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Local indirect stabilization of N–d system of two coupled wave equations under geometric conditions

Stabilisation locale indirecte d'un système N–d de deux équations d'ondes couplées sous conditions géométriques

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ABSTRACT

The purpose of this note is to investigate the stabilization of a system of two wave equations coupled by velocities with only one localized damping. The main novelty in this note is that the waves are not necessarily propagating at same speed and the coupling coefficient is not assumed to be positive and small. Assume that the coupling region and the damping region intersect. We prove that our system is strongly stable without geometric conditions. We then study the energy decay rate by distinguishing two cases. The first one is when the waves propagate at the same speed. In this case, under appropriate geometric conditions, we establish an exponential energy decay estimate for usual initial data. For the other case, we first show that our system is not uniformly stable. Next, under the same geometric conditions, we establish a polynomial energy decay of type $\frac{1}{t}$ for smooth initial data. Finally, in one space dimension, using the real part of the asymptotic expansion of eigenvalues of the system, we prove that the obtained polynomial decay rate is optimal.

RÉSUMÉ

Nous nous intéressons à la stabilisation d'un système multidimensionnel de deux équations d'ondes couplées par les termes de vitesse, et dont l'une seulement est localement amortie. La principale nouveauté contenue dans cette note est que les ondes ne se propagent pas forcément à la même vitesse et que le coefficient de couplage n'est pas supposé être positif et petit. Nous supposons que la zone de couplage et la zone d'amortissement s'intersectent. D'abord, nous montrons que notre système est fortement stable sans conditions géométriques. Puis nous étudions le taux de décroissance de l'énergie en distinguant deux cas. Dans le premier cas, nous supposons que les ondes se propagent à la même vitesse ; nous établissons alors, sous certaines conditions géométriques, un taux de décroissance exponentiel de l'énergie du système pour des données initiales usuelles.

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Dans le second cas, nous montrons d'abord que l'énergie ne décroît pas exponentiellement vers 0. Ensuite, sous les mêmes conditions géométriques, nous établissons un taux de décroissance polynomial de type $\frac{1}{t}$ pour des données initiales régulières. Finalement, dans le cas particulier où la dimension de l'espace est égale à 1, en utilisant la partie réelle du développement asymptotique des valeurs propres de système, nous montrons, de plus, que le taux polynomial obtenu est optimal.

1. Introduction

Let Ω be a nonempty bounded open set of $\mathbb{R}^N$ having a boundary Γ of class C^2. In [8], F. Alabau-Boussouira et al. considered the energy decay of a system of two wave equations coupled by velocities:

$$u_{tt} - a\Delta u + b(x)y_t + \rho(x, u_t) = 0 \qquad \text{in } \Omega \times \mathbb{R}_+^\star, \tag{1.1}$$

$$y_{tt} - \Delta y - b(x)u_t = 0 \qquad \text{in } \Omega \times \mathbb{R}_+^\star, \tag{1.2}$$

$$u = y = 0 \qquad \text{on } \Gamma \times \mathbb{R}_+^\star, \tag{1.3}$$

with the following initial data:

$$u(x, 0) = u_0, \;\; y(x, 0) = y_0, \;\; u_t(x, 0) = u_1 \text{ and } y_t(x, 0) = y, \quad x \in \Omega$$

where $a > 0$ is a constant and $b \in C^0(\overline{\Omega}; \mathbb{R})$ is a non-zero function. The damping term ρ is applied to the first equation and the second equation is indirectly damped through the coupling between the two equations. In [8], using an approach based on multiplier techniques, weighted nonlinear inequalities and the optimal-weight convexity method (developed in [5]), the authors established an explicit energy decay formula in terms of the behavior of the nonlinear feedback close to the origin. Their results are obtained in the case when the following three conditions are satisfied: the waves propagate at the same speed ($a = 1$), the coupling coefficient $b(x)$ is small positive ($0 \le b(x) \le b_0$, $b_0 \in (0, b^\star]$ where $b^\star$ is a constant depending on Ω and on the control region), and both the coupling and the damping regions satisfy an appropriated geometric conditions named Piecewise Multipliers Geometric Conditions (introduced in [23], used in [5] and denoted by PMGC, in short). Then the stabilization of the system (1.1)–(1.3) in the case where the waves are not assumed to propagate with equal speeds (a is not necessarily equal to 1) and/or when the coupling coefficient $b(x)$ is not assumed to be positive and small has been left an open problem even when the damping term ρ is linear with respect to the second variable. In this note, we are interested to answer this open question and to provide a stability analysis for the system (1.1)–(1.3) when the damping term ρ is linear with respect to the second variable *i.e.* $\rho(x, u_t) = c(x)u_t$ where $c \in C^0(\overline{\Omega}; \mathbb{R}_+)$. So, we consider the stability of the following system:

$$u_{tt} - a\Delta u + b(x)y_t + c(x)u_t = 0 \qquad \text{in } \Omega \times \mathbb{R}_+^\star, \tag{1.4}$$

$$y_{tt} - \Delta y - b(x)u_t = 0 \qquad \text{in } \Omega \times \mathbb{R}_+^\star, \tag{1.5}$$

$$u = y = 0 \qquad \text{on } \Gamma \times \mathbb{R}_+^\star, \tag{1.6}$$

with the following initial data:

$$u(x, 0) = u_0, \;\; y(x, 0) = y_0, \;\; u_t(x, 0) = u_1 \text{ and } y_t(x, 0) = y, \quad x \in \Omega.$$

The notion of indirect damping mechanisms has been introduced by D.L. Russell in [29], and since then, it attracted the attention of many authors. In particular, the stabilization of systems of two second-order equations coupled through displacements when only one equation is effectively damped by internal or boundary feedback has been initiated and studied in [1–3], and further studied by many authors, for instance [6,24,17]. Recalling that the exponential or polynomial energy decay rate occurs in many control problems, we quote [13,30] for the Timoshenko system in bounded or unbounded domains. Here, we focus our attention only on the literature of the indirect internal stability of coupled wave equations. In [16], B. Kapitonov studied the stabilization of a system of two coupled hyperbolic equations involving (1.4)–(1.6). He established an exponential energy decay rate for usual initial data in the case where the waves propagated at the same speed ($a = 1$) and the damping and coupling coefficients have the same support. For the other cases, when $a \neq 1$ and/or support of b does not coincide with that of c, no energy decay rate has been discussed. In [2], F. Alabau et al. studied the indirect stabilization of a system of two evolution equations coupled through displacements where the damping is effective in the whole domain. Using the method of higher-order energies initiated in [1], they established a polynomial energy decay depending on the smoothness of the initial data. These results have been generalized by F. Alabau and M. Léautaud in [6] to the case when the coupling and the damping coefficients are localized in Ω and both satisfy the PMGC conditions. In addition, without geometric conditions, using an interpolation inequality for elliptic system (see Proposition 5.1 in [18]) together with the resolvent estimates of G. Lebeau in [19], the authors proved that the energy decay of smooth initial data is at least logarithmic when the coupling and the damping regions intersect in a nonempty sub-domain $\omega \subset \Omega$. However, when $\omega = \emptyset$, the question of the stability or the null controllability of the system is still an open problem. Indeed, F. Alabau and M. Léautaud in [7] solved partially this problem by proving that the system is null controllable provided that both the coupling and the damping regions satisfy the optimal geometric condition named Geometric Control Condition introduced by Bardos et al. in [11]. Finally, we refer to [1,2,4,10,12,32,30,31,27,16,25,9] for the indirect stabilization and the indirect exact controllability of distributed systems with different kinds of damping.

In this note, we study the stability of the system (1.4)–(1.6) when the coupling and damping regions intersect in $\omega \subset \Omega$. First, we establish the strong stability without geometric conditions. We then study the energy decay rate of our system by distinguishing two cases. The first one is when the waves propagate at same speed, i.e. $a = 1$. In this case, under the hypothesis that ω satisfies the geometric conditions PMGC (see below), we establish an exponential energy decay rate for usual initial data. Next, in the general case, when $a \neq 1$, we prove the non-uniform (exponential) stability and, under the same geometric conditions, we establish a polynomial energy decay rate of type $\frac{1}{t}$ for smooth initial data. Finally, in one space dimension, using the real part of the asymptotic expansion of the eigenvalues of the system, we show that the obtained polynomial decay is optimal.

2. Well-posedness and strong stability

In this section, we will study the strong stability of the system (1.4)–(1.6) without additional geometric conditions. First, we will study the existence, uniqueness, and regularity of the solution to our system.

2.1. Well-posedness of the problem

Let $U = (u, u_t, y, y_t)$ be a regular solution to (1.4)–(1.6), its associated energy is defined by

$$E(t) = \frac{1}{2}\int_\Omega \left(|u_t|^2 + a|\nabla u|^2 + |y_t|^2 + |\nabla y|^2\right) \mathrm{d}x. \tag{2.1}$$

So, a direct computation gives

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\int_\Omega c(x)|u_t|^2 \,\mathrm{d}x \le 0. \tag{2.2}$$

Consequently, the system (1.4)–(1.6) is dissipative in the sense that its energy is non-increasing.

First, we define the energy space $\mathcal{H} = (H_0^1(\Omega) \times L^2(\Omega))^2$ equipped, for all $U = (u, v, y, z)$, $\widetilde{U} = (\widetilde{u}, \widetilde{v}, \widetilde{y}, \widetilde{z}) \in \mathcal{H}$, by the scalar product:

$$(U, \widetilde{U})_{\mathcal{H}} = a\int_\Omega (\nabla u \cdot \nabla \widetilde{u})\,\mathrm{d}x + \int_\Omega v\widetilde{v}\,\mathrm{d}x + \int_\Omega (\nabla y \cdot \nabla \widetilde{y})\,\mathrm{d}x + \int_\Omega z\widetilde{z}\,\mathrm{d}x.$$

Next, we define the unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ by:

$$D(\mathcal{A}) = (H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega))^2, \;\; \mathcal{A}U = (\,v, a\Delta u - bz - cv,\, z,\, \Delta y + bv\,).$$

Note that, using the fact that $c(x) \ge 0$, then $\mathcal{A}$ is m-dissipative and generates a C_0 semi-group of contractions $\mathrm{e}^{t\mathcal{A}}$ on the energy space $\mathcal{H}$. As the system (1.4)–(1.6) is equivalent to

$$U_t = \mathcal{A}U \text{ in } \mathcal{H},\; t > 0,\; U(0) = U_0 \tag{2.3}$$

with $U = (u, u_t, y, y_t)$, we deduce its well-posed character. So, we have the following existence results:

Theorem 2.1. *Let $U_0 \in \mathcal{H}$ then, problem (2.3) admits a unique weak solution U satisfying*

$$U(t) \in C^0\left(\mathbb{R}^+, \mathcal{H}\right).$$

Moreover, if $U_0 \in D(\mathcal{A})$ then, problem (2.3) admits a unique strong solution U satisfying

$$U(t) \in C^1\left(\mathbb{R}^+, \mathcal{H}\right) \cap C^0(\mathbb{R}^+, D(\mathcal{A})).$$

Secondly, we will study the strong stability of our system.

