Group theory/Number theory

Coset diagrams of the modular group and continued fractions

Diagrammes de classes du groupe modulaire et fractions continues

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ABSTRACT

The coset diagram for each orbit under the action of the modular group on \( \mathbb{Q}(\sqrt{n})^* = \mathbb{Q}(\sqrt{n}) \cup \{ \infty \} \) contains a circuit \( C_1 \). For any \( \alpha \in \mathbb{Q}(\sqrt{n}) \), the path leading to the circuit \( C_1 \) and the circuit itself are obtained through continued fractions in this paper. We show that the structure of the continued fractions of a reduced quadratic irrational element is weaved with the structure or type of the circuit. The three types of circuits of the action of \( V_4 \) on \( \mathbb{Q}(\sqrt{n})^* \) are also interconnected with the structure of continued fractions. The action of the modular group on \( \mathbb{Q}(\sqrt{5})^* \) is chosen specifically because a circuit of it is related to the ratio of the Fibonacci numbers being the solution to the continued fractions of the golden ratio.

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RÉSUMÉ

Le diagramme des classes de chaque orbite de l'action du groupe modulaire sur \( \mathbb{Q}(\sqrt{n})^* = \mathbb{Q}(\sqrt{n}) \cup \{ \infty \} \) contient un circuit \( C_1 \). Dans cette Note, pour tout \( \alpha \in \mathbb{Q}(\sqrt{n}) \), le chemin menant au circuit \( C_1 \) et le circuit lui-même sont décrits en termes de fractions continues. Nous montrons que la structure des fractions continues des nombres quadratiques irrationnels réduits est liée à la structure ou au type du circuit. Les trois types de circuits de l'action de \( V_4 \) sur \( \mathbb{Q}(\sqrt{5})^* \) sont également reliés à la structure des fractions continues. L'action du groupe modulaire sur \( \mathbb{Q}(\sqrt{5})^* \) est choisie précisément, car un de ses circuits est lié au fait que les rapports des nombres de Fibonacci sont les convergents de la fraction continue du nombre d'or.

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1. Introduction

If the upper half-plane model \( H \) of hyperbolic plane geometry is considered, which is a model of Lobachevsky plane \( \{ z = x + iy : x, y \in \mathbb{R} \text{ and } y > 0 \} \) and the motions in it preserve the orientation, then the group of all orientation-preserving isometries of \( H \) consists of all \( \text{M"obius transformations} \) of the form \( z \mapsto \frac{az + b}{cz + d} \), where \( a, b, c, \) and \( d \) are real numbers, and \( ad - bc = \pm 1 \).

From another point of view, the group \( \text{PSL}(2, \mathbb{R}) \) acts on the upper half-plane \( H \) according to the faithful (left-) action

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d},
\]

forming a group of isometries of the hyperbolic plane \( H \). The group, which comprises linear fractional transformations of \( H \) with integer coefficients, is a discrete group of motions and forms an important subgroup of \( \text{PSL}(2, \mathbb{R}) \). This group of transformations is isomorphic to the projective special linear group \( \text{PSL}(2, \mathbb{Z}) \), which is the quotient of the 2-dimensional special linear group over the integers by its centre \( \{ I, -I \} \). In other words, \( \text{PSL}(2, \mathbb{Z}) \) consists of all \( 2 \times 2 \) matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) where \( a, b, c, \) and \( d \) are integers, and \( ad - bc = 1 \), and the pairs of matrices \( A \) and \( -A \) are considered to be identical and the group operation is the usual multiplication of matrices.

The modular group \( \text{PSL}(2, \mathbb{Z}) \) has the finite presentation \( \langle x, y : x^2 = y^3 = 1 \rangle \), where \( x \) and \( y \) correspond to the linear fractional transformations \( z \mapsto \frac{1}{z} \) and \( z \mapsto -\frac{1}{z} \). A proof of this, using coset diagrams, is given in [6].

The concept of graphs was first introduced in 1878 by A. Cayley. A number of group theorists used Cayley’s diagrams to prove many important results on finitely generated groups. O. Schreier generalised the Cayley’s diagrams by introducing a graph whose vertices represent the cosets of any given subgroup.

