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Existence and Besov regularity of the density for a class of SDEs with Volterra noise

Existence et régularité de Besov de la densité pour une classe d'équations différentielles stochastiques avec bruit de Volterra

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A B S T R A C T

By using a simple method based on the fractional integration by parts, we prove the existence and the Besov regularity of the density for solutions to stochastic differential equations driven by an additive Gaussian Volterra process. We assume weak regularity conditions on the drift. Several examples of Gaussian Volterra noises are discussed.

R É S U M É

En utilisant une méthode simple basée sur l'intégration fractionnelle par parties, nous prouvons l'existence et la régularité de Besov de la densité des solutions des équations différentielles stochastiques dirigées par un bruit additif gaussien de type Volterra. Nous supposons des conditions de faible régularité sur le coefficient de dérive. Plusieurs exemples de bruits gaussiens de Volterra sont discutés.

1. Introduction

A new and simple method has been introduced in [2], [3] in order to obtain the absolute continuity of the law of random variables. In particular, this method, based on fractional integration by parts, allows us to obtain the existence of the density of solutions to stochastic differential equations (SDEs in the sequel), together with its Besov regularity, under low regularity assumptions on the coefficients of the equation. These new techniques avoid the use of the Malliavin calculus,

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which usually requires strong regularity of the coefficients of the SDE. We refer, among others, to [1], [2], [3], [8], [9], [10] for several applications of the fractional integration by parts methodology to concrete examples.

Here, our purpose is to employ this recent method beyond the case of the standard Wiener noise, i.e. to treat the case of SDE with additive Gaussian Volterra noise. Actually, we consider the following SDE in $\mathbb{R}^d$

$$X_t = x + \int_0^t b(s, X_s)\, \mathrm{d}s + B_t \tag{1}$$

with $x \in \mathbb{R}^d, b \in L^\infty([0,T], C_b^\beta(\mathbb{R}^d))$ and $(B_t)_{t\in[0,T]}$ a d-dimensional Gaussian Volterra process that can be expressed as a Wiener integral with respect to the Wiener process under the form (2). Although our toy example is when B is a d-dimensional fractional Brownian motion (fBm), we will show that many other examples of Volterra noises can be considered.

We will show that any strong solution to (1), when it exists, admits a density with respect to the Lebesque measure. Moreover, we give the Besov regularity of the density of the solution, i.e. we find the Besov space to which the density belongs. Our main results are obtained under rather general conditions on the noise (the class of examples includes the fractional Brownian motion and the Ornstein–Uhlenbeck process, among others), and under a non-Lipschitz conditions on the drift b, i.e. $b \in L^\infty([0,T], C_b^\beta(\mathbb{R}^d))$ with $0 < \beta \leq 1$. We will show that the method can be also applied to a more general situation, i.e. to prove the existence and to find the Besov regularity of the density for solutions to a certain class of path-dependent SDEs.

We organized our paper as follows. In Section 2, we describe our context and our main assumptions. In Section 3, we prove the existence and the Besov regularity of the density of the solution to the stochastic differential equation (1), while in Section 4 we extend our result to the path-dependent case. Section 5 contains several examples of Gaussian Volterra noises that fit our assumptions.

Finally, concerning the notation used throughout the paper, we denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^d$, C_b^α denotes the set of bounded Hölder continuous functions of order α, while C denotes throughout the paper a generic strictly positive constant that may change from line to line.

2. Preliminaries

Let us start by introducing the basic definitions and assumptions.

2.1. The context

Let $(W_t)_{t\in[0,T]} = \left(W_t^{(1)}, \ldots, W_t^{(d)}\right)_{t\in[0,T]}$ be a d-dimensional Wiener process on the probability space $(\Omega, \mathcal{F}, P)$. Denote by $(\mathcal{F}_t)_{t\in[0,T]}$ the filtration generated by W and consider a Gaussian Volterra process $(B_t)_{t\in[0,T]} = \left(B_t^{(1)}, \ldots, B_t^{(d)}\right)_{t\in[0,T]}$ that can be expressed as

$$B_t = \int_0^t K(t,s)\, \mathrm{d}W_s \tag{2}$$

i.e. $B_t^{(i)} = \int_0^t K(t,s)\, \mathrm{d}W_s^{(i)}$ for every $i = 1, \ldots, d$. We assume in the sequel that K is a deterministic kernel such that $\int_0^T K^2(t,s)\, \mathrm{d}s < \infty$. In particular, the kernel K may have singularities, for example $K(t,s)$ may behave as $|t-s|^\alpha$ with $\alpha > \frac{1}{2}$ on the diagonal (see also the examples from Section 5). We will consider the following SDE in $\mathbb{R}^d$

$$\mathrm{d}X_t = b(t, X_t)\, \mathrm{d}t + B_t \tag{3}$$

with initial condition $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, where $(B_t)_{t\geq 0}$ is a Gaussian Volterra process of the form (2), i.e. for every $i = 1, \ldots, d$,

$$X_t^{(i)} = x_i + \int_0^t b_i(s, X_s)\, \mathrm{d}s + B_t^{(i)}$$

where b_i are the components of the function b. We will assume that the drift coefficient in (3) satisfies

$$b \in L^\infty\left([0,T], C_b^\beta(\mathbb{R}^d, \mathbb{R}^d)\right) \text{ with } 0 < \beta \leq 1. \tag{4}$$