2.2. Strong stability

In this subsection, we study the asymptotic behavior of $E(t)$. For this aim, we assume that there exists a nonempty open $\omega_{c_+} \subset \Omega$ satisfying the following condition:

$$\{x \in \Omega : c(x) > 0\} \supset \overline{\omega}_{c_+} \tag{LH1}$$

On the other side, as $b(x)$ is a non-zero continuous function, then there exists a nonempty open $\omega_{b_+} \cup \omega_{b_-} \subset \Omega$ such that

$$\{x \in \Omega : b(x) > 0\} \supset \overline{\omega}_{b_+} \quad \text{and} \quad \{x \in \Omega : b(x) < 0\} \supset \overline{\omega}_{b_-} \tag{LH2}$$

Our main result in this part is the following.

Theorem 2.2 *(Strong stability). Assume that $a > 0$, that condition (LH1) holds and that $\omega = \omega_{c_+} \cap \omega_{b_+} \neq \emptyset$ or $\omega_{c_+} \cap \omega_{b_-} \neq \emptyset$. Then the semi group of contractions $\mathrm{e}^{t\mathcal{A}}$ is strongly stable on the energy space $\mathcal{H}$, i.e., for any $U_0 \in \mathcal{H}$, we have*

$$\lim_{t \to +\infty} \|\mathrm{e}^{t\mathcal{A}} U_0\|_{\mathcal{H}} = 0. \tag{2.4}$$

In [6], the authors considered the stabilization of a system of two wave equations coupled in displacements with one localized internal damping. They showed that, under the assumption that the damping region and the coupling region have a non-empty intersection in Ω, *i.e.* $\omega = \omega_{c_+} \cap \omega_{b_+} \neq \emptyset$ (or $\omega_{c_+} \cap \omega_{b_-} \neq \emptyset$), the energy of smooth solutions decays logarithmically to zero as t goes to infinity. This result still holds in the case where the two wave equations are coupled through the velocities. Indeed, following the method introduced by G. Lebeau and L. Robbiano in [20], M. Léautaud in [18] established an interpolation inequality for the associated elliptic system. This interpolation inequality implies the resolvent estimates of G. Lebeau in [19] (see also [21]) that provide the logarithmic energy decay rate for smooth initial data. So, using the density of $D(\mathcal{A})$ in $\mathcal{H}$ and the contraction property of the C_0 semigroup $\mathrm{e}^{t\mathcal{A}}$, we deduce that the energy of the system (1.4)–(1.6) decays asymptotically to zero as t goes to infinity for all usual initial data.

Then we are interested, in this paper, to study the energy decay rate by distinguishing two cases.

3. Exponential stability, the case $a = 1$

This section is devoted to the study of the exponential stability of system (1.4)–(1.6) in case the waves propagate at the same speed (in the case $a = 1$) and under appropriated geometric conditions. For that purpose, we will use a frequency domain approach combined with a piecewise multiplier technique.

Before presenting our main result of this section, we recall the piecewise multiplier geometric condition introduced by K. Liu in [23].

Definition 3.1. We say that ω satisfies the piecewise multiplier geometric condition (PMGC in short) if there exist $\Omega_j \subset \Omega$ having Lipschitz boundary $\Gamma_j = \partial\Omega_j$ and $x_j \in \mathbb{R}^N$, $j = 1, ..., J$ such that $\Omega_j \cap \Omega_i = \emptyset$ for $j \neq i$ and ω contains a neighborhood in Ω of the set $\cup_{j=1}^{J} \gamma_j(x_j) \cup \left(\Omega \setminus \cup_{j=1}^{J} \Omega_j\right)$ where $\gamma_j(x_j) = \{x \in \Gamma_j : (x - x_j) \cdot \nu_j(x) > 0\}$ and ν_j is the outward unit normal vector to Γ_j.

Remark 3.2. The PMGC is the generalization of the multipliers geometric condition (MGC in short) introduced by Lions in [22], saying that ω contains a neighborhood in Ω of the set $\{x \in \Gamma : (x - x_0) \cdot \nu(x) > 0\}$, for some $x_0 \in \mathbb{R}^N$, where ν is the outward unit normal vector to $\Gamma = \partial\Omega$.

Now, we are in a position to present the main result of this section.

Theorem 3.3 *(Exponential decay rate). Let $a = 1$. Assume that condition (LH1) holds. Assume also that the nonempty open set $\omega = \omega_{c_+} \cap \omega_{b_+}$ (or $\omega = \omega_{c_+} \cap \omega_{b_-}$) satisfies the geometric conditions* PMGC *and that $b, c \in W^{1,\infty}(\Omega)$. Then there exist positive constants $M \geq 1$, $\theta > 0$ such that, for all initial data $(u_0, u_1, y_0, y_1) \in \mathcal{H}$, the energy of the system (1.4)–(1.6) satisfies the following decay rate:*

$$E(t) \leq M\,\mathrm{e}^{-\theta t} E(0), \quad \forall t > 0. \tag{3.1}$$

Remark 3.4. Note that in Theorem 3.3, we have no restriction on the upper bound and the sign of the function b. This theorem is then a generalization in the linear case of the result of [8] where the coupling coefficient considered has to satisfy $0 \leq b(x) \leq b_0$, $b_0 \in (0, b^\star]$ where $b^\star$ is a constant depending on Ω and on the control region. Nevertheless, the problem is still open in the nonlinear case.

In order to prove the above theorem, we apply a result of F.L. Huang [15] and J. Pruss [28]: a C_0-semigroup of contraction $(e^{t\mathcal{A}})_{t\geqslant 0}$ in a Hilbert space $\mathcal{H}$ is uniformly stable if and only if

$$i\mathbb{R} \subseteq \rho(\mathcal{A}) \tag{H1}$$

and

$$\sup_{\beta\in\mathbb{R}} \| (i\beta I - \mathcal{A})^{-1} \| < +\infty \tag{H2}$$

hold.

Since the resolvent of $\mathcal{A}$ is compact and $0 \in \rho(\mathcal{A})$, then from Theorem 2.2, we deduce that condition (H1) is satisfied. We now prove that condition (H2) holds, using an argument of contradiction. To this aim, we suppose that there exist a real sequence β_n with $\beta_n \to +\infty$ and a sequence $U_n = (u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ such that

$$\| (u_n, v_n, y_n, z_n) \|_{\mathcal{H}} = 1, \tag{3.2}$$

and

$$\lim_{n\to\infty} \| (i\beta_n I - \mathcal{A})U_n \|_{\mathcal{H}} = 0. \tag{3.3}$$

Now, detailing Eq. (3.3), we get

$$i\beta_n u_n - v_n = f_n^1 \to 0 \quad \text{in} \quad H_0^1(\Omega), \tag{3.4}$$
$$i\beta_n v_n - \Delta u_n + b(x)z_n + c(x)v_n = g_n^1 \to 0 \quad \text{in} \quad L^2(\Omega), \tag{3.5}$$
$$i\beta_n y_n - z_n = f_n^2 \to 0 \quad \text{in} \quad H_0^1(\Omega), \tag{3.6}$$
$$i\beta_n z_n - \Delta y_n - b(x)v_n = g_n^2 \to 0 \quad \text{in} \quad L^2(\Omega). \tag{3.7}$$

Eliminating v_n and z_n from the previous system, we obtain the following reduced system

$$\beta_n^2 u_n + \Delta u_n - i\beta_n b(x)y_n - i\beta_n c(x)u_n = -g_n^1 - b(x)f_n^2 - i\beta_n f_n^1 - c(x)f_n^1, \tag{3.8}$$
$$\beta_n^2 y_n + \Delta y_n + i\beta_n b(x)u_n = -i\beta_n f_n^2 + b(x)f_n^1 - g_n^2. \tag{3.9}$$

On the other side, using Eq. (3.2) we deduce that z_n and v_n are uniformly bounded in $L^2(\Omega)$. It follows, from Eqs. (3.4) and (3.6), that

$$\int_\Omega |y_n|^2 \,dx = \frac{O(1)}{\beta_n^2} \quad \text{and} \quad \int_\Omega |u_n|^2 \,dx = \frac{O(1)}{\beta_n^2}. \tag{3.10}$$

Lemma 3.5. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ to the system (3.4)–(3.7) satisfies the following estimation*

$$\int_{\omega_{c_+}} |\beta_n u_n|^2 \,dx = o(1). \tag{3.11}$$

Proof. First, since U_n is uniformly bounded in $\mathcal{H}$, then from (3.3), we get

$$\mathrm{Re}\left\{ i\beta_n \| U_n \|^2 - (\mathcal{A}U_n, U_n) \right\} = \int_\Omega c(x)|v_n|^2 \,dx = o(1). \tag{3.12}$$