In 1978, G. Higman proposed coset diagrams for the modular and extended modular groups. These are called coset diagrams because here the vertices are identifiable with the right cosets in a permutation group \( G \) of the stabiliser \( N \) of any point of the \( G \)-space \( \Omega \), so that an edge \( x_i \) joins the coset \( Ng \) to the coset \( Ngx_i \) for each element \( g \) of \( G \). Since \( \text{PSL}(2, \mathbb{Z}) \) has two generators, the edges associated with the involution \( x \) are represented by small edges without any orientation attached to them. In the case of \( y \), which has order 3, there is a need to distinguish \( y \) from \( y^{-1} \). The 3-cycles of \( y \) are therefore represented by small triangles, with the convention that \( y \) permutes their vertices counter-clockwise. The fixed points of \( x \) and \( y \), if they exist, are denoted by heavy dots. More details can be found in [11].

A quadratic irrational field is a field extension of degree 2 over \( \mathbb{Q} \) denoted by \( \mathbb{Q}(\sqrt{m}) \). If an element \( \alpha \in \mathbb{Q}(\sqrt{m}) \), then \( \alpha = a + b\sqrt{m} \), where \( a, b \in \mathbb{Q} \). The algebraic conjugate of \( \alpha \) is \( \bar{\alpha} = a - b\sqrt{m} \). The trace and norm of \( \alpha \) are \( \text{Tr}(\alpha) = \alpha + \bar{\alpha} \) and \( N(\alpha) = \alpha \bar{\alpha} \), respectively. Every \( \alpha \in \mathbb{Q}(\sqrt{m}) \) is the root of a monic polynomial of degree 2 with rational coefficients \( (x - \alpha)(x - \bar{\alpha}) = x^2 - \text{Tr}(\alpha)x + N(\alpha) \), so an element is an integer of \( \mathbb{Q}(\sqrt{m}) \) if \( \text{Tr}(\alpha) \) and \( N(\alpha) \) belong to \( \mathbb{Z} \). When \( n > 0 \), \( \mathbb{Q}(\sqrt{n}) \) is called a real, and when \( n < 0 \) an imaginary, quadratic field [5], [14].

Every real quadratic irrational number \( \alpha \) can be uniquely written as \( (a + \sqrt{m})/c \), where \( n \) is a non-square positive integer, and \( a, (a^2 - n)/c, \) and \( c \) are relatively prime integers. If \( \alpha \) and its algebraic conjugate \( \bar{\alpha} \) have positive signs, then \( \alpha \) is called a totally positive number; if they have negative signs, then \( \alpha \) is called a totally negative number, and if they have different signs, then \( \alpha \) is called an ambiguous number. A formula for obtaining the ambiguous numbers is provided in [7], but this approach does not seem to have any connection with the continued fraction of the element.

Let the modular group \( \Gamma = \text{PSL}(2, \mathbb{Z}) = \langle x, y : x^2 = y^3 = 1 \rangle \) act on the extended real quadratic irrational field \( \mathbb{Q}(\sqrt{m})^* = \mathbb{Q}(\sqrt{m}) \cup \{ \infty \} \) [10]. The ambiguous numbers in the coset diagram of the orbit \( \alpha \Gamma \) form a single closed path called a circuit.

A circuit of type \( (q_1, q_2, \ldots, q_m) \) means a sequence of positive integers \( q_i \) representing alternatively \( q_j \) triangles with one vertex inside the circuit and \( q_{j+1} \) triangles with one vertex outside the circuit for all \( j = 1, 2, \ldots, m \). Algebraically the circuit corresponds to the word \( w = (xy)^{q_1}(xy)^{q_2} \cdots (xy)^{q_{m-1}}(xy)^{q_m} \), which fixes the element \( \alpha \). This is represented in Fig. 1.

The coset diagram of an orbit of the action of \( \Gamma \) on \( \mathbb{Q}(\sqrt{m})^* \) has only one circuit; let \( c_i \) be the circuit and \( \Gamma c_i \) be its coset diagram. If the set

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**Fig. 1.** A circuit of the action of the modular group on \( \mathbb{Q}(\sqrt{m})^* \).
is the collection of all circuits, then \( \Gamma_{G(\mathbb{Q})} \) denotes the complete coset diagram of the action of \( G \) on \( \mathbb{Q}(\sqrt{n})^* \).

By L. Euler, every real number has a continued fraction \( \alpha = [q_1, q_2, \ldots] \) that is finite for rational numbers and infinite for irrational numbers. The irrationals whose continued fractions repeat after a certain stage such as \( \alpha = [q_1; q_2, \ldots, q_m, q_{m+1}, q_{m+2}, \ldots, q_{m+k}] \) are the quadratic irrational numbers. Associated with these quadratic irrationals are the matrices \(
\begin{bmatrix}
A_1 & A_{i-1} \\
B_1 & B_{i-1}
\end{bmatrix}
\)
where \( A_i = [q_1, q_2, \ldots, q_i] \) and \( B_i = [q_2, q_3, \ldots, q_i] \) are continuants of the \( i \)th convergent \( \frac{A_i}{B_i} \) (see [2]).

