_e^\varepsilon) = \boldsymbol{g} & \text{on } \Gamma_N \\ \boldsymbol{T}_i(\boldsymbol{u}_i^\varepsilon) = \boldsymbol{T}_e(\boldsymbol{u}_e^\varepsilon) & \text{on } \Gamma \\ \boldsymbol{u}_i^\varepsilon = \boldsymbol{u}_e^\varepsilon & \text{on } \Gamma \\ \boldsymbol{u}_i^\varepsilon = \boldsymbol{0} & \text{on } \Gamma^\varepsilon. \end{cases} \tag{1.1}$$

The solution to such a problem exists, is unique and belongs to $\boldsymbol{H}^1_{\Gamma_D\cup\Gamma^\varepsilon}(\Omega\backslash\overline{\omega^\varepsilon})$ thanks to the Lax–Milgram theorem and Korn's inequality (see, e.g., [3, Theorem 6.3-4]).

To avoid instabilities in the numerical treatment of the transmission problem (1.1), we approximate the solution $\boldsymbol{u}_e^\varepsilon \in \boldsymbol{H}^1_{\Gamma_D}(\Omega\backslash\overline{\omega})$ by the solution $\boldsymbol{v}_{[N]}^\varepsilon$ to some boundary value problems of the form

$$\begin{cases} -\operatorname{div}\left(A_e e(\boldsymbol{v}_{[N]}^\varepsilon)\right) = \boldsymbol{f} & \text{in } \Omega\backslash\overline{\omega} \\ \boldsymbol{v}_{[N]}^\varepsilon = \boldsymbol{0} & \text{on } \Gamma_D \\ \boldsymbol{T}_e(\boldsymbol{v}_{[N]}^\varepsilon) = \boldsymbol{g} & \text{on } \Gamma_N \\ \mathbf{B}_N^\varepsilon\left(\varepsilon, \boldsymbol{v}_{[N]}^\varepsilon, \boldsymbol{T}_e(\boldsymbol{v}_{[N]}^\varepsilon)\right) = \boldsymbol{0} & \text{on } \Gamma, \end{cases} \tag{1.2}$$

where $||\boldsymbol{u}_e^\varepsilon - \boldsymbol{v}_{[N]}^\varepsilon||_{\boldsymbol{H}^1(\Omega\backslash\overline{\omega})} = O(\varepsilon^{N+1})$, for any $N \in \mathbb{N}$, and the last equation of (1.2) is a so-called *Generalized Impedance Boundary Condition* (GIBC). This approximate condition is obtained by expanding the Navier equation in ω_i^ε in terms of ε and surface derivatives on Γ. The transmission problems is then split into sequences of coupled boundary value problems in ω_i^ε (rescaled through a nondimensional variable) and in $\Omega\backslash\overline{\omega}$ (which will constitute the GIBCs). Both the exterior and interior solutions are expanded as power series of the thickness ε, whose coefficients functions are obtained iteratively. The GIBC of order $N \in \mathbb{N}$ is deduced from the boundary condition satisfied by the truncated series of the exterior field up to the index N. The method originates from [10] for $d = 2$ and leads to optimal error estimates. The results are now established for the Laplace, Helmholtz, and Maxwell equations, and more recently for the Lamé system when a Neumann condition is imposed on Γ^ε (see [2] and references therein for a bibliographical overview). It is the purpose of this short note to address the case when a Dirichlet condition is considered on Γ^ε. The coefficient functions and the GIBCs, for $N = 0, 1, 2$, are given in Proposition 2.1 and its proof. The expected optimal error estimates are stated in Theorem 3.1 and its proof for both the interior and exterior solutions.

Mechanical engineering applications of the proposed results include the modeling of delaminated elastic area with thin opening by Dirichlet and Neumann crack jumps [1,4] at multi-layered interfaces followed by the mathematical analysis of the associated inverse problem of delimination detection [1,8,9] for both the static and dynamic framework.