Notice that there is not a general result on the existence and uniqueness of the solution to (3) under the assumption (4) for a general Volterra noise of the form (2). In the sequel, we will work under the assumption that there exists a strong solution to (3). Nevertheless, as we will comment in the last section, there are concrete situations when there exists a unique strong solution to (3) under the assumptions (4) (for instance, this happens at least when the noise is a Wiener process or a fractional Brownian motion).

If we assume stronger assumptions on b (for instance, if the drift is Lipschitz continuous and satisfies a linear growth condition), then we can easily get the existence and uniqueness of a strong solution to (3) for a rather general Volterra noise B. In this case, the existence of the density of the solution to (3) can be also obtained by different techniques (i.e. via Malliavin's calculus). Even in this case, the method employed below has the advantage that it allows us to find the Besov regularity of the density.

2.2. Besov spaces

We refer to [11] for a complete exposition on Besov spaces. Here we only recall the definition of a particular Besov space, namely the space $\mathcal{B}^s_{1,\infty}$ with $s > 0$.

Consider a function $f : \mathbb{R}^d \to \mathbb{R}$ and, for every $x, h \in \mathbb{R}^d$, put $(\Delta^1_h f)(x) = f(x+h) - f(x)$ and, for $n \geq 1$ integer, define the n-th increment of the function f at lag h by

$$(\Delta^n_h f)(x) = \Delta^1_h \left(\Delta^{n-1}_h f\right))(x) = \sum_{j=0}^{n} (-1)^{n-j} f(x+jh).$$

For $0 < s < n$ we define the norm

$$\|f\|_{\mathcal{B}^s_{1,\infty}} = \|f\|_{L^1(\mathbb{R}^d)} + \sup_{|h|\leq 1} |h|^{-s} \|\Delta^n_h f\|_{L^1(\mathbb{R}^d)}. \tag{5}$$

It can be shown that, for any $n, m > s$, the norms obtained in (5) using n, m are equivalent. Therefore, one can define the Besov space $\mathcal{B}^s_{1,\infty}$ as the set of functions $f \in L^1(\mathbb{R}^d)$ such that $\|f\|_{\mathcal{B}^s_{1,\infty}} < \infty$.

2.3. Fractional integration by parts

Our main tool to get the existence and the regularity of the density of the solution to (3) is the following smoothing lemma from [8].

Lemma 1. *Let X be a $\mathbb{R}^d$-valued random variable. If there exist an integer $m \geq 1$, two real numbers $s > 0$, $\alpha > 0$, with $\alpha < s < m$, and a constant $K > 0$ such that, for every $\phi \in C^\alpha_b(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$, with $|h| \leq 1$,*

$$\left|\mathbf{E}\left[\Delta^m_h \phi(X)\right]\right| \leq K|h|^s \|\phi\|_{C^\alpha_b},$$

then X has density f_X with respect to Lebesgue measure on $\mathbb{R}^d$. Moreover, $f_X \in B^{s-\alpha}_{1,\infty}$ and $\|f\|_{B^{s-\alpha}_{1,\infty}} \leq C(1+K)$.

3. The existence and the Besov regularity of the density

We consider the setup from Section 2: the SDE (3) with Volterra noise of the form (2) and with drift coefficient satisfying (4). We assume that there exists a strong solution to (3).

Fix a deterministic function $\varphi \in C^\alpha_b(\mathbb{R}^d)$ with $\alpha \in (0,1)$ to be chosen later. We need to estimate the quantity $\mathbf{E}\left[\Delta^m_h \varphi(X_t)\right]$ for $h > 0$ and $m \geq 1$ integer.