Under condition (LH1), it follows that

$$\int_{\omega_{c_+}} |v_n|^2 \,dx = o(1). \tag{3.13}$$

So, using Eqs. (3.12) and (3.4), we get

$$\int_\Omega c(x)|\beta_n u_n|^2 \,dx = o(1). \tag{3.14}$$

Consequently, we have

$$\int_{\omega_{c_+}} |\beta_n u_n|^2 \,dx = o(1).$$

The proof is thus complete. □

Now, the subset ω satisfies the PMGC. Hence, denoting by Ω_j and x_j, $j = 1, ..., J$ the sets and the points given by the PMGC, we have $\omega \supset \mathcal{N}_\epsilon \left(\cup_{j=1}^J \gamma_j\left(x_j\right) \cup \left(\Omega \setminus \cup_{j=1}^J \Omega_j\right)\right) \cap \Omega$. In this expression, $\mathcal{N}_\epsilon(\mathcal{O}) = \{x \in \mathbb{R}^N : d(x, \mathcal{O}) < \varepsilon\}$ with $d(\cdot, \mathcal{O})$ is the usual Euclidean distance to the subset $\mathcal{O}$ of $\mathbb{R}^N$ and $\gamma_j(x_j) = \{x \in \Gamma_j : (x - x_j) \cdot \nu_j(x) > 0\}$ where ν_j is the outward unit normal vector to $\Gamma_j = \partial\Omega_j$. Let the reals $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \varepsilon$ and define

$$\mathcal{V}_i = \mathcal{N}_{\varepsilon_i}\left(\cup_{j=1}^J \gamma_j\left(x_j\right) \cup \left(\Omega \setminus \cup_{j=1}^J \Omega_j\right)\right), \quad i = 1, 2, 3.$$

Since $\left(\overline{\Omega} \setminus \mathcal{V}_3\right) \cap \overline{\mathcal{V}_2} = \emptyset$, then we may define the function $\eta \in C_0^\infty(\mathbb{R}^N)$ by

$$\eta(x) = 0 \quad \text{if } x \in \Omega \setminus \mathcal{V}_3, \quad 0 \le \eta(x) \le 1, \quad \eta(x) = 1 \quad \text{if } x \in \mathcal{V}_2.$$

Lemma 3.6. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ to the system (3.4)–(3.7) satisfies the following estimation*

$$\int_\Omega \eta(x)\, |\nabla u_n|^2 \,dx = o(1) \;\; \textit{and} \;\; \int_{\mathcal{V}_2 \cap \Omega} |\nabla u_n|^2 \,dx = o(1). \tag{3.15}$$

Proof. Multiplying Eq. (3.8) by $\eta \bar{u}_n$ and using Green's formula and the fact that $u_n = 0$ on Γ, we obtain

$$\begin{cases} \int_\Omega \eta(x) |\beta_n u_n|^2 \,dx - \int_\Omega \eta(x) \mid \nabla u_n \mid^2 dx - \int_\Omega \overline{u}_n (\nabla\eta \cdot \nabla u_n) \,dx - i\beta_n \int_\Omega b(x)\eta y_n \overline{u}_n \,dx \\ -i\beta_n \int_\Omega c(x)\eta(x)|u_n|^2 \,dx = \int_\Omega (-g_n^1 - b(x) f_n^2 - i\beta_n f_n^1 - c(x) f_n^1)\eta\overline{u}_n \,dx. \end{cases} \tag{3.16}$$

As f_n^1 and f_n^2 converge to zero in $H_0^1(\Omega)$, g_n^1 converges to zero in $L^2(\Omega)$, the sequences $(\beta_n u_n)$, $(\beta_n y_n)$, (∇u_n) are uniformly bounded in $L^2(\Omega)$ and $\| u_n \| = o(1)$, we get

$$\int_\Omega \eta(x) |\beta_n u_n|^2 dx - \int_\Omega \eta(x) \mid \nabla u_n \mid^2 dx = o(1). \tag{3.17}$$

By using the definition of η and Eqs. (3.11) and (3.17), we deduce

$$\int_\Omega \eta(x) \mid \nabla u_n \mid^2 dx = o(1) \;\; \text{and} \;\; \int_{\mathcal{V}_2 \cap \Omega} \mid \nabla u_n \mid^2 dx = o(1).$$

The proof is thus complete. □

Lemma 3.7. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ to the system (3.4)–(3.7) satisfies the following estimation*

$$\int_\Omega \eta(x) |\nabla y_n|^2 \,dx = o(1) \quad \textit{and} \quad \int_{\mathcal{V}_2 \cap \Omega} |\nabla y_n|^2 \,dx = o(1). \tag{3.18}$$

Proof. The proof contains three points.
(i) Notice that, from equation (3.9), $\frac{1}{\beta_n} \Delta \bar{y}_n$ is uniformly bounded in $L^2(\Omega)$. So, multiplying Eq. (3.8) by $\frac{1}{\beta_n} \eta \Delta \bar{y}_n$. Using Green's formula and the fact that $u_n = y_n = f_n^1 = 0$ on Γ, we get

$$\left\{\begin{array}{l} -\int_\Omega \beta_n \eta(x)(\nabla \overline{y}_n \cdot \nabla u_n)\,\mathrm{d}x - \int_\Omega \beta_n u_n(\nabla \eta \cdot \nabla \overline{y}_n)\,\mathrm{d}x + \int_\Omega \frac{1}{\beta_n}\eta(x)\Delta \overline{y}_n \Delta u_n \mathrm{d}x \\ +\mathrm{i}\int_\Omega b(x)\eta(x) \mid \nabla y_n \mid^2 \mathrm{d}x + \mathrm{i}\int_\Omega \eta(x) y_n(\nabla b \cdot \nabla \overline{y}_n)\,\mathrm{d}x + \mathrm{i}\int_\Omega b(x) y_n(\nabla \eta \cdot \nabla \overline{y}_n)\,\mathrm{d}x \\ +\mathrm{i}\int_\Omega c(x)\eta(x)(\nabla u_n \cdot \nabla \overline{y}_n)\,\mathrm{d}x + \mathrm{i}\int_\Omega c(x) u_n(\nabla \eta \cdot \nabla \overline{y}_n)\,\mathrm{d}x + \mathrm{i}\int_\Omega \eta(x) u_n(\nabla c \cdot \nabla \overline{y}_n)\,\mathrm{d}x = \\ +\int_\Omega (-g_n^1 - b(x) f_n^2 - c(x) f_n^1)(\frac{1}{\beta_n}\eta(x)\Delta \overline{y}_n)\,\mathrm{d}x + \mathrm{i}\int_\Omega \eta(x)(\nabla f_n^1 \cdot \nabla \overline{y}_n)\,\mathrm{d}x \\ +\mathrm{i}\int_\Omega f_n^1(\nabla \eta \cdot \nabla \overline{y}_n)\,\mathrm{d}x. \end{array}\right. \tag{3.19}$$

First, since f_n^1, f_n^2 converge to zero in $H_0^1(\Omega)$, g_n^1 converges to zero in $L^2(\Omega)$ and (∇y_n), $(\frac{1}{\beta_n}\Delta y_n)$ are uniformly bounded in $L^2(\Omega)$, then we have

$$\left\{\begin{array}{l} \int_\Omega (-g_n^1 - b f_n^2 - c f_n^1)(\frac{1}{\beta_n}\eta \Delta \overline{y}_n)\,\mathrm{d}x + \mathrm{i}\int_\Omega \eta(\nabla f_n^1 \cdot \nabla \overline{y}_n)\,\mathrm{d}x \\ +i\int_\Omega f_n^1(\nabla \eta \cdot \nabla \overline{y}_n)\,\mathrm{d}x = o(1). \end{array}\right. \tag{3.20}$$

Next, using the definition of η, the Eqs. (3.11), (3.15), and the fact that $\| u_n \| = o(1)$, $\| y_n \| = o(1)$ and (∇y_n) is uniformly bounded $L^2(\Omega)$, we get

$$\left\{\begin{array}{l} -\int_\Omega \beta_n u_n(\nabla \eta \cdot \nabla \overline{y}_n)\,\mathrm{d}x + \mathrm{i}\int_\Omega \eta(x) y_n(\nabla b \cdot \nabla \overline{y}_n)\,\mathrm{d}x \\ +\mathrm{i}\int_\Omega b(x) y_n(\nabla \eta \cdot \nabla \overline{y}_n)\,\mathrm{d}x + \mathrm{i}\int_\Omega c(x)\eta(x)(\nabla u_n \cdot \nabla \overline{y}_n)\,\mathrm{d}x \\ +\mathrm{i}\int_\Omega c(x) u_n(\nabla \eta \cdot \nabla \overline{y}_n)\,\mathrm{d}x + \mathrm{i}\int_\Omega \eta(x) u_n(\nabla c \cdot \nabla \overline{y}_n)\,\mathrm{d}x = o(1). \end{array}\right. \tag{3.21}$$

Finally, inserting (3.20) and (3.21) into (3.19), we get

$$-\int_\Omega \beta_n \eta(\nabla \overline{y}_n \cdot \nabla u_n)\,\mathrm{d}x + \int_\Omega \frac{1}{\beta_n}\eta \Delta \overline{y}_n \Delta u_n\,\mathrm{d}x + \mathrm{i}\int_\Omega b\eta \mid \nabla y_n \mid^2 \mathrm{d}x = o(1). \tag{3.22}$$

(ii) Multiplying (3.9) by the bounded sequence $\frac{1}{\beta_n}\eta \Delta \overline{u}_n$, integrating over Ω and using the fact that $u_n = y_n = f_n^2 = 0$ on Γ, we get

$$\left\{\begin{array}{l} -\int_\Omega \beta_n \eta(x)(\nabla y_n \cdot \nabla \overline{u}_n)\,\mathrm{d}x - \beta_n \int_\Omega y_n(\nabla \eta \cdot \nabla \overline{u}_n)\,\mathrm{d}x + \int_\Omega \frac{1}{\beta_n}\eta(x)\Delta y_n \Delta \overline{u}_n \mathrm{d}x \\ -\mathrm{i}\int_\Omega \eta(x) u_n(\nabla b \cdot \nabla \overline{u}_n)\,\mathrm{d}x - \mathrm{i}\int_\Omega \eta(x) b(x)|\nabla u_n|^2\,\mathrm{d}x - \mathrm{i}\int_\Omega b(x) u_n(\nabla \overline{u}_n \cdot \nabla \eta)\,\mathrm{d}x = \\ \mathrm{i}\int_\Omega \left(\eta(x)\left(\nabla f_n^2 \cdot \nabla \overline{u}_n\right) + f_n^2\left(\nabla \eta \cdot \nabla \overline{u}_n\right)\right)\mathrm{d}x + \int_\Omega \left(b(x) f_n^1 - g_n^2\right)\frac{1}{\beta_n}\eta(x)\Delta \overline{u}_n\,\mathrm{d}x. \end{array}\right. \tag{3.23}$$