2. Generalized impedance boundary conditions

The operator $\mathbf{B}_N^\varepsilon$ is composed of curvature operators and/or surface differential operators and depends on the interior Lamé parameters. Thus we use the classical surface differential operators: the tangential gradient ∇_Γ defined in [7, pages 68–75] and the surface divergence div_Γ defined as the trace of ∇_Γ applied to vector functions. Moreover, $\mathcal{R}$ and $\mathcal{H}$ represent the curvature operator of Γ and its trace, respectively.

To determine the approximate boundary condition, we follow the procedure described in [5]. For any $x \in \Gamma$ and $s \geq 0$, we set $\boldsymbol{u}(x + s\boldsymbol{n}(x)) =: \bar{\boldsymbol{u}}(x, s)$ and we use the change of variables $y = x + s\boldsymbol{n}(x) = x + \varepsilon S\boldsymbol{n}(x)$, with $S \in [0, 1]$. We set $\bar{\boldsymbol{u}}(x, s) = \bar{\boldsymbol{u}}(x, \varepsilon S) =: \mathbf{U}^\varepsilon(x, S)$. Firstly, we obtain the following asymptotic expansion when $\varepsilon \to 0$:

$$\mathrm{div}(A_i e(\boldsymbol{u}))\,(x + \varepsilon S\boldsymbol{n}(x)) = \frac{1}{\varepsilon^2}\left(\Lambda_0 \partial_S^2 + \sum_{n\geq 1} \varepsilon^n \Lambda_n\right)\mathbf{U}^\varepsilon(x, S), \tag{2.1}$$

where

$$\Lambda_0 := (\lambda_i + 2\mu_i)\boldsymbol{n} \otimes \boldsymbol{n} + \mu_i\big(\mathrm{I}_d - \boldsymbol{n} \otimes \boldsymbol{n}\big), \tag{2.2}$$

and

$$\Lambda_1\mathbf{U}^\varepsilon := \Lambda_{1,1}\partial_S\mathbf{U}^\varepsilon, \qquad \text{with} \quad \Lambda_{1,1}\mathbf{U}^\varepsilon := \mu_i\mathcal{H}\mathbf{U}^\varepsilon + (\lambda_i + \mu_i)\big(\boldsymbol{n}\,\mathrm{div}_\Gamma\,\mathbf{U}^\varepsilon + \nabla_\Gamma(\mathbf{U}^\varepsilon \cdot \boldsymbol{n})\big). \tag{2.3}$$

Moreover, the traction trace operator is defined on Γ, i.e. for $S = 0$, by

$$\boldsymbol{T}_i\mathbf{U}^\varepsilon := \frac{1}{\varepsilon}\Lambda_0\partial_S\mathbf{U}^\varepsilon + \lambda_i\boldsymbol{n}\,\mathrm{div}_\Gamma\,\mathbf{U}^\varepsilon + \mu_i[\nabla_\Gamma\mathbf{U}^\varepsilon]\boldsymbol{n} = \frac{1}{\varepsilon}\Lambda_0\partial_S\mathbf{U}^\varepsilon + \mathcal{B}_t^0\mathbf{U}^\varepsilon.$$

Secondly, we set $\boldsymbol{u}_e^\varepsilon := \sum_{n\geq 0}\varepsilon^n\boldsymbol{u}_e^n$ in $\Omega\backslash\overline{\varpi}$ and $\overline{\boldsymbol{u}_i^\varepsilon}(x, s) := \mathbf{U}_i^\varepsilon(x, S) = \sum_{n\geq 0}\varepsilon^n\mathbf{U}_i^n(x, S)$ in $\Gamma \times [0, 1]$, with the convention $\mathbf{U}_i^\ell = \boldsymbol{u}_e^\ell = \mathbf{0}$ for any integer $\ell < 0$. In the case of a Dirichlet interior boundary condition, the original transmission problem (1.1) can then be rewritten as a couple of two boundary value problems for every coefficient functions $(\mathbf{U}_i^\ell, \boldsymbol{u}_e^\ell)$:

$$\begin{cases} \partial_S^2\Lambda_0\mathbf{U}_i^\ell = -\sum_{k=1}^{\ell}\Lambda_k\mathbf{U}_i^{\ell-k} & \text{in } \Gamma \times (0, 1) \\ \partial_S\Lambda_0\mathbf{U}_i^\ell = \boldsymbol{T}_e(\boldsymbol{u}_e^{\ell-1}) - \mathcal{B}_t^0\mathbf{U}_i^{\ell-1} & \text{on } \Gamma \times \{0\} \\ \Lambda_0\mathbf{U}_i^\ell = \mathbf{0} & \text{on } \Gamma \times \{1\}, \end{cases}$$

and

$$\begin{cases} \mathrm{div}\left(A_e e(\boldsymbol{u}_e^\ell)\right) = \delta_\ell^0\boldsymbol{f} & \text{in } \Omega\backslash\overline{\varpi} \\ \boldsymbol{u}_e^\ell = \mathbf{0} & \text{on } \Gamma_D \\ \boldsymbol{T}_e(\boldsymbol{u}_e^\ell) = \delta_\ell^0\boldsymbol{g} & \text{on } \Gamma_N \\ \Lambda_0\boldsymbol{u}_e^\ell = \Lambda_0\mathbf{U}_i^\ell(\cdot, 0) & \text{on } \Gamma, \end{cases} \tag{2.4}$$

where δ_i^j is the Kronecker delta. For $\ell = 0, 1, 2$, we solve iteratively the new systems to compute first $\mathbf{U}_i^\ell$ and then recover the boundary condition satisfied by $\boldsymbol{u}_e^\ell$. The truncated fields are denoted by $\boldsymbol{u}_{e[N]}^\varepsilon := \sum_{k=0}^{N}\varepsilon^k\boldsymbol{u}_e^k$ and $\mathbf{U}_{i,[N]}^\varepsilon := \sum_{k=0}^{N}\varepsilon^k\mathbf{U}_i^k$. From these results, we deduce the GIBC satisfied by $\boldsymbol{v}_{[N]}^\varepsilon$, which is an approximation of $\boldsymbol{u}_{e,[N]}^\varepsilon$ up to $O(\varepsilon^{N+1})$. The results are stated in the following proposition.

Proposition 2.1. *The GIBC, defined on Γ, modeling thin layer effects for $N = 0$, corresponds to the homogeneous Dirichlet condition. For $N = 1, 2$ it can be written in the form*

$$\varepsilon\,\boldsymbol{T}_e\boldsymbol{v}_{[N]}^\varepsilon + \mathbf{C}_N^\varepsilon(\boldsymbol{v}_{[N]}^\varepsilon) = \mathbf{0},$$

with

$$\mathbf{C}_1^\varepsilon(\boldsymbol{w}) := \Lambda_0\boldsymbol{w}_{|\Gamma},$$

$$\mathbf{C}_2^\varepsilon(\boldsymbol{w}) := \Lambda_0\boldsymbol{w}_{|\Gamma} + \tfrac{1}{2}\varepsilon\,(\mu_i\mathcal{H} + (\lambda_i + \mu_i)\mathcal{R} + (\lambda_i - \mu_i)\mathcal{M})\,\boldsymbol{w}_{|\Gamma},$$

where $\mathcal{M}$ represents the tangential Gunter derivative defined by [6, Chapter V, §1] $\mathcal{M}\boldsymbol{w} = [\nabla_\Gamma\boldsymbol{w}]\boldsymbol{n} - \boldsymbol{n}\,\mathrm{div}_\Gamma\,\boldsymbol{w}$.