The core idea is to use the auxiliary process

$$Y^\epsilon_s = \begin{cases} X_s, & s \leq t-\epsilon \\ X_{t-\epsilon} + \int_{t-\epsilon}^s b(r, X_{t-\epsilon})\,\mathrm{d}r + (B_s - B_{t-\epsilon}), & s \geq t-\epsilon\,. \end{cases} \tag{6}$$

In order to estimate the quantity $\mathbf{E}\left[\Delta^m_h \varphi(X_t)\right]$, we will decompose it as follows

$$\mathbf{E}\left[\Delta^m_h \varphi(X_t)\right] = (\mathbf{Pe})_{m,h,\varepsilon,t} + (\mathbf{Ae})_{m,h,\varepsilon,t}$$

where *the probability estimate* $(\mathbf{Pe})_{m,h,\varepsilon,t}$ is given by

$$(\mathbf{Pe})_{m,h,\varepsilon,t} = \mathbf{E}\left[\Delta^m_h \varphi(Y^\varepsilon_t)\right] \tag{7}$$

and *the approximation error* $(\mathbf{Ae})_{m,h,\varepsilon,t}$ is

$$(\mathbf{Ae})_{m,h,\varepsilon,t} = \mathbf{E}\left[\Delta^m_h \varphi(X_t)\right] - \mathbf{E}\left[\Delta^m_h \varphi(Y^\varepsilon_t)\right]. \tag{8}$$

We will deal separately with the summands $(\mathbf{Pe})_{m,h,\varepsilon,t}$ and $(\mathbf{Ae})_{m,h,\varepsilon,t}$, by using the ideas from [8] and the properties of the Volterra noise B.

3.1. The probabilistic estimate

To get a suitable estimate for $(\mathbf{Pe})_{m,h,\varepsilon,t}$, we will express it in terms of two independent random variables. First notice that from (6),

$$Y_t^\varepsilon = X_{t-\varepsilon} + \int_{t-\varepsilon}^{t} b(u, X_{t-\varepsilon})\,\mathrm{d}u + B_t - B_{t-\varepsilon}$$

and by writting $B_t - B_{t-\varepsilon} = \int_0^{t-\varepsilon} (K(t,s) - K(t-\varepsilon, s))\,\mathrm{d}W_s + \int_{t-\varepsilon}^t K(t,s)\,\mathrm{d}W_s$, we obtain

$$\begin{aligned} Y_t^\varepsilon &= X_{t-\varepsilon} + \int_{t-\varepsilon}^{t} b(u, X_{t-\varepsilon})\,\mathrm{d}u + \int_0^{t-\varepsilon} (K(t,s) - K(t-\varepsilon, s))\,\mathrm{d}W_s + \int_{t-\varepsilon}^{t} K(t,s)\,\mathrm{d}W_s \\ &= Z_t^\varepsilon + I_t^\varepsilon \end{aligned} \tag{9}$$

where

$$Z_t^\varepsilon = X_{t-\varepsilon} + \int_{t-\varepsilon}^{t} b(u, X_{t-\varepsilon})\,\mathrm{d}u + \int_0^{t-\varepsilon} (K(t,s) - K(t-\varepsilon, s))\,\mathrm{d}W_s \tag{10}$$

and $I_t^{i,\varepsilon} = (I_t^{1,\varepsilon}, \ldots, I_t^{d,\varepsilon})$ with

$$I_t^{i,\varepsilon} = \int_{t-\varepsilon}^{t} K(t,s)\,\mathrm{d}W_{i,s} \text{ for every } i = 1, \ldots, d. \tag{11}$$

The key observation is that Z_t^ε is a $\mathcal{F}_{t-\varepsilon}$ measurable random variable in $\mathbb{R}^d$ while I_t^ε is a centered Gaussian random variable independent of $\mathcal{F}_{t-\varepsilon}$. Using the above decomposition (9), we obtain the following estimate for the probabilistic estimate.

Proposition 1. *Assume (4) and suppose that, for every* $0 < \varepsilon < t$,

$$\mathrm{Var}(I_t^\varepsilon) \geq C\varepsilon^{2A} c(\varepsilon, t) \text{ with some } A \in (0,1) \tag{12}$$

where $c(\varepsilon, t)$ *is a strictly positive constant that may depend on* ε, t *and satisfies* $c(\varepsilon, t) \leq C$ *for every* $0 < \varepsilon < t$. *Then for every real* $h > 0$ *and for every integer* $m \geq 1$

$$\left|(\mathbf{Pe})_{m,h,\varepsilon,t}\right| \leq C\|\varphi\|_\infty \left(\frac{|h|}{\varepsilon^A}\right)^m.$$

Proof. From the decomposition (9), with $\varphi \in C_b^\alpha(\mathbb{R}^d)$,

$$\begin{aligned} (\mathbf{Pe})_{m,h,\varepsilon,t} &= \mathbf{E}\left[\Delta_h^m \varphi(Y_t^\varepsilon)\right] = \mathbf{E}\left[\Delta_h^m \varphi(Z_t^\varepsilon + I_t^\varepsilon)\right] \\ &= \mathbf{E}\left[\mathbf{E}\left[\Delta_h^m \varphi(Z_t^\varepsilon + I_t^\varepsilon)/\mathcal{F}_{t-\varepsilon}\right]\right] = \mathbf{E} f(Z_t^\varepsilon) \end{aligned} \tag{13}$$

with $f(y) = \mathbf{E}\left[\Delta_h^m \varphi(y + I_t^\varepsilon)\right]$. Denote by $g_{t,\varepsilon}$ the density of the Gaussian random variable I_t^ε, i.e.