The conjunction of modular surfaces and continued fractions is not new (see [15]); their connection has been exploited in several directions by [1], [3] and [9], to name a few. In this study, the connection of coset diagrams of the action of \( G \) on \( \mathbb{Q}(\sqrt{n})^* \), as defined by the second author in [10], to continued fractions is explored. Some authors [8] have obtained two proper \( G \)-subsets of \( \mathbb{Q}(\sqrt{n})^* \) corresponding to each odd prime divisor of \( n \). R. Qureshi and T. Nakahara also presented the process to reach an ambiguous number from a totally negative or totally positive number through continued fractions in [13].

We show that the structure of circuits in the orbits of the action \( G \) on \( \mathbb{Q}(\sqrt{n})^* \) and the structure of the continued fractions of the \( \mathbb{Q}(\sqrt{n}) \) are intertwined. Thus, several results of continued fractions give information about the circuits and ambiguous numbers, and vice versa.

2. Continued fractions and coset diagrams of \( \text{PSL}(2, \mathbb{Z}) \)

We start by giving a few simple equalities.

**Lemma 1.** Let \( x, y \in G \), with \( x : z \to \frac{a}{z} \), \( y : z \to \frac{b}{z} \) and \( \alpha \in \mathbb{Q}(\sqrt{n}) \), then the following equalities hold:

1. \( \alpha (xy) = \frac{\alpha}{\frac{1}{\alpha + 1}} \cdot \frac{1}{\alpha + 1} = (\alpha^{-1})(xy)^{\frac{1}{\alpha + 1}}. \)
2. \( (\alpha)(xy) = \frac{\alpha}{\frac{1}{\alpha + 1}} \cdot \frac{1}{\alpha + 1} = (\alpha^{-1})(xy) \).
3. \( (\alpha)(xy) = \frac{\alpha}{\frac{1}{\alpha + 1}} \cdot \frac{1}{\alpha + 1} = (\alpha^{-1})(xy) \).

**Proof.** 1. As \( (\alpha)(xy) = \frac{\alpha}{\frac{1}{\alpha + 1}} \cdot \frac{1}{\alpha + 1} = (\alpha^{-1})(xy) \).
2. As \( (\alpha)(xy) = \frac{\alpha}{\frac{1}{\alpha + 1}} \cdot \frac{1}{\alpha + 1} = (\alpha^{-1})(xy) \).
3. By the previous two equalities, \( (\alpha)(xy) = \frac{\alpha}{\frac{1}{\alpha + 1}} \cdot \frac{1}{\alpha + 1} = (\alpha^{-1})(xy) \).

A Möbius transformation of the form \( (z) = \frac{az + b}{cz + d} \) may be represented in a matrix notation

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\begin{bmatrix}
z \\
1
\end{bmatrix}
= \frac{az + b}{cz + d}.
\]

So, \( x \) and \( y \) have matrix representations of the form:

\[
X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.
\]

The following theorem connects the structure of continued fraction of an element \( \alpha \in \mathbb{Q}(\sqrt{n}) \) to the structure of words with \( xy \) and \( xy^2 \) of \( G \), which will later be used for the coset diagrams.

**Theorem 1.** The continued fraction \( [q_1; q_2, \ldots, q_m] \) of \( \alpha \in \mathbb{Q}(\sqrt{n}) \) gives a path \((xy)q_1(xy)^q_2 \ldots (xy)^q_m \) from \( \alpha \) to another \( \alpha' \) in the coset diagram of the action of \( G \) on \( \mathbb{Q}(\sqrt{n})^* \).

**Proof.** Let \( x, y \in \text{PSL}(2, \mathbb{Z}) \) with matrix representations:

\[
X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.
\]

Then,
are the respective matrices of the transformations \((x y)^{q_i} = z + q_i\) and \((x y^2)^{q_j} = \frac{z}{q_j z + 1}\).

By [16], which goes back at least to [4], a result of [17] states that if
\[
\begin{bmatrix}
q_1 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
q_2 & 1 \\
1 & 0
\end{bmatrix}
\cdots
\begin{bmatrix}
q_m & 1 \\
1 & 0
\end{bmatrix}
= \begin{bmatrix}
A_m & A_{m-1} \\
B_m & B_{m-1}
\end{bmatrix}, \ m = 1, 2, 3,\
\]
then \(\frac{A_m}{B_m} = [q_1; q_2, \ldots, q_m]\), where \(\frac{A_m}{B_m}\) is the \(m\)th convergent of \(\alpha\) and \(\left|\begin{array}{cc}
A_m & A_{m-1} \\
B_m & B_{m-1}
\end{array}\right| = (-1)^m\).