Proof. First, we obtain

$$\Lambda_0 \mathbf{U}_i^0 = \mathbf{0}\,, \quad \Lambda_0 \mathbf{U}_i^1(\,\cdot\,, S) = (S-1)\boldsymbol{T}_e \boldsymbol{u}_e^0\,.$$

Using $\mathbf{U}_i^1(\,\cdot\,,0) = -\Lambda_0^{-1}\boldsymbol{T}_e\boldsymbol{u}_e^0 = \boldsymbol{u}^1_{e|\Gamma}$, we rewrite $\mathbf{U}_e^1(\,\cdot\,,S) = -(S-1)\boldsymbol{u}^1_{e|\Gamma}$ and we get

$$\Lambda_0 \mathbf{U}_i^2(\,\cdot\,,S) = \tfrac{S^2-1}{2}\Lambda_{1,1}\boldsymbol{u}^1_{e|\Gamma} + (S-1)\left(\boldsymbol{T}_e\boldsymbol{u}_e^1 - \mathcal{B}_t^0\boldsymbol{u}^1_{e|\Gamma}\right).$$

When $S=0$, we simplify $\Lambda_0\boldsymbol{u}^2_{e|\Gamma} = \Lambda_0\mathbf{U}_i^2(\,\cdot\,,0) = -\boldsymbol{T}_e\boldsymbol{u}_e^1 - \frac{1}{2}\Lambda_{1,1}\boldsymbol{u}^1_{e|\Gamma} + \mathcal{B}_t^0\boldsymbol{u}^1_{e|\Gamma}$ and

$$\tfrac{1}{2}\Lambda_{1,1} - \mathcal{B}_t^0 = \tfrac{1}{2}\Big(\mu_i\mathcal{H} + (\lambda_i+\mu_i)\mathcal{R} + (\lambda_i-\mu_i)\left([\nabla_\Gamma\cdot]\boldsymbol{n} - \boldsymbol{n}\,\mathrm{div}_\Gamma\right)\Big).$$

Then, substituting these results in (2.4), we get the following boundary conditions for $\boldsymbol{u}^\varepsilon_{e[N]}$ on Γ:

$$\begin{aligned}
\Lambda_0\boldsymbol{u}^\varepsilon_{e[0]} &= \Lambda_0\boldsymbol{u}_e^0 = \mathbf{0}\\
\Lambda_0\boldsymbol{u}^\varepsilon_{e[1]} &= \Lambda_0(\boldsymbol{u}_e^0 + \varepsilon\boldsymbol{u}_e^1) = -\varepsilon\boldsymbol{T}_e\boldsymbol{u}_e^0 = -\varepsilon\boldsymbol{T}_e\boldsymbol{u}^\varepsilon_{e[1]} + \varepsilon^2\boldsymbol{T}_e\boldsymbol{u}_e^1\\
\Lambda_0\boldsymbol{u}^\varepsilon_{e[2]} &= \Lambda_0(\boldsymbol{u}_e^0 + \varepsilon\boldsymbol{u}_e^1 + \varepsilon^2\boldsymbol{u}_e^2)\\
&= -\varepsilon\boldsymbol{T}_e(\boldsymbol{u}_e^0 + \varepsilon\boldsymbol{u}_e^1) - \tfrac{1}{2}\varepsilon^2\Big(\mu_i\mathcal{H} + (\lambda_i+\mu_i)\mathcal{R} + (\lambda_i-\mu_i)\mathcal{M}\Big)\boldsymbol{u}_e^1\\
&= -\varepsilon\boldsymbol{T}_e\boldsymbol{u}^\varepsilon_{e[2]} - \tfrac{1}{2}\varepsilon^2\Big(\mu_i\mathcal{H} + (\lambda_i+\mu_i)\mathcal{R} + (\lambda_i-\mu_i)\mathcal{M}\Big)\boldsymbol{u}^\varepsilon_{e[2]}\\
&\quad + \varepsilon^3\boldsymbol{T}_e\boldsymbol{u}_e^2 + \tfrac{1}{2}\varepsilon^3\Big(\mu_i\mathcal{H} + (\lambda_i+\mu_i)\mathcal{R} + (\lambda_i-\mu_i)\mathcal{M}\Big)\boldsymbol{u}_e^2.
\end{aligned}$$