$$g_{t,\varepsilon}(x) = \frac{1}{\sqrt{2\pi \mathrm{Var}(I_t^\varepsilon)}^d} \mathrm{e}^{-\frac{|x|^2}{2\,\mathrm{Var}(I_t^\varepsilon)}}. \tag{14}$$

We compute $f(y)$ via a trivial change of variables

$$\begin{aligned} |f(y)| &= \left|\int_{\mathbb{R}^d} \Delta_h^m \varphi(y+x) g_{t,\varepsilon}(x)\,\mathrm{d}x\right| = \left|\int_{\mathbb{R}^d} \varphi(y+x)\left(\Delta_{-h}^m g_{t,\varepsilon}(x)\right)\mathrm{d}x\right| \\ &\leq \|\varphi\|_\infty \|\Delta_{-h}^m g_{t,\varepsilon}(x)\|_{L^1(\mathbb{R}^d)}. \end{aligned}$$

It follows from [8] (see page 5, two lines after (2.7)) that assumption (12) implies that

$$\|\Delta_{-h}^m g_{t,\varepsilon(x)}\|_{L^1(\mathbb{R}^d)} \le C\left(\frac{|h|}{\sqrt{\mathrm{Var}I_t^\varepsilon}}\right)^m \le C\left(\frac{|h|}{\varepsilon^A}\right)^m \tag{15}$$

for every $h > 0$ and for any integer $m \ge 1$. Then the conclusion is obtained from (13) and (15). $\square$

We will see in the last section that (12) is satisfied for many Gaussian processes, including the fractional Brownian motion.

3.2. The approximation error

In order to handle the term $(\mathbf{Ae})_{m,h,\varepsilon,t}$ given by (8), we need the following hypothesis on the Gaussian noise B: there exists $C > 0$ such that

$$\mathbf{E}\,|B_t - B_s|^2 \le C|t-s|^{2H} \text{ with some } H \in (0,1). \tag{16}$$

Remark 1. In particular, assumption (16) implies that the process B has Hölder continuous paths of order δ for every $\delta \in (0, H)$.

We have the following result for the approximation error $(\mathbf{Ae})_{m,h,\varepsilon,t}$.

Proposition 2. *Assume (4) and (16). Then for every* $0 < \varepsilon < t$, $h > 0$, $m \ge 1$

$$\left|(\mathbf{Ae})_{m,h,\varepsilon,t}\right| \le C\|\varphi\|_{C_b^\alpha}\varepsilon^{(\beta H+1)\alpha}. \tag{17}$$

Proof. Since φ is α-Hölder continuous, clearly

$$(\mathbf{Ae})_{m,h,\varepsilon,t} = \mathbf{E}\left[\Delta_h^m\varphi(X_t)\right] - \mathbf{E}\left[\Delta_h^m\varphi(Y_t^\varepsilon)\right] \le C\|\varphi\|_{C_b^\alpha}\mathbf{E}\left|X_t - Y_t^\varepsilon\right|^\alpha.$$

Now, the difference $X_t - Y_t^\varepsilon$ can be written as $X_t - Y_t^\varepsilon = \int_{t-\varepsilon}^t (b(u, X_u) - b(u, X_{t-\varepsilon}))\,\mathrm{d}u$. Thus

$$\mathbf{E}\left|X_t - Y_t^\varepsilon\right|^\alpha = \mathbf{E}\left|\int_{t-\varepsilon}^t (b(u, X_u) - b(u, X_{t-\varepsilon}))\,\mathrm{d}u\right|^\alpha \le C\left|\int_{t-\varepsilon}^t \mathbf{E}\,|X_u - X_{t-\varepsilon}|^\beta\,\mathrm{d}u\right|^\alpha. \tag{18}$$

Using (16), for every $u > t - \varepsilon$

$$\mathbf{E}\,|X_u - X_{t-\varepsilon}|^\beta = \mathbf{E}\left|\int_{t-\varepsilon}^u b(v, X_v)\,\mathrm{d}v + B_u - B_{t-\varepsilon}\right|^\beta \le C\left((u-t+\varepsilon)^\beta + (u-t+\varepsilon)^{\beta H}\right).$$

So, by plugging the above inequality into (18),

$$\mathbf{E}\left|X_t - Y_t^\varepsilon\right|^\alpha \le C\left|\int_{t-\varepsilon}^t \left((u-t+\varepsilon)^\beta + (u-t+\varepsilon)^{\beta H}\right)\mathrm{d}u\right|^\alpha \le C\varepsilon^{(\beta H+1)\alpha}$$

and this implies (17). $\square$

3.3. The density of the solution

We are now ready to apply the smoothing Lemma 1. From Proposition 1 and 2 we obtain the following theorem.