First, since f_n^1, f_n^2 converge to zero in $H_0^1(\Omega)$, g_n^2 converges to zero in $L^2(\Omega)$ and ∇u_n, $\frac{1}{\beta_n}\Delta u_n$ are uniformly bounded in $L^2(\Omega)$, then we have

$$\mathrm{i}\int_\Omega \left(\eta(x)\left(\nabla f_n^2\cdot\nabla\overline{u}_n\right)+f_n^2\left(\nabla\eta\cdot\nabla\overline{u}_n\right)\right)\mathrm{d}x+\int_\Omega \left(b(x)f_n^1-g_n^2\right)\frac{1}{\beta_n}\eta(x)\Delta\overline{u}_n\,\mathrm{d}x=o(1). \tag{3.24}$$

Next, using Eq. (3.15) and the fact that $\| u_n \|= o(1)$ and $(\beta_n y_n)$ is uniformly bounded in $L^2(\Omega)$, we deduce that

$$\begin{cases} -\beta_n\int_\Omega y_n(\nabla\eta\cdot\nabla\overline{u}_n)\,\mathrm{d}x-\mathrm{i}\int_\Omega \eta(x)u_n(\nabla b\cdot\nabla\overline{u}_n)\,\mathrm{d}x \\ -\mathrm{i}\int_\Omega \eta(x)b(x)|\nabla u_n|^2\,\mathrm{d}x-\mathrm{i}\int_\Omega b(x)u_n(\nabla\overline{u}_n\cdot\nabla\eta)\,\mathrm{d}x=o(1). \end{cases} \tag{3.25}$$

Finally, inserting (3.24) and (3.25) into (3.23), we get

$$-\int_\Omega \beta_n\eta(x)(\nabla y_n\cdot\nabla\overline{u}_n)\,\mathrm{d}x+\int_\Omega \frac{1}{\beta_n}\eta(x)\Delta y_n\Delta\overline{u}_n\,\mathrm{d}x=o(1). \tag{3.26}$$

(iii) Summing the imaginary parts of Eqs. (3.22) and (3.26), we obtain

$$\int_\Omega b(x)\eta(x)|\nabla y_n|^2\,\mathrm{d}x=o(1). \tag{3.27}$$

Using the definition of the function η and condition (LH2), we deduce that

$$\int_\Omega \eta(x)|\nabla y_n|^2\,\mathrm{d}x=o(1) \quad \text{and} \quad \int_{\mathcal{V}_2\cap\Omega} |\nabla y_n|^2\,\mathrm{d}x=o(1).$$

The proof is thus complete. □

Lemma 3.8. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ to the system (3.4)-(3.7) satisfies the following estimation*

$$\int_\Omega \eta(x)|\beta_n y_n|^2\,\mathrm{d}x=o(1) \quad \textit{and} \quad \int_{\mathcal{V}_2\cap\Omega} |\beta_n y_n|^2\,\mathrm{d}x=o(1). \tag{3.28}$$

Proof. Multiplying Eq. (3.9) by $\eta\bar{y}_n$ and integrating over Ω. Then, using Green's formula and the fact that $\| y_n \|= o(1)$ and $y_n = 0$ on Γ, we get

$$\begin{cases} \int_\Omega \eta(x)|\beta_n y_n|^2\,\mathrm{d}x-\int_\Omega \overline{y}_n(\nabla y_n\cdot\nabla\eta)\,\mathrm{d}x \\ -\int_\Omega \eta(x)\mid\nabla y_n\mid^2\,\mathrm{d}x+\ \mathrm{i}\beta_n\int_\Omega b(x)\eta(x)y_n\overline{u}_n\,\mathrm{d}x=o(1). \end{cases} \tag{3.29}$$

Combining Eqs. (3.11), (3.18) and (3.29) and using the fact that $\| y_n \|= o(1)$, we get

$$\int_\Omega \eta(x)|\beta_n y_n|^2\,\mathrm{d}x=o(1) \quad \text{and} \quad \int_{\mathcal{V}_2\cap\Omega} |\beta_n y_n|^2\,\mathrm{d}x=o(1).$$

The proof is thus complete. □

Now, since $(\overline{\Omega}_j\setminus\mathcal{V}_2)\cap\overline{\mathcal{V}_1}=\emptyset$, we can define the function $\psi_j\in C_0^\infty(\mathbb{R}^N)$ by:

$$\psi_j(x)=0 \ \ \text{if}\, x\in\mathcal{V}_1, \quad 0\leqslant\psi_j\leqslant 1, \quad \psi_j(x)=1 \ \ \text{if}\, x\in\overline{\Omega_j}\setminus\mathcal{V}_2.$$

For $m_j(x) = (x - x_j)$, we define $h_j(x) = \psi_j(x)m_j(x)$.

Lemma 3.9. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ to the system (3.4)–(3.7) satisfies the following estimation*

$$N \int_{\Omega\setminus(\mathcal{V}_2\cap\Omega)} |\beta_n u_n|^2 \,\mathrm{d}x + (2-N) \int_{\Omega\setminus(\mathcal{V}_2\cap\Omega)} |\nabla u_n|^2 \,\mathrm{d}x$$
$$+2\mathrm{Re}\left\{ i \sum_{j=1}^{J} \int_{\Omega_j\setminus(\mathcal{V}_2\cap\Omega_j)} \beta_n b(x) y_n (m_j \cdot \nabla \overline{u}_n) \,\mathrm{d}x \right\} \leqslant o(1). \tag{3.30}$$

Proof. Multiplying Eq. (3.8) by $2(h_j \cdot \nabla \overline{u}_n)$ and integrating over Ω_j, we obtain

$$2\beta_n^2 \int_{\Omega_j} u_n (h_j \cdot \nabla \overline{u}_n) \,\mathrm{d}x + 2 \int_{\Omega_j} \Delta u_n (h_j \cdot \nabla \overline{u}_n) \,\mathrm{d}x - 2\,\mathrm{i} \int_{\Omega_j} \beta_n b(x) y_n (h_j \cdot \nabla \overline{u}_n) \,\mathrm{d}x =$$
$$2 \int_{\Omega_j} (-g_n^1 - b(x) f_n^2 - c(x) f_n^1)(h_j \cdot \nabla \overline{u}_n) \,\mathrm{d}x - 2\,\mathrm{i} \int_{\Omega_j} \beta_n f_n^1 (h_j \cdot \nabla \overline{u}_n) \,\mathrm{d}x. \tag{3.31}$$

(i) **Estimation of the second member of (3.31).** First, using Green's formula, the fact that $u_n = 0$ on $(\Gamma_j \setminus \gamma_j) \cap \Gamma$ and that $h_j = 0$ on γ_j, we get

$$-2\,\mathrm{i} \int_{\Omega_j} \beta_n f_n^1 (h_j \cdot \nabla \overline{u}_n) \,\mathrm{d}x = 2\,\mathrm{i} \int_{\Omega_j} \beta_n \overline{u}_n (h_j \cdot \nabla f_n^1) \,\mathrm{d}x + 2\,\mathrm{i} \int_{\Omega_j} \beta_n \overline{u}_n f_n^1 \,(\mathrm{div} h_j) \,\mathrm{d}x. \tag{3.32}$$

It follows, from the convergence of f_n^1 to zero in $H_0^1(\Omega)$ and the uniformly boundedness in $L^2(\Omega)$ of $\beta_n u_n$, that

$$-2\,\mathrm{i} \int_{\Omega_j} \beta_n f_n^1 (h_j \cdot \nabla \overline{u}_n) \,\mathrm{d}x = o(1). \tag{3.33}$$

Next, as f_n^1, f_n^2 converge to zero in $H_0^1(\Omega)$, g_n^1 converges to zero in $L^2(\Omega)$ and (∇u_n) is uniformly bounded in $L^2(\Omega)$, we deduce that

$$2 \int_{\Omega_j} (-g_n^1 - b(x) f_n^2 - c(x) f_n^1)(h_j \cdot \nabla \overline{u}_n) \,\mathrm{d}x = o(1). \tag{3.34}$$

Finally, we deduce that the second member of Eq. (3.31) is $o(1)$.
(ii) **Estimation of the first integral of Eq. (3.31).** Using Green's formula, we get

$$\mathrm{Re}\left\{ 2 \int_{\Omega_j} \beta_n^2 u_n (h_j \cdot \nabla \overline{u}_n) \,\mathrm{d}x \right\} = -\int_{\Omega_j} (\mathrm{div} h_j) |\beta_n u_n|^2 \,\mathrm{d}x + \int_{\Gamma_j} (h_j \cdot \nu_j) |\beta_n u_n|^2 \,\mathrm{d}\Gamma_j. \tag{3.35}$$

Since $\Psi_j = 0$ on γ_j and $u_n = 0$ on $(\Gamma_j \setminus \gamma_j) \cap \Gamma$, then we have

$$\mathrm{Re}\left\{ 2 \int_{\Omega_j} \beta_n^2 u_n (h_j \cdot \nabla \overline{u}_n) \,\mathrm{d}x \right\} = -\int_{\Omega_j} (\mathrm{div} h_j) |\beta_n u_n|^2 \,\mathrm{d}x. \tag{3.36}$$

(iii) **Estimation of the second integral of Eq. (3.31).** Using Green's formula, we get

$$\mathrm{Re}\left\{ 2 \int_{\Omega_j} \Delta u_n (h_j \cdot \nabla \overline{u}_n) \right\} = -2\,\mathrm{Re}\left\{ \sum_{i,k=1}^{N} \int_{\Omega_j} \partial_i h_j^k \partial_i u_n \partial_k \bar{u}_n \mathrm{d}x \right\} + \int_{\Omega_j} (\mathrm{div} h_j) |\nabla u_n|^2 \,\mathrm{d}x$$
$$- \int_{\Gamma_j} (h_j \cdot \nu_j) |\nabla u_n|^2 \,\mathrm{d}\Gamma_j + 2\mathrm{Re}\left\{ \int_{\Gamma_j} (\partial_{\nu_j} u_n)(h_j \cdot \nabla \overline{u}_n) \,\mathrm{d}\Gamma_j \right\}. \tag{3.37}$$