We observe that the truncated series up to the index N satisfy $\varepsilon\,\boldsymbol{T}_e\boldsymbol{u}^\varepsilon_{e[N]} + \mathbf{C}_N^\varepsilon(\boldsymbol{u}^\varepsilon_{e[N]}) = O(\varepsilon^{N+1})$. Then we choose to approach the total exterior field $\boldsymbol{u}_e^\varepsilon$ by a new field $\boldsymbol{v}^\varepsilon_{[N]}$ that satisfies $\varepsilon\,\boldsymbol{T}_e\boldsymbol{v}^\varepsilon_{[N]} + \mathbf{C}_N^\varepsilon(\boldsymbol{v}^\varepsilon_{[N]}) = \mathbf{0}$. $\square$

Remark 2.2. Even if we provide the formula for $N = 0, 1, 2$ only, we guess that, for any $N \in \mathbb{N}^*$, the impedance operator $\mathbf{C}_N^\varepsilon$ is a surface differential operator of order $(N-1)$ for the three components of the state. Although it differs a bit from the Laplace equation case, the procedure to prove optimal error estimates in $\boldsymbol{H}^1$-norm presented in [10, Chapter 1] extends to the elastic case at any order $N \in \mathbb{N}$ and is sketched in Section 3 for the sake of completeness.

For $N = 1, 2$, the associated weak variational formulations of the GIBC problems (1.2) with non-vanishing right-hand side $\varepsilon\boldsymbol{T}_e\boldsymbol{v} + \mathbf{C}_N^\varepsilon(\boldsymbol{v}) = \boldsymbol{h}$ when $\boldsymbol{h} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ read: find $\boldsymbol{v} \in \boldsymbol{H}^1_{\Gamma_D}(\Omega\backslash\overline{\varpi})$ satisfying

$$\boldsymbol{a}_N^\varepsilon(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{\ell}(\boldsymbol{w}), \quad \forall \boldsymbol{w} \in \boldsymbol{H}^1_{\Gamma_D}(\Omega\backslash\overline{\varpi}) \tag{2.5}$$

where

$$\boldsymbol{a}_1^\varepsilon(\boldsymbol{v}, \boldsymbol{w}) := \lambda_e \int_{\Omega\backslash\overline{\varpi}} (\mathrm{div}\,\boldsymbol{v})\,(\mathrm{div}\,\boldsymbol{w}) + 2\mu_e \int_{\Omega\backslash\overline{\varpi}} e(\boldsymbol{v}) : e(\boldsymbol{w}) + \varepsilon^{-1}\int_\Gamma \Lambda_0\boldsymbol{v}\cdot\boldsymbol{w},$$

$$\boldsymbol{a}_2^\varepsilon(\boldsymbol{v}, \boldsymbol{w}) := \boldsymbol{a}_1^\varepsilon(\boldsymbol{v}, \boldsymbol{w}) + \frac{1}{2}\int_\Gamma (\mu_i\mathcal{H} + (\lambda_i+\mu_i)\mathcal{R} + (\lambda_i-\mu_i)\mathcal{M})\,\boldsymbol{v}\cdot\boldsymbol{w},$$

and

$$\boldsymbol{\ell}(\boldsymbol{w}) := \int_{\Omega\backslash\overline{\varpi}} \boldsymbol{f}\cdot\boldsymbol{w} + \int_{\Gamma_N} \boldsymbol{g}\cdot\boldsymbol{w} + \varepsilon^{-1}\int_\Gamma \boldsymbol{h}\cdot\boldsymbol{w}.$$

The bilinear forms a_N^ε are symmetric and continuous on $\boldsymbol{H}^1_{\Gamma_D}(\Omega\backslash\overline{\varpi}) \times \boldsymbol{H}^1_{\Gamma_D}(\Omega\backslash\overline{\varpi})$ (see [6, Chapter V, §1] for the properties of $\mathcal{M}$). The coercivity of a_1^ε is obvious since Λ_0 is a positive definite matrix.