Theorem 1. *Assume (4), (12) and (16). Let* $(X_t)_{t\in[0,T]}$ *be a strong solution to (3). Then, for every* $t \in [0, T]$, *the random variable* X_t *admits a density* ρ_t *with respect to the Lebesque measure. Moreover,*

$$\rho_t \in \mathcal{B}_{1,\infty}^\eta \text{ for any } \eta < \frac{1 - A + \beta H}{A}.$$

Proof. From Propositions 1 and 2, for $h>0, m\geq 1$,

$$\left|\mathbf{E}\Delta_h^m\varphi(X_t)\right| \leq \left|(\mathbf{Pe})_{m,h,\varepsilon,t}\right| + \left|(\mathbf{Ae})_{m,h,\varepsilon,t}\right| \leq C\|\varphi\|_{C_b^\alpha}\left(\left(\frac{|h|}{\varepsilon^A}\right)^m + \varepsilon^{(\beta H+1)\alpha}\right).$$

Let us choose $\varepsilon = h^{\frac{m}{\alpha(\beta H+1)+Am}}$. Then we get

$$\left|\mathbf{E}\Delta_h^m\varphi(X_t)\right| \leq C\|\varphi\|_{C_b^\alpha}|h|^s$$

with $s=\frac{m\alpha(1+\beta H)}{\alpha(1+\beta H)+Am}$.

Note that, for m large enough, the exponent of $|h|$ is about $\frac{\alpha(1+\beta H)}{A}$. Therefore, by Lemma 1, for every $t\in[0,T]$, the random variable X_t has a density ρ_t belonging to the Besov space $\mathcal{B}^\eta_{1,\infty}$, with $\eta<s-\alpha=\frac{\alpha(1+\beta H)}{A}-\alpha$. Since we can choose α to be arbitrary close to 1, we obtain the conclusion. □

Let us finish this section but some comments around Theorem 1.

Remark 2.

- In the case of the Wiener noise (i.e. $K(t,s)=1_{[0,t]}(s)$ for every $s,t\in[0,T]$, conditions (12) and (16) hold with $A=H=\frac{1}{2}$. On the other hand, a unique strong solution to (3) exists under (4). Indeed, the existence and uniqueness of the strong solution is assured for every measurable function $b\in L^\infty([0,T]\times\mathbb{R}^d)$ (see [13] for $d=1$ and [12] for general dimensional $d\geq 1$). It follows from Theorem 1, that the solution to (3) admits a density in the Besov space $\mathcal{B}^\eta_{1,\infty}$ for every $\eta<1+\beta$. We retrieve a result in Section 2 of [8].
- We notice that both the noise in (3) and the variance of I_t^ε affect the regularity of the density. The more regular are the paths of the noise B (i.e. H increases), the more regular is the density of solution (i.e. η increases). Also, as the variance of I_t^ε increases, then A decreases and therefore the regularity of the solution increases.

4. The path-dependent case

The argument from the previous section can be easily adapted to treat the path-dependent case. By "path-dependent", we mean that a new process V is introduced in the expression on the drift b (and not that b depends on the whole trajectory of X as in, e.g., [8], Section 7.1). As before, we will consider $(W_t)_{t\in[0,T]}$ a d-dimensional $\mathcal{F}_t$-Brownian motion on the probability space $(\Omega,\mathcal{F},P)$, where $(\mathcal{F}_t)t_{\geq 0}$ is a filtration that satisfies the usual conditions. Let $(B_t)_{t\in[0,T]}$ be a Volterra process of the form (2). We consider the SDE

$$X_t(x) = x + \int_0^t b(r,V_r,X_r)\,\mathrm{d}r + B_t \tag{19}$$

with $t\in[0,T]$, $x\in\mathbb{R}^d$. In this section, the drift b is assumed to satisfy

$$b\in L^\infty\left([0,T],C_b^\beta\left(\mathbb{R}^d\times\mathbb{R}^d,\mathbb{R}^d\right)\right),\quad 0<\beta\leq 1, \tag{20}$$

while $(V_t)_{t\in[0,T]}$ is a $\mathcal{F}_t$-adapted process such that

$$\mathbf{E}|V_t-V_s|^\beta \leq C|t-s|^\delta \text{ for some } \delta>0. \tag{21}$$

We assume, as before, that there exists a strong solution to (19). For $\varepsilon>0$, we define the auxiliary process Y_t^ε by

$$Y_s^\epsilon = \begin{cases} X_s, & s\leq t-\epsilon \\ X_{t-\epsilon}+\int_{t-\epsilon}^s b(r,V_{t-\epsilon},X_{t-\epsilon})\,\mathrm{d}r + (B_s-B_{t-\epsilon}), & s\geq t-\epsilon\,. \end{cases} \tag{22}$$