According to the choice of Ψ_j, only the boundary terms over $(\Gamma_j \setminus \gamma_j) \cap \Gamma$ are non-vanishing in (3.37). But on this part of the boundary, $u_n = 0$, and consequently $\nabla u_n = (\partial_\nu u_n) \cdot \nu = (\partial_{\nu_j} u_n) \nu_j$. Then, we have

$$-\int_{\Gamma_j}(h_j\cdot\nu_j)|\nabla u_n|^2\,\mathrm{d}\Gamma_j+2\,\mathrm{Re}\left\{\int_{\Gamma_j}(\partial_{\nu_j}u_n)(h_j\cdot\nabla\overline{u}_n)\,\mathrm{d}\Gamma_j\right\}=\int_{(\Gamma_j\backslash\gamma_j)\cap\Gamma}(\psi_j m_j\cdot\nu_j)|\partial_{\nu_j}u_n|^2\,\mathrm{d}\Gamma_j\leqslant 0. \tag{3.38}$$

Inserting (3.38) in (3.37), we get

$$\mathrm{Re}\left\{2\int_{\Omega_j}\Delta u_n(h_j\cdot\nabla\overline{u}_n)\right\}\leq -2\,\mathrm{Re}\left\{\sum_{i,k=1}^{N}\int_{\Omega_j}\partial_i h_j^k\partial_i u_n\partial_k\bar{u}_n\mathrm{d}x\right\}+\int_{\Omega_j}(\mathrm{div}h_j)|\nabla u_n|^2\,\mathrm{d}x. \tag{3.39}$$

(iv) **The main estimation.** Inserting Eqs. (3.33), (3.34), (3.36), and (3.39) in (3.31) and using the fact that $\psi_j=0$ on $\mathcal{V}_1$, we get

$$\begin{aligned}&\int_{\Omega_j\backslash(\mathcal{V}_1\cap\Omega_j)}\mathrm{div}(\psi_j m_j)(|\beta_n u_n|^2-|\nabla u_n|^2)\,\mathrm{d}x+2\,\mathrm{Re}\int_{\Omega_j\backslash(\mathcal{V}_1\cap\Omega_j)}\sum_{i,k=1}^{N}\partial_i(\psi_j m_j^k)\partial_i u_n\partial_k\bar{u}_n\mathrm{d}x\\&+2\,\mathrm{i}\,\mathrm{Re}\int_{\Omega_j\backslash(\mathcal{V}_1\cap\Omega_j)}\beta_n b(x)y_n(\psi_j m_j\cdot\nabla\overline{u}_n)\,\mathrm{d}x\leqslant o(1).\end{aligned} \tag{3.40}$$

Thus, summing over j and using the fact that $\psi_j=1$ on $\overline{\Omega_j}\setminus\mathcal{V}_2$, we get

$$\begin{aligned}&N\int_{\Omega\backslash(\mathcal{V}_2\cap\Omega)}|\beta_n u_n|^2\,\mathrm{d}x+(2-N)\int_{\Omega\backslash(\mathcal{V}_2\cap\Omega)}|\nabla u_n|^2\,\mathrm{d}x+2\,\mathrm{Re}\left\{\mathrm{i}\sum_{j=1}^{J}\int_{\Omega_j\backslash(\mathcal{V}_2\cap\Omega_j)}\beta_n b(x)y_n(m_j\cdot\nabla\overline{u}_n)\,\mathrm{d}x\right\}\\&\leqslant-\sum_{j=1}^{J}\int_{\mathcal{V}_2\cap\Omega_j}\left[\mathrm{div}(\psi_j m_j)(|\beta_n u_n|^2-|\nabla u_n|^2)\,\mathrm{d}x+2\sum_{i,k=1}^{N}\partial_i(\psi_j m_j^k)\partial_i u_n\partial_k u_n\right]\mathrm{d}x\\&-2\,\mathrm{i}\sum_{j=1}^{J}\int_{\mathcal{V}_2\cap\Omega_j}\beta_n b y_n(\psi_j m_j\cdot\nabla\overline{u}_n)\,\mathrm{d}x+o(1).\end{aligned} \tag{3.41}$$

Using Eqs. (3.11), (3.15), and (3.28), we deduce

$$\begin{aligned}&-\sum_{j=1}^{J}\int_{\mathcal{V}_2\cap\Omega_j}\left[\mathrm{div}(\psi_j m_j)(|\beta_n u_n|^2-|\nabla u_n|^2)\,\mathrm{d}x+2\sum_{i,k=1}^{N}\partial_i(\psi_j m_j^k)\partial_i u_n\partial_k u_n\right]\mathrm{d}x\\&-2\,\mathrm{i}\sum_{j=1}^{J}\int_{\mathcal{V}_2\cap\Omega_j}\beta_n b(x)y_n(\psi_j m_j\cdot\nabla\overline{u}_n)\,\mathrm{d}x=o(1).\end{aligned} \tag{3.42}$$

Finally, inserting (3.42) into (3.41), we obtain the desired Eq. (3.30) and the proof is thus complete. □

Lemma 3.10. *The solution $(u_n,v_n,y_n,z_n)\in D(\mathcal{A})$ to the system (3.4)–(3.7) satisfies the following estimation*

$$\begin{aligned}&N\int_{\Omega\backslash\mathcal{V}_2\cap\Omega}|\beta_n y_n|^2\,\mathrm{d}x+(2-N)\int_{\Omega\backslash\mathcal{V}_2\cap\Omega}|\nabla y_n|^2\,\mathrm{d}x\\&+2\mathrm{Re}\left\{\mathrm{i}\sum_{j=1}^{J}\int_{\Omega_j\backslash\mathcal{V}_2\cap\Omega_j}\beta_n b(x)\overline{y}_n(m_j\cdot\nabla u_n)\,\mathrm{d}x\right\}\leqslant o(1).\end{aligned} \tag{3.43}$$

Proof. Multiplying Eq. (3.9) by $2(h_j\cdot\nabla\overline{y}_n)$, we obtain

$$\begin{aligned}&2\beta_n^2\int_{\Omega_j}y_n(h_j\cdot\nabla\overline{y}_n)\,\mathrm{d}x+2\int_{\Omega_j}\Delta y_n(h_j\cdot\nabla\overline{y}_n)\,\mathrm{d}x+2\,\mathrm{i}\int_{\Omega_j}\beta_n b(x)u_n(h_j\cdot\nabla\overline{y}_n)\,\mathrm{d}x=\\&-2\,\mathrm{i}\int_{\Omega_j}\beta_n f_n^2(h_j\cdot\nabla\overline{y}_n)\,\mathrm{d}x+2\int_{\Omega_j}(b(x)f_n^1-g_n^2)(h_j\cdot\nabla\overline{y}_n)\,\mathrm{d}x.\end{aligned} \tag{3.44}$$

(i) **Estimation of the second member in** (3.44)**.** Using Green's formula and the fact that $y_n = 0$ on $(\Gamma_j \setminus \gamma_j) \cap \Gamma$ and $h_j = 0$ on $\gamma_j \cap \Gamma$, we obtain

$$-2\,\mathrm{i}\int_{\Omega_j} \beta_n f_n^2 (h_j \cdot \nabla \overline{y}_n)\,\mathrm{d}x = 2\,\mathrm{i}\int_{\Omega_j} \beta_n \overline{y}_n (h_j \cdot \nabla f_n^2)\,\mathrm{d}x + 2\,\mathrm{i}\int_{\Omega_j} \beta_n \overline{y}_n f_n^2\,(\mathrm{div} h_j)\,\mathrm{d}x.$$

It follows, since f_n^2 converges to zero in $H_0^1(\Omega)$ and $\beta_n y_n$ is uniformly bounded in $L^2(\Omega)$, that

$$-2\,\mathrm{i}\int_{\Omega_j} \beta_n f_n^2 (h_j \cdot \nabla \overline{y}_n)\,\mathrm{d}x = o(1). \tag{3.45}$$

As f_n^1 converges to zero in $H_0^1(\Omega)$, g_n^2 converges to zero in $L^2(\Omega)$ and (∇y_n) is uniformly bounded in $L^2(\Omega)$, we deduce that

$$2\int_{\Omega_j} (b(x) f_n^1 - g_n^2)(h_j \cdot \nabla \overline{y}_n)\,\mathrm{d}x = o(1). \tag{3.46}$$

So, combining (3.45) and (3.46) with (3.44), we get

$$2\beta_n^2 \int_{\Omega_j} y_n (h_j \cdot \nabla \overline{y}_n)\,\mathrm{d}x + 2\int_{\Omega_j} \Delta y_n (h_j \cdot \nabla \overline{y}_n)\,\mathrm{d}x + 2\,\mathrm{i}\int_{\Omega_j} \beta_n b(x) u_n (h_j \cdot \nabla \overline{y}_n)\,\mathrm{d}x = o(1). \tag{3.47}$$

(ii) **Estimation of first member of** (3.44)**.** Using Green's formula in (3.47) and following the same technique used in Lemma 3.9, we get

$$\begin{aligned} &N \int_{\Omega \setminus \mathcal{V}_2 \cap \Omega} |\beta_n y_n|^2\,\mathrm{d}x + (2-N) \int_{\Omega \setminus \mathcal{V}_2 \cap \Omega} |\nabla y_n|^2\,\mathrm{d}x \\ &-2\mathrm{Re}\left\{ i \sum_{j=1}^{J} \int_{\Omega_j \setminus \mathcal{V}_2 \cap \Omega_j} \beta_n b(x) u_n (m_j \cdot \nabla \overline{y}_n)\,\mathrm{d}x \right\} \leqslant o(1). \end{aligned} \tag{3.48}$$

(iii) **Estimation of the third integral of** (3.48)**.** Integrating by parts and using the fact that $u_n = y_n = 0$ on $\partial(\Omega_j \setminus \mathcal{V}_2 \cap \Omega_j) \subset \Gamma$, we obtain

$$\begin{aligned} 2\,\mathrm{i}\sum_{j=1}^{J} \int_{\Omega_j \setminus \mathcal{V}_2 \cap \Omega_j} \beta_n b(x) u_n (m_j \cdot \nabla \overline{y}_n)\,\mathrm{d}x &= -2\,\mathrm{i}\sum_{j=1}^{J} \int_{\Omega_j \setminus \mathcal{V}_2 \cap \Omega_j} \beta_n b \overline{y}_n (m_j \cdot \nabla u_n)\,\mathrm{d}x \\ -2\,\mathrm{i}\sum_{j=1}^{J} \int_{\Omega_j \setminus \mathcal{V}_2 \cap \Omega_j} \beta_n u_n \overline{y}_n (m_j \cdot \nabla b)\,\mathrm{d}x &- 2\,\mathrm{i}\sum_{j=1}^{J} \int_{\Omega_j \setminus \mathcal{V}_2 \cap \Omega_j} b(x)\,\mathrm{div}(m_j) \beta_n u_n \overline{y}_n \mathrm{d}x. \end{aligned} \tag{3.49}$$