3. Convergence analysis

Assuming the existence and uniqueness of the solution $\boldsymbol{v}^\varepsilon_{[N]}$ to GIBC problems at any order $N \in \mathbb{N}$, one can establish optimal error estimates between $\boldsymbol{u}_e^\varepsilon$ and its approximate field $\boldsymbol{v}^\varepsilon_{[N]}$ as done below.

Theorem 3.1. *Let $N \in \mathbb{N}$ and $\boldsymbol{f} \in \mathscr{C}^\infty(\overline{\Omega\backslash\overline{\varpi}})$. Then there exists a constant $C_{\Omega\backslash\overline{\varpi}}$ independent of ε such that*

$$||\boldsymbol{v}^\varepsilon_{[N]} - \boldsymbol{u}_e^\varepsilon||_{\boldsymbol{H}^1(\Omega\backslash\overline{\varpi})} \le C_{\Omega\backslash\overline{\varpi}}\,\varepsilon^{N+1}.$$

Proof. For any $N \in \mathbb{N}$, the assumption $\boldsymbol{f} \in \mathscr{C}^\infty(\overline{\Omega\backslash\varpi})$ allows us to ensure higher order local regularity for $\boldsymbol{u}_e^\ell$, with $\ell = 0, \dots, N-1$ in the neighborhood of Γ so that the right-hand side to problems (2.4) belong to $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ and the solution satisfies $u_e^N \in \boldsymbol{H}^1_{\Gamma_D}(\Omega\backslash\overline{\varpi})$.

- We decompose the remainder as $\boldsymbol{v}^\varepsilon_{[N]} - \boldsymbol{u}^\varepsilon_e := (\boldsymbol{v}^\varepsilon_{[N]} - \boldsymbol{u}^\varepsilon_{e,[N]}) + (\boldsymbol{u}^\varepsilon_{e,[N]} - \boldsymbol{u}^\varepsilon_e)$.
- Firstly, we set $\boldsymbol{r}^\varepsilon_{[N]} := \boldsymbol{u}^\varepsilon - \sum_{n=0}^{N} \varepsilon^n \boldsymbol{u}^n$ and we denote by $\boldsymbol{r}^\varepsilon_{e,[N]}$ and $\boldsymbol{r}^\varepsilon_{i,[N]}$ the restriction of $\boldsymbol{r}^\varepsilon_{[N]}$ respectively to $\Omega\backslash\overline{\varpi}$ and to ω_i^ϵ. The remainders $(\boldsymbol{r}^\varepsilon_{e,[N]}, \mathbf{R}^\varepsilon_{i,[N]})$ solve the thin layer transmission problem with right-hand sides up to $O(\varepsilon^{N-1})$. Using a judicious rewritting of $(\boldsymbol{r}^\varepsilon_{e,[N]}, \mathbf{R}^\varepsilon_{i,[N]})$ (see [2, subsection 2.2] or [10, Subsection 1.3.5]) and a change of variable formula for $s = \varepsilon S$ to evaluate $\boldsymbol{H}^1$-norms, we get the estimates

$$\|\boldsymbol{r}^\varepsilon_{e,[N]}\|_{\boldsymbol{H}^1(\Omega\backslash\overline{\varpi})} \le C\,\varepsilon^{N+1} \quad \text{and} \qquad \sqrt{\varepsilon}\|\boldsymbol{r}^\varepsilon_{i,[N]}\|_{\boldsymbol{H}^1(\omega_i^\varepsilon)} \le C\,\varepsilon^{N+1}, \tag{3.1}$$

with C depending on N but independent of ε (thanks to the uniform coercivity of the bilinear form associated with the transmission problem).