It is observed that
\[
\begin{bmatrix}
q_1 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
q_j & 1 \\
1 & 0
\end{bmatrix} = \begin{bmatrix}
q_i q_j + 1 & q_i \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
q_j & 1
\end{bmatrix}.
\]

By equations (2.2) and (2.1), we get:
\[
\begin{bmatrix}
1 & q_i \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
1 & q_m
\end{bmatrix} = \begin{bmatrix}
1 & q_i \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
A_m & A_{m-1} \\
B_m & B_{m-1}
\end{bmatrix}.
\]

Since these matrices are powers of linear fractional transformations of \(x y\) and \(x y^2\), we have
\[
\begin{bmatrix}
1 & q_i \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
1 & q_m
\end{bmatrix} = (x y)^{q_i} (x y^2)^{q_j}.
\]

which implies that Eq. (2.3) can be written as:
\[
(x y)^{q_1} (x y^2)^{q_2} \cdots (x y^2)^{q_m} = \begin{bmatrix}
A_m & A_{m-1} \\
B_m & B_{m-1}
\end{bmatrix}.
\]

This sets up a correspondence between the words of powers of \(x y\) and \(x y^2\) of \(G\) and continued fractions. □

We state the following important theorem.

**Theorem 2.** If \(\alpha \in \mathbb{Q}(\sqrt{n})\), where \(n\) is a square-free positive integer, then for the continued fraction \([q_1; q_2, \ldots, q_m, q_{m+1}, q_{m+2}, \ldots, q_{m+d}]\) of \(\alpha\), the period of the continued fraction forms a circuit in the coset diagram of the action of \(G\) on \(\mathbb{Q}(\sqrt{n})^*\). The word \(w_\alpha = (x y^{q_1}) (x y^{q_2}) \cdots (x y^{q_{m-1}}) (x y)^{q_m}\), where \(m\) is even, leads \(\alpha\) to the reduced quadratic irrational number \(\alpha'\), and the word \(w_{C_\alpha} = (x y^{q_1}) (x y)^{q_{m-1}} \cdots (x y^{q_2}) (x y)^{q_1}\) of \(G\) and \(x y^{q_1}\) fixes \(\alpha'\).

**Proof.** By Eq. (2.3), for the continued fraction \([q_1; q_2, \ldots, q_m, q_{m+1}, q_{m+2}, \ldots, q_{m+d}]\) of \(\alpha,\)
\[
\begin{bmatrix}
1 & q_1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
1 & q_m
\end{bmatrix} = \begin{bmatrix}
A & 1 \\
B & 1
\end{bmatrix}
\]
which leads from \(\alpha'\) to \(\alpha\), that is, it is a path from \(\alpha'\) to \(\alpha\) in the coset diagram of the action of \(PSL(2, \mathbb{Z})\) on \(\mathbb{Q}(\sqrt{n})^*\). Denote this path by a word \(w = (x y^{q_1}) (x y)^{q_{m-1}} \cdots (x y^{q_2}) (x y)^{q_1}\) such that
\[
(\alpha') w = \alpha.
\]

Then \(\alpha' = (\alpha) w^{-1} = (\alpha) w_\alpha\), which gives the path from \(\alpha\) to \(\alpha'\) such that
\[
\alpha' = (\alpha) \left( (x y)^{q_1} (x y) \cdots (x y^{q_{m-1}}) (x y)^{q_m} \right).
\]

The periodic part of the continued fraction gives
\[
\begin{bmatrix}
1 & q_{m+1} \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
q_{m+2} & 1
\end{bmatrix} \cdots
\begin{bmatrix}
1 & 0 \\
q_{m+d} & 1
\end{bmatrix}
\begin{bmatrix}
\alpha' \\
1
\end{bmatrix} = \begin{bmatrix}
\alpha' \\
1
\end{bmatrix}.
\]

Denote this path by a word \(w_{C_\alpha} = (x y)^{q_{m+1}} (x y)^{q_{m+d-1}} \cdots (x y)^{q_{m+2}} (x y)^{q_{m+1}}\) such that
\[
(\alpha') w_{C_\alpha} = \alpha'.
\]

As \(w_{C_\alpha} \in G\) and fixes \(\alpha'\), by [10] such a word is a circuit \((q_{m+1}, q_{m+2}, \ldots, q_{m+d})\) in the respective coset diagram, where \(x\) is represented by \(-\) and the three cycles of \(y\) by \(\Delta\) permuted anticlockwise for the action of \(G\) on \(\mathbb{Q}(\sqrt{n})^*\) with \(q_{m+1}\)
triangles inside the circuit and \( q_{m+2} \) triangles outside the circuit, alternatively. So, the period of an element of \( \mathbb{Q}(\sqrt{n}) \) provides the only circuit residing in the coset diagram of the orbit to which \( \alpha \) belongs.