Using the fact that $\beta_n u_n$ is uniformly bounded in $L^2(\Omega)$ and $\| y_n \| = o(1)$ in the right-hand side of (3.49), we deduce

$$2\,\mathrm{i}\sum_{j=1}^{J} \int_{\Omega_j \setminus \mathcal{V}_2 \cap \Omega_j} \beta_n b(x) u_n (m_j \cdot \nabla \overline{y}_n)\,\mathrm{d}x = -2\,\mathrm{i}\sum_{j=1}^{J} \int_{\Omega_j \setminus \mathcal{V}_2 \cap \Omega_j} \beta_n b(x) \overline{y}_n (m_j \cdot \nabla u_n)\,\mathrm{d}x + o(1). \tag{3.50}$$

(iii) **The main estimation.** Combining Eqs. (3.48) and (3.50), we obtain

$$\begin{aligned} &N \int_{\Omega \setminus \mathcal{V}_2 \cap \Omega} |\beta_n y_n|^2\,\mathrm{d}x + (2-N) \int_{\Omega \setminus \mathcal{V}_2 \cap \Omega} |\nabla y_n|^2\,\mathrm{d}x \\ &+2\,\mathrm{Re}\left\{ i \sum_{j=1}^{J} \int_{\Omega_j \setminus \mathcal{V}_2 \cap \Omega_j} \beta_n b(x) \overline{y}_n (m_j \cdot \nabla u_n)\,\mathrm{d}x \right\} \leqslant o(1). \end{aligned}$$

The proof is thus complete. □

Lemma 3.11. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ to the system (3.4)–(3.7) satisfies the following estimation*

$$\int_{\Omega\setminus(\mathcal{V}_2\cap\Omega)} \left(|\nabla u_n|^2 + |\beta_n u_n|^2 + |\nabla y_n|^2 + |\beta_n y_n|^2\right) dx = o(1). \tag{3.51}$$

Proof. By combining Eqs. (3.30) and (3.43), we conclude that

$$N \int_{\Omega\setminus(\mathcal{V}_2\cap\Omega)} \left(|\beta_n u_n|^2 + |\beta_n y_n|^2\right) dx + (2-N) \int_{\Omega\setminus(\mathcal{V}_2\cap\Omega)} \left(|\nabla u_n|^2 + |\nabla y_n|^2\right) dx \le o(1). \tag{3.52}$$

Now, multiplying (3.8) by $(1-N)\overline{u}_n$, integrating on Ω, using Green's formula and the fact that $(\beta_n u_n)$ is uniformly bounded in $L^2(\Omega)$, $\| u_n \| = o(1)$ and $\| y_n \| = o(1)$, we obtain

$$(1-N)\int_{\Omega} |\beta_n u_n|^2 dx - (1-N)\int_{\Omega} |\nabla u_n|^2 dx = o(1). \tag{3.53}$$

Using (3.11) and (3.15) in (3.53), we deduce

$$(1-N)\int_{\Omega\setminus(\mathcal{V}_2\cap\Omega)} |\beta_n u_n|^2 dx - (1-N)\int_{\Omega\setminus(\mathcal{V}_2\cap\Omega)} |\nabla u_n|^2 dx = o(1). \tag{3.54}$$

Similarly, multiplying (3.9) by $(1-N)\overline{y}_n$, integrating on Ω, using Green's formula, the fact that $(\beta_n y_n)$ is uniformly bounded in $L^2(\Omega)$, $\| u_n \| = o(1)$, and $\| y_n \| = o(1)$, we obtain

$$(1-N)\int_{\Omega} |\beta_n y_n|^2 dx - (1-N)\int_{\Omega} |\nabla y_n|^2 dx = o(1). \tag{3.55}$$

Using (3.18) and (3.28) in (3.55), we deduce

$$(1-N)\int_{\Omega\setminus(\mathcal{V}_2\cap\Omega)} |\beta_n y_n|^2 dx - (1-N)\int_{\Omega\setminus(\mathcal{V}_2\cap\Omega)} |\nabla y_n|^2 dx = o(1). \tag{3.56}$$

Finally, combining (3.52), (3.54) and (3.56), we obtain

$$\int_{\Omega\setminus(\mathcal{V}_2\cap\Omega)} \left(|\nabla u_n|^2 + |\beta_n u_n|^2 + |\nabla y_n|^2 + |\beta_n y_n|^2\right) dx = o(1). \quad \square$$

Remark 3.12. It is easy to see that the condition $a = 1$ is only used in the proof of estimation (3.18) in Lemma 3.7. So, if one may get this estimation in the case $a \neq 1$, then the results of Lemmas 3.8, 3.9, 3.10, and 3.11 are still also true for $a \neq 1$.

Proof of Theorem (3.1). It follows, from (3.11), (3.15), (3.18), (3.28) and (3.51), that $\| U_n \|_{\mathcal{H}} = o(1)$, which is a contradiction with (3.2). Consequently, condition (H2) holds and the energy of the system (1.4)–(1.6) decays exponentially to zero as t goes to infinity. The proof has been completed. $\square$

4. Non-uniform stability in the case $a \neq 1$

The aim of this section is to show that the system (1.4)–(1.6) is not uniformly (*i.e.* not exponentially) stable when the waves propagate with different speeds (*i.e.* $a \neq 1$), since it is already the case when c and b are constants in the whole domain, as shown below. Our result is the following.

Theorem 4.1. *Assume that $c = c_0 > 0$, $b = b_0 \neq 0$ in Ω and that $a \neq 1$. Then the energy of the system (1.4)–(1.6) does not decrease exponentially to zero as t goes to infinity.*

Proof. We will show that the resolvent of the operator $\mathcal{A}$ is not uniformly bounded on the imaginary axis, *i.e.* condition (H2) does not hold. So, to prove the non-uniform stability, it suffices to construct a sequence $\beta_n \in \mathbb{R}$ and a sequence $U_n = (u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ such that

$$\beta_n \to +\infty, \tag{4.1}$$

$$\| U_n \|_{\mathcal{H}} \to +\infty \tag{4.2}$$

and

$$\| (\mathrm{i}\beta_n I - \mathcal{A})U_n \|_{\mathcal{H}} \leq C < +\infty. \tag{4.3}$$

For this aim, let $\mu_n^2 > 0$ be an eigenvalue of the Laplacian with Dirichlet boundary condition and φ_n its associated eigenfunction:

$$\begin{cases} -\Delta\varphi_n = \mu_n^2\varphi_n, & \text{in } \Omega, \\ \varphi_n = 0, & \text{on } \Gamma. \end{cases} \tag{4.4}$$

Set $\beta_n = \mu_n$ and $U_n = (u_n, v_n, y_n, z_n) = (b_n\varphi_n, \mathrm{i}\beta_n b_n\varphi_n, a_n\varphi_n, \mathrm{i}\beta_n a_n\varphi_n)$, where

$$a_n = \frac{(a-1)}{b^2} + \frac{\mathrm{i}c}{b^2\mu_n} + \frac{\mathrm{i}}{b\mu_n} \quad \text{and} \quad b_n = \frac{1}{\mathrm{i}\mu_n b}. \tag{4.5}$$

This implies that $U_n = (u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ and

$$v_n - \mathrm{i}\mu_n u_n = 0, \tag{4.6}$$

$$a\Delta u_n - bz_n - cv_n - \mathrm{i}\mu_n v_n = \varphi_n, \tag{4.7}$$

$$z_n - \mathrm{i}\mu_n y_n = 0, \tag{4.8}$$

$$\Delta y_n + bv_n - \mathrm{i}\mu_n z_n = \varphi_n. \tag{4.9}$$

It follows that U_n is a solution to the equation

$$\mathcal{A}U_n - \mathrm{i}\mu_n U_n = V_n, \tag{4.10}$$

where $V_n = (0, \varphi_n, 0, \varphi_n) \in \mathcal{H}$.

Finally, from (4.10), we deduce that

$$\| \mathrm{i}\mu_n U_n - \mathcal{A}U_n \|_{\mathcal{H}}^2 = \| (0, \varphi_n, 0, \varphi_n) \|_{\mathcal{H}}^2 = 2. \tag{4.11}$$

On the other side, we have

$$\| U_n \|_{\mathcal{H}}^2 = 2(|a_n|^2 + |b_n|^2)\mu_n^2 \sim \frac{(a-1)^2}{b^4}\mu_n^2 \to +\infty. \tag{4.12}$$

Consequently, the sequences $\beta_n = \mu_n$ and $U_n = (b_n\varphi_n, \mathrm{i}\beta_n b_n\varphi_n, a_n\varphi_n, \mathrm{i}\beta_n a_n\varphi_n)$ satisfy the conditions (4.1)–(4.3). So, using the results of Huang [15] and Pruss [28], the system (1.4)–(1.6) is not uniformly stable in the energy space $\mathcal{H}$. The proof is thus complete. □

5. Polynomial stability in the case $a \neq 1$

The condition of equal speed is then a necessary and sufficient condition for the exponential stability of our system. Therefore, we look for a polynomial energy decay rate. Our second main result when the waves propagate at different speeds ($a \neq 1$) can be stated as follows.