We decompose again the quantity $\mathbf{E}[\Delta_h^m\varphi(X_t)]$ into two terms, the approximation error

$$(\mathbf{Ae})_{m,h,\varepsilon,t} = \mathbf{E}\left[\Delta_h^m\varphi(X_t)\right] - \mathbf{E}[\Delta_h^m\varphi(Y_t^\epsilon)] \tag{23}$$

and the probabilistic estimate

$$(\mathbf{Pe})_{m,h,\varepsilon,t} = \mathbf{E}\left[\Delta_h^m\varphi(Y_t^\epsilon)\right]. \tag{24}$$

Concerning the summand $(\mathbf{Pe})_{m,h,\varepsilon,t}$, we have the following estimate:

Lemma 2. *Assume (20) and (12). Then we have*

$$\left|(\mathbf{Pe})_{m,h,\varepsilon,t}\right| = \left|\mathbf{E}\left[\Delta_h^m \varphi(Y_t^\epsilon)\right]\right| \le C\|\varphi\|_\infty \left(\frac{|h|}{\varepsilon^A}\right)^m. \tag{25}$$

Proof. From (22), we can write $(\mathbf{Pe})_{m,h,\varepsilon,t} = Z_t^\varepsilon + I_t^\varepsilon$, where I_t^ε is given by (11) and

$$Z_t^\epsilon = X_{t-\epsilon} + \int_{t-\epsilon}^{t} b\left(r, V_{t-\epsilon}, X_{t-\epsilon}\right) \mathrm{d}s + \int_0^{t-\epsilon} \left(K(t,s) - K(t-\epsilon, s)\right) \mathrm{d}W_s,$$

since Z_t^ε is $\mathcal{F}_{t-\varepsilon}$ measurable and I_t^ε is independent by $\mathcal{F}_{t-\varepsilon}$, we can write

$$\mathbf{E}[\Delta_h^m \varphi(Y_t^\epsilon)] = \mathbf{E}[\Delta_h^m \varphi(Z_t^\epsilon + I_t^\varepsilon)] = \mathbf{E}\big[\mathbf{E}\left[\Delta_h^m \varphi(y + I_t^\varepsilon)\right]_{y=Z^\epsilon}\big],$$

and using the inequality (15) and the assumption (12), we get (recall that $g_{t,\varepsilon}$ is given by (14))

$$\begin{aligned}\left|\mathbf{E}\left[\Delta_h^m \varphi(y + I_t^\varepsilon)\right]\right| &= \left|\int_{\mathbb{R}^d} \varphi(y+x) \Delta_{-h}^m g_{t,\epsilon}(x)\, \mathrm{d}x\right| \\ &\le \|\varphi\|_\infty \|\Delta_{-h}^m g_{t,\epsilon}(x)\|_{L^1(\mathbb{R}^d)} \le C\|\varphi\|_\infty (\frac{|h|}{\epsilon^A})^m. \qquad \square\end{aligned}$$

For the approximation error term $(\mathbf{Ae})_{m,h,\varepsilon,t}$, we have the next result.

Proposition 3. *Assume (20), (21), and (16). Then, with $\mu = \min(\beta H, \delta)$,*

$$\left|(\mathbf{Ae})_{m,h,\varepsilon,t}\right| \le C_H \|\varphi\|_\alpha \epsilon^{\alpha(\mu+1)}. \tag{26}$$

Proof. We write as in the proof of Proposition 2

$$\begin{aligned}\left|(\mathbf{Ae})_{m,h,\varepsilon,t}\right| &= \left|\mathbf{E}\left[\Delta_h^m \varphi(X_t)\right] - \mathbf{E}\left[\Delta_h^m \varphi(Y_t^\epsilon)\right]\right| \\ &\le C\|\varphi\|_{C_b^\alpha} \mathbf{E}\left|\int_{t-\epsilon}^t \left(b(r, V_r, X_r) - b(r, V_{t-\epsilon}, X_{t-\epsilon})\right) \mathrm{d}r\right|^\alpha \\ &\le C\|\varphi\|_\alpha \mathbf{E}\Big|\int_{t-\epsilon}^t |b\left(r, V_r, X_r\right) - b\left(r, V_r, X_{t-\epsilon}\right)| + |b(r, V_r, X_{t-\epsilon}) - b(r, V_{t-\epsilon}, X_{t-\epsilon})\, \mathrm{d}r\Big|^\alpha \\ &\le C\|\varphi\|_\alpha \left(\mathbf{E}\int_{t-\epsilon}^t |X_r - X_{t-\epsilon}|^\beta + |V_r - V_{t-\epsilon}|^\beta\, \mathrm{d}r\right)^\alpha.\end{aligned} \tag{27}$$