For the circuit, if \( d \) is even, then word \( w_{C_1} \) is associated with the matrix \( \begin{bmatrix} A_{m+d} & A_{m+d-1} \\ B_{m+d} & B_{m+d-1} \end{bmatrix} \), which has determinant 1. Hence, \( w_{C_1} \) belongs to \( G \). But if \( d \) is odd, the determinant of the associated matrix is \(-1\), which does not belong to \( G \). Taking \( w_{C_1} \) twice makes it even, that is,

\[
(\alpha') (x^2)^{qm+d} (xy)^{qm+d-1} \cdots (xy^2)^{qm} (xy)^{qm+d} \cdots (xy^2)^{q_{m+1}} (xy)^{qm} = \alpha'
\]

which yields a matrix with determinant 1, which thus belongs to \( G \). \( \square \)

**Corollary 1.** The uniquely represented reduced quadratic irrational numbers, whose continued fractions are cyclically not equivalent, reveal the circuits in the orbits of the action of \( G \) on \( \mathbb{Q}(\sqrt{n})^* \).

**Proof.** The uniquely represented reduced quadratic irrational numbers \( \alpha \) are elements that satisfy \( \gcd \left( a, \frac{d^2 - n}{c} \right) = 1 \), \( \alpha > 1 \) and \(-1 < \bar{\alpha} < 0 \). If two elements have cyclically equivalent continued fractions, either both belong in the same circuit or one of them belongs in the circuit of its algebraic conjugate. Hence, all the circuits of the action are obtained by the continued fractions of uniquely represented reduced quadratic irrational numbers, whose continued fractions are cyclically not equivalent. \( \square \)

We illustrate this by considering the following example.

**Example 1.** Let \( \alpha = \frac{-24 - \sqrt{15}}{15} \in \mathbb{Q}(\sqrt{15}) \). The continued fraction of \( \alpha \) is \([1; 5, 2, 3]\) which implies \( (\alpha) (y^2 x) (yx)^5 = \alpha' \), and \((\alpha') (x^2)^3 (xy)^2 = \alpha' \), where \( \alpha' \) is either \( \frac{3 + \sqrt{15}}{2} \) or its algebraic conjugate \( \frac{3 - \sqrt{15}}{2} \).

By the theorem (2), the continued fraction of an element of \( \mathbb{Q}(\sqrt{n}) \) is related to the path that leads from that element to an ambiguous number, and also to the type of circuit of the orbit to which the element belongs in the action of \( G \) on \( \mathbb{Q}(\sqrt{n})^* \). So, the structure of the continued fraction is interwoven with the structure or type of the circuit. Thus, obtaining all the reduced quadratic irrational numbers is equivalent to obtaining all the ambiguous numbers.

**Lemma 2.** If \( (\alpha') (x^2)^{q_0} (xy)^{q_{m-1}} \cdots (xy^2)^{q_2} (xy)^{q_1} = \alpha \), then, for \( m \) even,

\[
(\alpha) (y^2 x)^{q_1} (yx)^{q_2} \cdots (y^2 x)^{q_{m-1}} (yx)^{q_m} = \alpha',
\]

and, for \( m \) odd,

\[
(\alpha) (y^2 x)^{q_1} (yx)^{q_2} \cdots (yx)^{q_{m-1}} (y^2 x)^{q_m} = (\alpha')^{-1}.
\]

**Theorem 3.** For every distinct periodic part of the reduced quadratic irrational number in \( \mathbb{Q}(\sqrt{n}) \), there are two orbits of the action with the same type of circuit, and only one orbit if the circuit is of type \((q_{m+1}, q_{m+2}, \ldots, q_{m+d}, q_m, \ldots, q_{m+2}, q_{m+1})\).