Theorem 5.1 *(Polynomial decay rate). Let $a \neq 1$. Assume that all assumptions of Theorem 3.3 are satisfied. Then there exists a positive constant $C > 0$ independent of U_0 such that, for all initial data $U_0 = (u_0, u_1, y_0, y_1) \in D(\mathcal{A})$, the energy of the system (1.4)–(1.6) satisfies the following decay rate:*

$$E(t) \leq C\frac{1}{t}\|U(0)\|_{D(\mathcal{A})}^2, \quad \forall t > 0. \tag{5.1}$$

According to Theorem 2.4 of Borichev–Tomilov in [14], a C^0-semigroup of contractions $\mathrm{e}^{t\mathcal{A}}$ on a Hilbert space $\mathcal{H}$ verifies (5.1) if the following conditions

$$\mathrm{i}\mathbb{R} \subseteq \rho(\mathcal{A}) \tag{H1}$$

and

$$\sup_{|\beta|\geq 1} \frac{1}{\beta^2} \| (\mathrm{i}\beta I - (A))^{-1} \| < +\infty \tag{H3}$$

hold. As condition (H1) was already checked in Theorem 2.2, we now prove that condition (H3) holds, using an argument of contradiction. For this aim, we suppose that there exist a real sequence (β_n), with $\beta_n \longrightarrow +\infty$, and a sequence $U_n = (u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ such that

$$\| U_n \|_{\mathcal{H}} = 1 \tag{5.2}$$

and

$$\lim_{n\to\infty} \| \beta_n^2 (\mathrm{i}\beta_n I - \mathcal{A})U_n \|_{\mathcal{H}} = 0. \tag{5.3}$$

Next, by detailing Eq. (5.3), we obtain

$$\mathrm{i}\beta_n^3 u_n - \beta_n^2 v_n = f_n^1 \to 0 \quad \text{in} \quad H_0^1(\Omega), \tag{5.4}$$

$$\mathrm{i}\beta_n^3 v_n - a\beta_n^2 \Delta u_n + b(x)\beta_n^2 z_n + c(x)\beta_n^2 v_n = g_n^1 \to 0 \quad \text{in} \quad L^2(\Omega), \tag{5.5}$$

$$\mathrm{i}\beta_n^3 y_n - \beta_n^2 z_n = f_n^2 \to 0 \quad \text{in} \quad H_0^1(\Omega), \tag{5.6}$$

$$\mathrm{i}\beta_n^3 z_n - \beta_n^2 \Delta y_n - b(x)\beta_n^2 v_n = g_n^2 \to 0 \quad \text{in} \quad L^2(\Omega). \tag{5.7}$$

Eliminating v_n and z_n from (5.4)–(5.7), we get

$$\beta_n^2 u_n + a\Delta u_n - \mathrm{i}\beta_n b(x) y_n - \mathrm{i}\beta_n c(x) u_n = \frac{-g_n^1 - bf_n^2 - \mathrm{i}\beta_n f_n^1 - cf_n^1}{\beta_n^2}, \tag{5.8}$$

$$\beta_n^2 y_n + \Delta y_n + \mathrm{i}\beta_n b(x) u_n = \frac{-\mathrm{i}\beta_n f_n^2 + bf_n^1 - g_n^2}{\beta_n^2}. \tag{5.9}$$

In addition, from Eqs. (5.2), (5.4), and (5.6), we deduce that

$$\| u_n \|_{L^2(\Omega)} = \frac{O(1)}{\beta_n} \quad \text{and} \quad \| y_n \|_{L^2(\Omega)} = \frac{O(1)}{\beta_n}. \tag{5.10}$$

Lemma 5.2. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ to the system (5.4)–(5.7) satisfies the following estimations*

$$\int_{\omega_{c_+}} |u_n|^2 \,\mathrm{d}x = \frac{o(1)}{\beta_n^4}. \tag{5.11}$$

Proof. Multiplying equation (5.3) by U_n in $\mathcal{H}$, we get

$$\mathrm{Re}\left\{\mathrm{i}\beta_n^3 \| U_n \|^2 - \beta_n^2 (\mathcal{A}U_n, U_n)\right\} = \beta_n^2 \int_\Omega c(x)|v_n|^2 \,\mathrm{d}x = o(1). \tag{5.12}$$

Under condition (LH1), it follows

$$\int_{\omega_{c_+}} |v_n|^2 \,\mathrm{d}x = \frac{o(1)}{\beta_n^2}. \tag{5.13}$$

So, using Eqs. (5.4) and (5.12), we get

$$\int_\Omega c(x)|u_n|^2 \,\mathrm{d}x = \frac{o(1)}{\beta_n^4}. \tag{5.14}$$

This yields

$$\int_{\omega_{c_+}} |u_n|^2 \,\mathrm{d}x = \frac{o(1)}{\beta_n^4}. \quad \square$$

Lemma 5.3. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ to the system (5.4)–(5.7) satisfies the following estimation:*

$$\int_\Omega \eta |\beta_n \nabla u_n|^2 \,\mathrm{d}x = o(1) \quad \text{and} \quad \int_{\mathcal{V}_2 \cap \Omega} |\beta_n \nabla u_n|^2 \,\mathrm{d}x = o(1). \tag{5.15}$$

Proof. First, we multiply Eq. (5.8) by $\beta_n^2\eta\overline{u}_n$. Later, using Green's formula, (5.10), (5.11), and the fact that the sequences f_n^1, f_n^2, g_n^1 converge to zero, respectively, in $H_0^1(\Omega)$, $H_0^1(\Omega)$ and $L^2(\Omega)$, we get

$$\int_\Omega \eta|\beta_n^2 u_n|^2\,\mathrm{d}x - a\beta_n^2\int_\Omega \eta\mid\nabla u_n\mid^2\,\mathrm{d}x - \beta_n^2\int_\Omega a\overline{u}_n(\nabla\eta\cdot\nabla u_n)\,\mathrm{d}x - \mathrm{i}\beta_n^3\int_\Omega \eta b y_n\overline{u}_n\mathrm{d}x = o(1). \tag{5.16}$$

Using (5.11) and the fact that the sequences $(\beta_n y_n)$ and (∇u_n) are uniformly bounded in $L^2(\Omega)$, we deduce from (5.16) that

$$\int_\Omega \eta\mid\beta_n\nabla u_n\mid^2\,\mathrm{d}x = o(1) \text{ and } \int_{\mathcal{V}_2\cap\Omega}\mid\beta_n\nabla u_n\mid^2\,\mathrm{d}x = o(1). \quad \square$$

Lemma 5.4. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ to the system (5.4)–(5.7) satisfies the following estimation*

$$\int_\Omega \eta|\beta_n y_n|^2\,\mathrm{d}x = o(1) \quad \textit{and} \quad \int_{\mathcal{V}_2\cap\Omega}|\beta_n y_n|^2\,\mathrm{d}x = o(1). \tag{5.17}$$

Proof. Multiplying Eq. (5.8) by $\beta_n\eta\bar{y}_n$. Using Green's formula and the fact that $y_n = 0$ on Γ, we obtain

$$\begin{aligned}&\int_\Omega \beta_n^3\eta\overline{y}_n u_n\mathrm{d}x - a\int_\Omega \beta_n\eta(\nabla u_n\cdot\nabla\overline{y}_n)\,\mathrm{d}x - a\int_\Omega \beta_n\overline{y}_n(\nabla u_n\cdot\nabla\eta)\,\mathrm{d}x\\ &-i\textstyle\int_\Omega b(x)\eta|\beta_n y_n|^2\,\mathrm{d}x - \mathrm{i}\beta_n^2\displaystyle\int_\Omega c(x)\eta u_n\overline{y}_n\mathrm{d}x = o(1).\end{aligned} \tag{5.18}$$

Using (5.10), (5.11), (5.15), and the fact that the sequences $(\beta_n y_n)$ and (∇y_n) are uniformly bounded in $L^2(\Omega)$ in (5.18), we deduce

$$\int_\Omega b(x)\eta|\beta_n y_n|^2\,\mathrm{d}x = o(1). \tag{5.19}$$

Using the definition of function η and condition (LH2), we deduce

$$\int_\Omega \eta|\beta_n y_n|^2\,\mathrm{d}x = o(1) \quad \text{and} \quad \int_{\mathcal{V}_2\cap\Omega}|\beta_n y_n|^2\,\mathrm{d}x = o(1). \quad \square$$

Lemma 5.5. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of the system (5.4)–(5.7) satisfies the following estimation:*

$$\int_\Omega \eta\mid\nabla y_n\mid^2\,\mathrm{d}x = o(1) \quad \textit{and} \quad \int_{\mathcal{V}_2\cap\Omega}\mid\nabla y_n\mid^2\,\mathrm{d}x = o(1). \tag{5.20}$$

Proof. We multiply Eq. (5.9) by $\eta\overline{y}_n$. Then, using Green's formula and the condition $y_n = 0$ on Γ, we get

$$\int_\Omega \eta|\beta_n y_n|^2\,\mathrm{d}x - \int_\Omega \eta|\nabla y_n|^2\,\mathrm{d}x - \int_\Omega (\nabla\eta\cdot\nabla y_n)\overline{y}_n\mathrm{d}x + \mathrm{i}\beta_n\int_\Omega b u_n\eta\overline{y}_n = o(1) \tag{5.21}$$

Using (5.10) and the fact that $(\beta_n y_n)$ and (∇y_n) are bounded in $L^2(\Omega)$ in (5.21), we get

$$\int_\Omega \eta|\beta_n y_n|^2\,\mathrm{d}x - \int_\Omega \eta\mid\nabla y_n\mid^2\,\mathrm{d}x = o(1). \tag{5.22}$$

Finally, from (5.17), we deduce

$$\int_\Omega \eta|\nabla y_n|^2\,\mathrm{d}x = o(1) \quad \text{and} \quad \int_{\mathcal{V}_2\cap\Omega}\mid\nabla y_n\mid^2\,\mathrm{d}x = o(1). \quad \square$$

Proof of Theorem 5.1. As we mention in Remark 3.12, using Lemmas 5.2, 5.3, 5.4 and 5.5, we deduce that the estimation (3.51) is also true in the case $a \neq 1$. It follows, from the estimations (5.11), (5.15), (5.17), (5.20), and (3.51) that $\| U_n \|_{\mathcal{H}} = o(1)$, which is a contradiction with (5.2). Consequently, the condition (H3) holds and the energy of the smooth solution to the system (1.4)–(1.6) decays polynomially to zero as t goes to infinity. $\square$

Remark 5.6. Note that our results in Theorem 5.1 might be more general because the waves are not assumed to propagate with the same speed. So, this theorem generalizes the results of [8] and [16].