By inserting the following two bounds

$$\mathbf{E}\,|X_r - X_{t-\epsilon}|^\beta \le \|b\|_{L^\infty} (r - t + \epsilon)^\beta + (r - t + \epsilon)^{\beta H}$$

and $\mathbf{E}\,|V_r - V_{t-\epsilon}|^\beta \le C\,(r - t + \epsilon)^\delta$ into (27), we get

$$\left|(\mathbf{Ae})_{m,h,\varepsilon,t}\right| \le C\|\varphi\|_\alpha \Big(\int_{t-\epsilon}^t (r-t+\epsilon)^\beta + (r-t+\epsilon)^{\beta H} + (r-t+\epsilon)^\delta\, \mathrm{d}r\Big)^\alpha \le C_H \|\varphi\|_\alpha \epsilon^{\alpha(\mu+1)}$$

where $\mu = \min(\beta H, \delta)$. $\square$

We obtain the following results concerning the density of the solution to (19).

Theorem 2. *We assume the conditions (12), (16), (20) and (21). Then the law of X_t has a density $\rho_{t,x}$ respect to the Lebesgue measure and $\rho \in B_{1,\infty}^\eta$ with $\eta < \frac{\mu+1-A}{A}$, where $\mu = \min(\delta, H\beta)$.*

Proof. From the estimates (26) and (25) we get for $h > 0$ and $m \geq 1$

$$\left|\mathbf{E}\left[\Delta_h^m(X_t)\right]\right| \leq C_H \|\varphi\|_\alpha \epsilon^{\alpha(\mu+1)} + C_H \|\varphi\|_\alpha \left(\frac{|h|}{\epsilon^A}\right)^m$$

Now, choosing $\epsilon = h^{\frac{m}{\alpha(\mu+1)+Am}}$ and proceeding as in the proof of Theorem 1, we obtain the desired conclusion. □

Remark 3. Notice that the Besov regularity of the density is affected by the regularity of the process V since the exponent δ from (21) appears in the above result. By taking a regular process V with $\delta > H\beta$, we retrieve the result in Theorem 1, but for a process V such that $\delta < H\beta$, the Besov regularity of the density will change.

5. Examples

We discuss several examples where our main results stated in Theorems 1 and 2 apply.

5.1. Fractional Brownian motion

Let $(B_t)_{t\in[0,T]}$ be a fractional Brownian motion with Hurst index $H \in (0,1)$. Recall that B is a centered Gaussian process with covariance

$$\mathbf{E}B_t B_s = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \text{ for every } s,t \in [0,T].$$

The fBm admits the following integral representation

$$B_t = \int_0^t K_H(t,s)\,\mathrm{d}W_s \tag{28}$$

where $(W_t)_{t\in[0,T]}$ is a Wiener process, and $K_H(t,s)$ is the kernel

$$K_H(t,s) = d_H\,(t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}} F_1\left(\frac{t}{s}\right), \tag{29}$$

d_H being a constant and $F_1(z) = d_H\left(\frac{1}{2} - H\right)\int_0^{z-1} \theta^{H-\frac{3}{2}}\left(1 - (\theta+1)^{H-\frac{1}{2}}\right)\mathrm{d}\theta$. If $H > \frac{1}{2}$, the kernel K_H has the simpler expression

$$K_H(t,s) = c_H s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}}\,\mathrm{d}u \tag{30}$$

where $t > s$ and $c_H = \left(\frac{H(H-1)}{\beta(2-2H, H-\frac{1}{2})}\right)^{\frac{1}{2}}$.

The SDE (3) with fBm noise has been treated in [6], [7], [4], among others. The following facts have been proven (for $d = 1$):

- if $H > \frac{1}{2}$, then there exists a unique strong solution to (3) if the drift b is Hölder continuous in time of order $\gamma > H - \frac{1}{2}$ and it is Hölder continuous in space of order $\alpha > 1 - \frac{1}{2H}$, i.e.

  $$|b(t,x) - b(s,y)| \leq C(|x-y|^\alpha + |t-s|^\gamma)$$

  with $\alpha > 1 - \frac{1}{2H}$ and $\gamma > H - \frac{1}{2}$;
- if $H < \frac{1}{2}$, then there exists a unique strong solution to (3) if b satisfies the linear growth condition

  $$|b(t,x)| \leq C(1+|x|) \tag{31}$$

  for every $t \in [0,T], x \in \mathbb{R}$;
- If $H = \frac{1}{2}$, see Remark 2.

Notice that the assumption (4) clearly implies the linear growth condition (31), thus we always have existence and uniqueness of the solution to (3) under (4). When $H > \frac{1}{2}$, we need to assume $\beta \geq 1 - \frac{1}{2H}$ in (4) and also

$$|b(t,x) - b(s,x)| \leq C|t-s|^\gamma$$

with $\gamma > H - \frac{1}{2}$ for every $s, t \in [0, T], x \in \mathbb{R}$.