**Proof.** Let \( \alpha \) and \( \bar{\alpha} \) be roots of the quadratic equation of the word \( w_2 \) in Eq. (2.6). This means that they belong to a circuit of type \((q_{m+1}, q_{m+2}, \ldots, q_{m+d})\). If \( \alpha \) and \( \bar{\alpha} \) belong in the same circuit, that is, the circuit is of type

\[
(q_{m+1}, q_{m+2}, \ldots, q_{m+d}, q_m, \ldots, q_{m+2}, q_{m+1})
\]

then there is only one orbit with circuit \((q_{m+1}, q_{m+2}, \ldots, q_{m+d})\); otherwise there are two orbits with circuit

\[
(q_{m+1}, q_{m+2}, \ldots, q_{m+d})
\]

one orbit containing the circuit of \( \alpha \) and the other containing the conjugates of the aforementioned orbit. \( \square \)

**Corollary 2.** There are \( 4 \sum_{i=1}^{m} q_i \) ambiguous numbers for the circuit of type \((q_1, q_2, \ldots, q_m, q_m, \ldots, q_2, q_1)\) and, for any other circuit of type \((q_1, q_2, \ldots, q_{2n})\), there are two orbits, so that there are \( 4 \sum_{i=1}^{2n} q_i \) of them.
So, obtaining all the circuits of the action of $PSL(2, Z)$ on $\mathbb{Q}(\sqrt{m})^*$ gives all the ambiguous numbers of the action.

In the following table, we list a few relations between the periodic continued fractions of a real quadratic irrational number and the path and the circuit in the coset diagram of the action:

<table>
<thead>
<tr>
<th>Continued fraction</th>
<th>Path</th>
<th>Circuit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[\alpha]$</td>
<td>$\pm \sqrt{\alpha^2 + 1}$</td>
<td>$(\alpha') (xy^2)^{\alpha} (xy) = \alpha'$</td>
</tr>
<tr>
<td>$[1; \alpha]$</td>
<td>$\pm \pm \sqrt{1 + \alpha^2}$</td>
<td>$(\alpha') (xy^2)^{\alpha} (xy) = \alpha'$</td>
</tr>
<tr>
<td>$\alpha \frac{a}{b}$</td>
<td>$\pm \pm \pm \sqrt{ab(ab + 4)}$</td>
<td>$(\alpha') (xy^2)^{\alpha} (xy) = \alpha'$</td>
</tr>
<tr>
<td>$\alpha \frac{a}{b}$</td>
<td>$\pm \pm \pm \pm \sqrt{b_1 \ldots b_m}$</td>
<td>$(\alpha') (xy^2)^{\alpha} (xy) = \alpha'$</td>
</tr>
</tbody>
</table>

Finally, we state that the uniquely represented reduced quadratic irrational numbers reveal the circuits of the orbits of the action of $PSL(2, Z)$ on $\mathbb{Q}(\sqrt{m})^*$.

### 3. Continued fractions of three types

By [12], in the coset diagram for the action of $PSL(2, Z)$ on $\mathbb{Q}(\sqrt{m})^*$, a point $\alpha$ is on a circuit if and only if it is fixed by some element $g = (xy)^{q_1} (xy)^{q_2} \ldots (xy)^{q_m}$, which means that the circuits are permuted by any permutation $g$ of $\mathbb{Q}(\sqrt{m})^*$ that normalises the set $\{xy, xy^{-1}\}$. Two such permutations are $s: z \rightarrow z$ and $t: z \rightarrow 1/z$. Since $s^2 = t^2 = (st)^2 = 1$, a 4-permutation group permutes the circuits.

It is given that for the action of $V_4 = \{s, t: s^2 = t^2 = (st)^2 = 1\}$, under $s$ the circuits that contain $\alpha$ and its image $\bar{\alpha}$ are of type $(q_1, q_2, \ldots, q_m, q_m, \ldots, q_1)$, under $t$ the circuits that contain $\alpha$ and its image $1/\alpha$ are of type $(q_1, q_2, \ldots, q_m, q_1, q_2, \ldots, q_k)$, and under $st$ the circuits that contain $\alpha$ and its image $1/\bar{\alpha}$ are of type $(q_1, q_2, \ldots, q_m, q_m+1, q_m, \ldots, q_2)$.

Thus, there are three types of circuits in the orbits due to the action of $V_4$ on $\mathbb{Q}(\sqrt{m})^*$. We explain these circuits with respect to continued fractions.

1. For the continued fractions $[q_1; q_2, \ldots, q_m]$ of the reduced quadratic $\alpha$, it is known that

$$\alpha = \frac{A_m \alpha + A_{m-1}}{B_m \alpha + B_{m-1}}$$

and that the Eq. (2.3) and reverse of the cycle $[q_m; \ldots, q_1]$ yields

$$\beta = \frac{A_m \beta + B_m}{A_{m-1} \beta + B_{m-1}}$$

If $\beta = -1/\alpha$, then Eq. (3.2) converts into equation (3.1), which means that $-1/\beta$ is a root of this equation. Since $\alpha$ and $\beta$ are positive, so the other root of Eq. (3.1) is $\bar{\alpha} = -1/\alpha$.