6. Optimality of the polynomial energy decay rate

We study here the optimality of the polynomial decay rate obtained for the N-dimensional coupled wave system in Theorem 5.1. To this aim, we will study the asymptotic behavior of the eigenvalues of the operator $\mathcal{A}$ in the one-dimensional case for b and c being constants. Indeed, we consider the 1-dimensional version of the system (1.4)–(1.6):

$$u_{tt} - au_{xx} + by_t + cu_t = 0, \qquad \text{in } (0,1)\times(0,+\infty), \tag{6.1}$$

$$y_{tt} - y_{xx} - bu_t = 0, \qquad \text{in } (0,1)\times(0,+\infty), \tag{6.2}$$

$$u(0,t) = u(1,t) = y(0,t) = y(1,t) = 0, \qquad \text{in } (0,+\infty) \tag{6.3}$$

with the following initial data:

$$u(x,0) = u_0, \quad y(x,0) = y_0, \quad u_t(x,0) = u_1 \text{ and } y_t(x,0) = y_1, \quad x \in (0,1)$$

where $1 \neq a > 0$, $c > 0$ and $b \in \mathbb{R}^\star$. From subsection 2.1, we express the system (6.1)–(6.3) as an evolution equation of type (2.3) with $\mathcal{H} = \left(H_0^1(0,1)\times L^2(0,1)\right)^2$ and $\mathcal{A} : \left(H^2(0,1)\cap H_0^1(0,1)\times H_0^1(0,1)\right)^2 \to \mathcal{H}$, defined by $\mathcal{A}(u,v,y,z) := (v, au_{xx} - bz - cv, z, y_{xx} + bv)$. The aim of this section is to obtain the following result.

Theorem 6.1. *Assume that $N = 1$, $a \neq 1$, $b = b_0 \neq 0$ and $c = c_0 > 0$. The energy decay rate (5.1) is optimal in the sense that, for any $\varepsilon > 0$, we can not expect the decay rate $\frac{1}{t^{1+\varepsilon}}$ for all initial data $U_0 \in D(\mathcal{A})$.*

For the proof of the Theorem 6.1, we first study the asymptotic behavior of the eigenvalues of the operator $\mathcal{A}$. Since $\mathcal{A}$ is dissipative, we fix $\alpha_0 > 0$ small enough and we study the asymptotic behavior of the eigenvalues λ of $\mathcal{A}$ in the strip

$$S = \{\lambda \in \mathbb{C} : -\alpha_0 \leq \operatorname{Re}(\lambda) \leq 0\}.$$

So, let $\lambda \in \mathbb{C}^\star$ be an eigenvalue of $\mathcal{A}$ with its associated eigenvector $U = (u,v,y,z) \in D(\mathcal{A})$. Then $\mathcal{A}U = \lambda U$ and equivalently

$$\begin{cases} v = \lambda u, \\ au_{xx} - bz - cv = \lambda v, \\ z = \lambda y, \\ y_{xx} + bv = \lambda z, \\ u(0) = u(1) = y(0) = y(1) = 0. \end{cases} \tag{6.4}$$

Eliminating v and z from (6.4), we get

$$\begin{cases} au_{xx} - \lambda(\lambda + c)u - b\lambda y &= 0, \\ y_{xx} - \lambda^2 y + b\lambda u &= 0, \\ u(0) = u(1) &= 0, \\ y(0) = y(1) &= 0. \end{cases} \tag{6.5}$$

From the second equation of (6.5), we have

$$u = \frac{1}{b\lambda}[\lambda^2 y - y_{xx}]. \tag{6.6}$$

Substituting (6.6) in the first equation of (6.5), we get

$$\begin{cases} ay_{xxxx} - [\lambda^2(a+1) + c\lambda]y_{xx} + \lambda^2(\lambda^2 + c\lambda + b^2)y &= 0, \\ y(0) = y(1) &= 0, \\ y_{xx}(0) = y_{xx}(1) &= 0. \end{cases} \tag{6.7}$$

The characteristic equation associated with the system (6.7) is given by

$$Q(r) := ar^4 - [\lambda^2(a+1) + c\lambda]r^2 + \lambda^2(\lambda^2 + c\lambda + b^2) = 0.$$

In order to proceed, we set the following notation. Here and below, in the case where z is a non-zero non-real number, we define (and denote) by $\sqrt{z}$ the square root of z; *i.e.* the unique complex number with positive real part whose square is equal to z.

The general solution to the first equation of (6.7) is given by

$$y(x)=\sum_{i=1}^{4} c_i\, e^{r_i(\lambda)x},$$

where

$$r_1(\lambda)=\frac{1}{\sqrt{2a}}\sqrt{\lambda\left[\lambda(a+1)+c+\sqrt{(a-1)^2\lambda^2-2c(a-1)\lambda-4ab^2+c^2}\right]},\qquad r_2(\lambda)=-r_1(\lambda), \tag{6.8}$$

$$r_3(\lambda)=\frac{1}{\sqrt{2a}}\sqrt{\lambda\left[\lambda(a+1)+c-\sqrt{(a-1)^2\lambda^2-2c(a-1)\lambda-4ab^2+c^2}\right]},\qquad r_4(\lambda)=-r_3(\lambda). \tag{6.9}$$

For simplicity, here and below, we denote $r_i(\lambda)$ by r_i.
Hence, the general solution is given by

$$y(x)=A_1\sinh(r_1x)+A_2\cosh(r_1x)+A_3\sinh(r_3x)+A_4\cosh(r_3x).$$

Using the boundary condition $y(0)=y_{xx}(0)=0$, we get

$$\begin{cases} A_2+A_4 &= 0,\\ A_2r_1^2+A_4r_3^2 &= 0,\end{cases} \tag{6.10}$$

which implies $A_2=A_4=0$, since $r_3^2-r_1^2\neq 0$.
Therefore,

$$y(x)=A_1\sinh(r_1x)+A_3\sinh(r_3x).$$

The boundary conditions $y(1)=y_{xx}(1)=0$ may be written as the following system

$$M(\lambda)c(\lambda)=\begin{pmatrix}\sinh(r_1) & \sinh(r_3)\\ r_1^2\sinh(r_1) & r_3^2\sinh(r_3)\end{pmatrix}\begin{pmatrix}A_1\\ A_3\end{pmatrix}=0. \tag{6.11}$$

So, set $F(\lambda)=det(M(\lambda))$. We have the following results.

Proposition 6.1. *Assume that $N=1$, $a\neq 1$, $b=b_0\neq 0$ and $c=c_0>0$. Then, there exist $n_0\in\mathbb{N}$ sufficiently large and two sequences $\left(\lambda_n^{(0)}\right)$ and $\left(\lambda_{n'}^{(1)}\right)$ of simple roots of F (which are also simple eigenvalues of $\mathcal{A}$) satisfying the following asymptotic behavior*

$$\lambda_n^{(0)}=in\pi-\frac{ib^2}{2(a-1)n\pi}-\frac{cb^2}{2(a-1)^2n^2\pi^2}+O\left(\frac{1}{n^3}\right),\quad \forall\,|n|\geq n_0 \tag{6.12}$$

and

$$\lambda_{n'}^{(1)}=in'\pi\sqrt{a}-\frac{c}{2}+O\left(\frac{1}{n'}\right),\qquad \forall\,|n'|\geq n_0. \tag{6.13}$$

Proof. It is easy to see that the system (6.11) has a non-trivial solution $(A_1,A_2)\neq(0,0)$ if and only if λ is solution to the following equation:

$$F(\lambda)=(r_3^2-r_1^2)\sinh(r_1)\sinh(r_3)=0.$$

Since $r_3^2-r_1^2\neq 0$, then $\sinh(r_1)=0$ or $\sinh(r_3)=0$. Thus

$$r_1=in\pi\quad\text{or}\quad r_3=in'\pi,\quad n,n'\in\mathbb{Z}.$$

It follows from the asymptotic expansion in (6.8) and (6.9) that

$$\lambda-\frac{b^2}{2(a-1)\lambda}-\frac{cb^2}{2(a-1)^2\lambda^2}+O(\frac{1}{\lambda^3})=in\pi,$$

or

$$\frac{\lambda}{\sqrt{a}}+\frac{c}{2\sqrt{a}}+O(\frac{1}{\lambda})=in'\pi.$$

Hence, since $n\sim n'\sim\lambda$, we obtain two branches $\lambda_n^{(0)}$ and $\lambda_{n'}^{(1)}$ of eigenvalues of the operator $\mathcal{A}$ that satisfy the asymptotic behaviors (6.12) and (6.13). The proof is thus complete. □

Remark 6.2. The operator $\mathcal{A}$ has two branches of eigenvalues. The energy corresponding to the first branch $\lambda_n^{(0)}$ decays polynomially, while the energy corresponding to the second branch of eigenvalues $\lambda_{n'}^{(1)}$ decays exponentially.

Proof of Theorem 6.1. Let $\epsilon > 0$ and set $\hat{l} = \frac{\epsilon}{1+\epsilon}$. First, let $\lambda_n^{(0)}$, with $n \geq n_0$, be the sequence of eigenvalues of the operator $\mathcal{A}$ described in Proposition 6.1, and let $U_n \in D(\mathcal{A})$ be the associated normalized eigenfunction. Moreover, we introduce the following sequence

$$\beta_n = \Im(\lambda_n^{(0)}), \quad \forall n \geq n_0.$$

Next, using (6.12), we have

$$(\mathrm{i}\beta_n I - \mathcal{A})U_n = \left(\frac{cb^2}{2(a-1)^2 n^2 \pi^2} + O\left(\frac{1}{n^3}\right) \right) U_n, \quad \forall n \geq n_0,$$

and therefore

$$\beta_n^{2-2\hat{l}} \|(\mathrm{i}\beta_n I - \mathcal{A})U_n\|_{\mathcal{H}} \sim \frac{cb^2}{2(a-1)^2\pi^2} \times \frac{1}{n^{\frac{2\epsilon}{1+\epsilon}}}, \quad \forall n \geq n_0.$$

Thus, we deduce that

$$\lim_{n\to+\infty} \beta_n^{2-2\hat{l}} \|(\mathrm{i}\beta_n I - \mathcal{A})U_n\|_{\mathcal{H}} = 0.$$

Thanks to Theorem 2.4 in Borichev–Tomilov [14], we deduce that for $U_0 \in D(\mathcal{A})$, $\|e^{t\mathcal{A}}U_0\|_{\mathcal{H}}$ decays more slowly than $\frac{1}{t^{\frac{1}{2-2\hat{l}}}}$ as time $t \to +\infty$. The proof is thus complete. □

Remark 6.3. We can use Theorem 3.4.1 in [26] and Eq. (6.12), to deduce the optimality of the polynomial energy decay rate (5.1) in the case $N = 1$.

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