- Secondly, we focus on $\boldsymbol{d}_{[N]} := \boldsymbol{u}^\varepsilon_{e,[N]} - \boldsymbol{v}^\varepsilon_{[N]}$. For $N = 0$, we have $\boldsymbol{u}^\varepsilon_{e,[0]} = \boldsymbol{u}^0_e = \boldsymbol{v}^\varepsilon_{[0]}$, i.e. $\boldsymbol{d}_{[0]} = \boldsymbol{0}$. Then the estimate is deduced from the previous result. To get optimal results, for $N \ge 1$, the trick is different than in the Neumann case [2, subsection 2.2] and is well explained in [10, page 36]. Indeed, using the bounds provided by the Lax–Milgram theorem applied to the variational formulations (2.5) does not lead to optimal estimates due to the factor ε^{-1} in the right-hand side. Instead, we use an asymptotic expansion of the solution $\boldsymbol{v}^\varepsilon_{[N]}$ to the GIBC problems (1.2). We write $\boldsymbol{v}^\varepsilon_{[N]} = \sum_{\ell\ge0} \varepsilon^\ell \boldsymbol{v}^\ell_N$, where the coefficient functions $\boldsymbol{v}^\ell_N$ are defined iteratively as the solution to mixed boundary value problems with Dirichlet-type condition on Γ. For example, when $N = 1, 2$, we have:

$$\left\{\begin{array}{rcll} \operatorname{div}\left(A_e e(\boldsymbol{v}_1^\ell)\right) &=& \delta^0_\ell \boldsymbol{f} & \text{in } \Omega\backslash\overline{\varpi} \\ \boldsymbol{v}_1^\ell &=& \boldsymbol{0} & \text{on } \Gamma_D \\ \boldsymbol{T}_e(\boldsymbol{v}_1^\ell) &=& \delta^0_\ell \boldsymbol{g} & \text{on } \Gamma_N \\ \boldsymbol{v}_1^\ell &=& -\Lambda_0^{-1}\boldsymbol{T}_e\boldsymbol{v}_1^{\ell-1} & \text{on } \Gamma\,, \end{array}\right. \tag{3.2}$$

and

$$\left\{\begin{array}{rcll} \operatorname{div}\left(A_e e(\boldsymbol{v}_2^\ell)\right) &=& \delta^0_\ell \boldsymbol{f} & \text{in } \Omega\backslash\overline{\varpi} \\ \boldsymbol{v}_2^\ell &=& \boldsymbol{0} & \text{on } \Gamma_D \\ \boldsymbol{T}_e(\boldsymbol{v}_1^\ell) &=& \delta^0_\ell \boldsymbol{g} & \text{on } \Gamma_N \\ \boldsymbol{v}_2^\ell &=& -\Lambda_0^{-1}\left(\boldsymbol{T}_e\boldsymbol{v}_2^{\ell-1} + \frac{1}{2}\Big(\mu_i\mathcal{H} + (\lambda_i+\mu_i)\mathcal{R} + (\lambda_i-\mu_i)\mathcal{M}\Big)\boldsymbol{v}_2^{\ell-1}\right) & \text{on } \Gamma\,. \end{array}\right. \tag{3.3}$$

We observe that

$$\forall\,\ell = 0, \dots, N, \quad \boldsymbol{v}^\ell_N = \boldsymbol{u}^\ell_e.$$

Higher-order coefficient functions $\boldsymbol{v}^\ell_N$, with $\ell \ge N+1$, surely differ from $\boldsymbol{u}^\ell_e$. Thus, by applying the Lax–Milgram theorem to the mixed boundary value problem with a Dirichlet-type boundary condition on Γ, we get the estimates:

$$\|\boldsymbol{v}^\epsilon_{[1]} - \boldsymbol{u}^\varepsilon_{e,[1]}\|_{\boldsymbol{H}^1(\Omega\backslash\overline{\omega})} = \|\boldsymbol{v}^\epsilon_{[1]} - \boldsymbol{v}^0_1 - \varepsilon\boldsymbol{v}^1_1\|_{\boldsymbol{H}^1(\Omega\backslash\overline{\omega})} \le C_1\varepsilon^2,$$