We observe that, for the circuit of type $(q_1, q_2, \ldots, q_m, q_m, \ldots, q_1)$ in the coset diagram, the reverse circuit does not change, which means that, under the transformation $s$, $\alpha$ and $\bar{\alpha} = -1/\alpha$ lie in the same circuit.

2. For a cycle of odd length $[q_1; q_2, \ldots, q_{m+1}]$, the matrix $\begin{bmatrix} A_m & A_{m-1} \\ B_m & B_{m-1} \end{bmatrix}$ has determinant $-1$. To obtain a matrix with determinant 1, continue the series of convergents until an even length is reached. Take an odd cycle

$$(\alpha) (xy^2)^{q_1} (xy)^{q_2} (xy)^{q_3} = \alpha^{-1},$$

by lemma (1) we get $(\alpha^{-1}) (xy)^{q_1} (xy)^{q_2} (xy)^{q_3} = \alpha$. Conjoining both of the equations, we get

$$(\alpha) (xy^2)^{q_1} (xy)^{q_2} (xy)^{q_3} = \alpha.$$

Hence if the continued fraction is of type $[q_1; q_2, \ldots, q_{m+1}, q_1, q_2, \ldots, q_{m+1}]$ then $\alpha$ and $\alpha^{-1}$ belong to the same circuit where the position of $\alpha^{-1}$ is determined by equation (3.3).

3. Combining the above two facts, if the cycle is of type $[q_1; q_2, \ldots, q_m, q_{m+1}, q_m, \ldots, q_2]$, then $\alpha$ and $1/\bar{\alpha}$ belong to the same orbit.
4. Circuits of the action of \( PSL(2, \mathbb{Z}) \) on \( \mathbb{Q}(\sqrt{5})^* \)

In this section, we determine all the circuits of the action of \( G \) on \( \mathbb{Q}(\sqrt{5})^* \) and investigate their structure because a circuit of this action is related to the ratio of the Fibonacci numbers that are the solutions to the continued fractions of the golden ratio.

We denote the fact \((\infty)^{0}(xy)^{1}=\infty+1=\infty\) as a circuit of the type \((1,0)\). This is represented in Fig. 2.

**Theorem 4.** The action of \( G \) on \( \mathbb{Q}(\sqrt{5})^* \) has only three orbits with circuits \((1,0),(1,1)\) and \((4,4)\) up to the unique representation of the quadratic irrational numbers.

**Proof.** Let \( \alpha_{ij}=\frac{a_{i}+\sqrt{5}}{c_{j}} \) be a uniquely represented reduced quadratic irrational number of \( \mathbb{Q}(\sqrt{5}) \), with \( \gcd\left(a_{i}, \frac{a_{i}^{2}-5}{c_{j}}, c_{j}\right) = 1 \) and \( a_{i}, c_{j} \in \mathbb{Z}^+ \). By the properties of the reduced quadratic irrational numbers \( \alpha_{ij} \geq 1 \) and \( -1 < \alpha_{ij} < 0 \), we have

\[
c_{j} < a_{i} + \sqrt{5} \quad \text{and} \quad \sqrt{5} - a_{i} < c_{j}.
\]

The inequality \( a_{i} < \sqrt{5} \) implies two possible values, 1 and 2, of \( a_{i} \). For \( a_{i} = 1 \), \( c_{j} \) has two possible values 2 and 3, and for \( a_{i} = 2 \), \( c_{j} \) has four possible values 1, 2, 3, and 4 implying 6 reduced quadratic irrational numbers, so that:

<table>
<thead>
<tr>
<th>( a_{i} )</th>
<th>( c_{j} )</th>
<th>( \alpha_{ij} )</th>
<th>continued fractions</th>
<th>circuits</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>(1+\sqrt{5}/2)</td>
<td>[1; 1, 1, 2, 2, 2]</td>
<td>(1, 1)</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>(1+\sqrt{5}/3)</td>
<td>[1; 1, 2, 1, 2, 2]</td>
<td>(2, 2, 1, 1, 2, 1, 2)</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>(2+\sqrt{5}/2)</td>
<td>[2; 8]</td>
<td>(4, 4)</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>(2+\sqrt{5}/2)</td>
<td>[1; 2, 2, 1, 12]</td>
<td>(2, 8)</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>(2+\sqrt{5}/4)</td>
<td>[1; 16]</td>
<td>(2, 2, 1, 1, 2, 2)</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>(2+\sqrt{5}/4)</td>
<td>[1; 16]</td>
<td>(1, 16)</td>
</tr>
</tbody>
</table>

As the two reduced real quadratic irrational numbers \( \frac{1+\sqrt{5}}{2} \) and \( \frac{2+\sqrt{5}}{2} \) have continued fractions that are cyclically equivalent, they belong to the same orbit. Since the circuits \((1,1)\) and \((4,4)\) are of type \((q_{m+1}, \ldots q_{m+d}, q_{m+d}, \ldots, q_{m+1})\), by theorem \((3)\) only one orbit of the form exists.