In order to apply Theorems 1 and 2, we need to check (12) and (16). Assumption (16) clearly holds for every $H \in (0, 1)$. To check (12), we notice that this condition is obviously satisfied when $H = \frac{1}{2}$ and we discuss separately the cases $H > \frac{1}{2}$ and $H < \frac{1}{2}$.

The case $H > \frac{1}{2}$. We claim that $I_t^\epsilon = \int_{t-\epsilon}^t K_H(t,s)\,\mathrm{d}B_s$ is Gaussian with expectation equal to zero and variance bigger than $c_H \epsilon^{2H}$. We have:

$$\mathrm{Var}(I_t^\epsilon) = \mathbf{E}|\int_{t-\epsilon}^t K_H(t,s)\,\mathrm{d}B_s|^2 = \int_{t-\epsilon}^t |K_H(t,s)|^2\,\mathrm{d}s$$

and from formula (30), since for $H - \frac{1}{2} > 0$ we have $\left(\frac{u}{s}\right)^{H-\frac{1}{2}} \geq 1$, we can write

$$K_H(t,s) \geq c_H \int_s^t (u-s)^{H-\frac{3}{2}}\,\mathrm{d}u = C_H (t-s)^{H-\frac{1}{2}}.$$

Then

$$\int_{t-\epsilon}^t |K_H(t,s)|^2\,\mathrm{d}s \geq C_H^2 \int_{t-\epsilon}^t (t-s)^{2H-1}\,\mathrm{d}s = C_H \epsilon^{2H}$$

so (12) holds with $A = H$ and $c(\varepsilon, t) = C_H$.

The case $H < \frac{1}{2}$. From Proposition 5.12 in [5], we have $K_H(t,s) \geq C_H (\frac{t}{s})^{H-\frac{1}{2}}\ (t-s)^{H-\frac{1}{2}}$. Hence,

$$\int_{t-\epsilon}^t |K_H(t,s)|^2\,\mathrm{d}s \geq C_H t^{2H-1} \int_{t-\epsilon}^t s^{1-2H} (t-s)^{2H-1}\,\mathrm{d}s$$

$$\geq C_H t^{2H-1} (t-\epsilon)^{1-2H} \int_{t-\epsilon}^t (t-s)^{2H-1}\,\mathrm{d}s = C_H t^{2H-1} (t-\epsilon)^{1-2H} \epsilon^{2H}$$

$$= C_H t^{2H-1} (1 - \frac{\epsilon}{t})^{1-2H} \epsilon^{2H}.$$

Consequently, (12) holds with $A = H$ and $c(\varepsilon, t) = t^{2H-1}(1 - \frac{\epsilon}{t})^{1-2H}$, which is larger than a constant for $0 < \varepsilon < t$.

5.2. The Riemann–Liouville process

The Riemann-Liouville process is defined as

$$B_t = \int_0^t (t-s)^{H-\frac{1}{2}}\,\mathrm{d}W_s, \text{ for every } t \in [0, T] \tag{32}$$

with $H \in (0, 1)$. It shares many properties with the fBm (it is self-similar of index H, its paths are Hölder continuous of order $\delta \in (0, H)$), but it has no stationary increments. Notice that

$$\mathrm{Var}(I_t^\varepsilon) = \int_{t-\varepsilon}^t (t-s)^{2H-1}\,\mathrm{d}s = \frac{1}{2H} \varepsilon^{2H}$$

and it is well known that

$$\mathbf{E}\,|B_t - B_s|^2 \leq C|t-s|^{2H}.$$

Therefore assumptions (12) and (16) are fulfilled with $A = H$ and $c(t, \varepsilon) = 1$.

5.3. The Ornstein–Uhlenbeck process

The Ornstein-Uhlenbeck process $(B_t)_{t\in[0,T]}$ can be expressed as

$$B_t = \int_0^t e^{-(t-s)}\, dW_s.$$

It represents the unique solution to the SDE $dB_t = -B_t\, dt + dW_t$ with vanishing initial condition. It is well known that $\mathbf{E}|B_t - B_s|^2 \leq C|t-s|$, so (16) is satisfied with $H = \frac{1}{2}$. On the other hand, if I_t^ε is given by (11),

$$\mathrm{Var}(I_t^\varepsilon) = \mathrm{Var}\left(\int_{t-\varepsilon}^t e^{-(t-s)}\, dW_s\right) = \int_{t-\varepsilon}^t e^{-2(t-s)}\, ds = \frac{1}{2}(1 - e^{-2\varepsilon}) \geq c\varepsilon$$

so (12) holds with $A = \frac{1}{2}$ and $c(\varepsilon, t) = 1$.

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