and

$$\|\boldsymbol{v}^\epsilon_{[2]} - \boldsymbol{u}^\varepsilon_{e,[2]}\|_{\boldsymbol{H}^1(\Omega\backslash\overline{\omega})} = \|\boldsymbol{v}^\epsilon_{[2]} - \boldsymbol{v}^0_2 - \varepsilon\boldsymbol{v}^1_2 - \varepsilon^2\boldsymbol{v}^2_2\|_{\boldsymbol{H}^1(\Omega\backslash\overline{\omega})} \le C_2\varepsilon^3.$$

And so on, we get for any $N \in \mathbb{N}$, $\|\boldsymbol{v}^\epsilon_{[N]} - \boldsymbol{u}^\varepsilon_{e,[N]}\|_{\boldsymbol{H}^1(\Omega\backslash\overline{\omega})} \le C_N\varepsilon^{N+1}$, with C_N independent of ε. Using the triangular inequality and (3.1), we deduce the announced estimates. □

Remark 3.2. The convergence estimates at any order $N \in \mathbb{N}$ are obtained assuming analyticity of the boundary Γ and smoothness of the data around the thin layer. However, a boundary Γ of class $\mathscr{C}^{1,1}$ and $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega\backslash\overline{\varpi})$ are sufficient to get the previous estimates for $N = 1$.

References

[1] F. Cakoni, I. de Teresa, H. Haddar, P. Monk, Nondestructive testing of the delaminated interface between two materials, SIAM J. Appl. Math. 76 (6) (2016) 2306–2332.

[2] F. Caubet, D. Kateb, F. Le Louër, Shape sensitivity analysis for elastic structures with generalized impedance boundary conditions of the Wentzell type – application to compliance minimization, J. Elast. 136 (1) (2019) 17–53.

[3] P.G. Ciarlet, Mathematical Elasticity. Vol. I, Three-Dimensional Elasticity, Studies in Mathematics and its Applications, vol. 20, North-Holland Publishing Co., Amsterdam, 1988.
[4] J. Hendrickx, P.H. Geubelle, N.R. Sottosc, A spectral scheme for the simulation of dynamic mode 3 delamination of thin films, Eng. Fract. Mech. 72 (12) (2005) 1866–1891.
[5] D. Kateb, F. Le Louër, Generalized impedance boundary conditions and shape derivatives for 3D Helmholtz problems, Math. Models Methods Appl. Sci. 26 (10) (2016) 1995–2033.
[6] V.D. Kupradze, T.G. Gegelia, M.O. Basheleĭshvili, T.V. Burchuladze, in: V.D. Kupradze (Ed.), Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity, in: North-Holland Series in Applied Mathematics and Mechanics, vol. 25, North-Holland Publishing Co., Amsterdam, 1979, Russian edition.
[7] J.-C. Nédélec, Acoustic and Electromagnetic Equations, Integral Representations for Harmonic Problems, Applied Mathematical Sciences, vol. 144, Springer-Verlag, New York, 2001.
[8] M. Pasquali, W. Lacarbonara, Delamination detection in composite laminates using high-frequency P- and S-waves. Part I: theory and analysis, Compos. Struct. 134 (2015) 1095–1108.
[9] C. Soutis, S.H. Díaz Valdés, Delamination detection in laminated composites using lamb waves, in: E.E. Gdoutos, Z.P. Marioli-Riga (Eds.), Recent Advances in Composite Materials: In Honor of S.A. Paipetis, Springer Netherlands, 2003, pp. 109–126.
[10] G. Vial, Analyse multi-échelle et conditions aux limites approchées pour un problème avec couche mince dans un domaine à coin, PhD thesis, Université de Rennes-1, France, 2003.