Recall that every real quadratic irrational number can be written uniquely as \((a_{i} + \sqrt{n})/c_{j}\), \( n \) is a non-square positive integer, and \( a_{i}, b_{ij} \) and \( c_{j} \) are relatively prime integers, where \( b_{ij} = (a_{i}^2 - n)/c_{j} \):

(1) for \( a = 1, c = 2, \) and \( b = \frac{1^2-5}{2} = -2, \) \( \gcd(a, b, c) = 1 \), implies that \( \frac{1+\sqrt{5}}{2} \) belongs to the circuit \((1, 1)\) of \( \mathbb{Q}(\sqrt{5}) \);  
(2) \( \frac{1+\sqrt{5}}{2} \) (or \( \frac{2+\sqrt{5}}{2} \)) \( \in (2, 2, 1, 12, 1, 2) \). For \( a = 1 \) or \( 2, c = 3 \) and \( b = \frac{1^2-5}{3} = -\frac{1}{3} \) and \( \frac{2^2-5}{3} = -\frac{1}{3} \), \( \gcd(a, b, c) = \frac{1}{3} \), Converting \( \frac{1+\sqrt{5}}{3} \) in \( \frac{1+\sqrt{5}}{\sqrt{45}} \), which belongs to \( \mathbb{Q}(\sqrt{45}) \), one has that \( \gcd(3, \frac{4-20}{4}, \frac{4}{4}) = 1 \);  
(3) \( 2 + \sqrt{5} \) belongs to the circuit \((4,4)\) of \( \mathbb{Q}(\sqrt{5}) \), one has that \( \gcd(2, \frac{2^2-5}{1}, 1) = 1 \);  
(4) \( \frac{2+\sqrt{5}}{2} \) belongs to the circuit \((2, 8)\), so that \( \gcd(2, \frac{2^2-5}{2}, 2) = \frac{1}{2} \neq 1 \), Converting \( \frac{2+\sqrt{5}}{2} \) into \( \frac{4+\sqrt{20}}{4} \), which belongs to \( \mathbb{Q}(\sqrt{20}) \), one has that \( \gcd(4, \frac{4^2-20}{4}, 4) = 1 \);
(5) \(\frac{2+\sqrt{5}}{4} \in (1, 16)\), so that \(\gcd(2, \frac{2^2-5}{4}, 4) = \frac{1}{4} \neq 1\). Converting \(\frac{2+\sqrt{5}}{4}\) into \(\frac{8+\sqrt{80}}{16}\), which belongs to \(\mathbb{Q}(\sqrt{80})\), one has that \(\gcd(8, \frac{8^2-80}{16}, 16) = 1\).

Out of six circuits, only two satisfy the condition of unique representation. Hence, the action has only three orbits containing the circuit (1, 0) (Fig. 3), the circuit (1, 1) (Fig. 4) and the circuit (4, 4) (Fig. 5) unique up to the representation of the quadratic irrational numbers, and gives 20 ambiguous numbers of the action, whereas the rest of the circuits belong to the orbits of the action of \(G\) on \(\mathbb{Q}(s\sqrt{5})\) for the positive integers \(s = 2, 3, 4\). \(\square\)

**Conclusion.** By theorem (2), the structure of the continued fraction of an element of \(\mathbb{Q}(\sqrt{n})\) is interwoven with the structure or the type of the circuit. Thus, obtaining all the reduced quadratic irrational numbers is equivalent to obtaining all the circuits and thus all the ambiguous numbers. The continued fraction of an element \(a_{ij}\) of \(\mathbb{Q}(\sqrt{n})\) gives the path that leads to the ambiguous number and the
period of $\alpha_{ij}$ gives the circuit of the orbit to which $\alpha_{ij}$ belongs. Hence, the uniquely represented reduced quadratic irrational numbers reveal the circuits of the orbits of the action of $\text{PSL}(2, \mathbb{Z})$ on $\mathbb{Q}(\sqrt{n})^*$. 

References