Geometry/Differential geometry

A centro-projective inequality

Une inégalité centro-projective

Constantin Vernicos\textsuperscript{a}, Deane Yang\textsuperscript{b}

\textsuperscript{a} IMAG, Université de Montpellier, case courrier 051, place Eugène-Bataillon, 34395 Montpellier cedex, France
\textsuperscript{b} Department of Mathematics, Tandon School of Engineering, New York University, Six Metrotech Center, Brooklyn NY 11201, USA

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\section*{A B S T R A C T}
We give a new integral formula for the centro-projective area of a convex body, which was first defined by Berck–Bernig–Vernicos. We then use the formula to prove that it is bounded from above by the centro-projective area of an ellipsoid and that equality occurs if and only if the convex set is an ellipsoid.

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\section*{0. Introduction and statement of results}

Let $V$ be an $n$-dimensional vector space with origin $o$. Given a convex body $K$ containing $o$ in the interior, we define the function $a : \partial K \to (0, \infty)$ such that, for each $p \in \partial K$, $-a(p)p \in \partial K$. The letter $a$ stands for \textit{antipodal}. Given a Euclidean scalar product $\langle \cdot, \cdot \rangle$ on $V$, let $k(p)$ be the Gauss curvature and $v_{K}(p)$ the outer unit normal at each $p \in \partial K$, whenever they are well and uniquely defined (which, by A.D. Alexandroff [1], holds almost everywhere).

\textbf{Definition 1.} The \textit{centro-projective} area of $K$ is

$$C_0(K) := \int_{\partial K} \frac{\sqrt{k}}{\langle v_{K}(p), p \rangle^{n-1}} \left( \frac{2a}{1+a} \right)^{\frac{n-1}{2}} dA(p),$$

where $dA$ is the $(n-1)$-dimensional Hausdorff measure on $\partial K$. 

\textit{E-mail addresses:} Constantin.Vernicos@umontpellier.fr (C. Vernicos), deane.yang@nyu.edu (D. Yang). 

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As shown in [2], this does not depend on the choice of the scalar product. In fact, the centro-projective area is invariant under projective transformations fixing the origin. It is also upper semi-continuous with respect to the Hausdorff topology.

It is worth comparing the definitions of centro-affine surface area with those of centro-projective area. The two are similar, except that the latter has an additional factor (containing the function α). A reader familiar with the theory of valuations will recognize that centro-projective area is not a valuation, but is in a sense the closest possible projective analogue of a valuation. In particular, if the body is origin-symmetric, then centro-projective area equals centro-affine surface area and therefore is a valuation on the space of origin-symmetric convex bodies.

We will prove the following centro-projective inequality.

**Theorem 2.** Let $K$ be a convex body containing the origin in its interior, then one has

$$C_0(K) \leq C_0(B),$$

where $B \subset \mathbb{R}^n$ is the standard unit ball. Equality holds if and only if $K$ is an ellipsoid that contains the origin in its interior, but is not necessarily centered at the origin.

Let us point out that $C_0(B)$ does not depend on the particular choice of the origin inside the ellipsoid $B$. Indeed, if $p, q$ are points inside $B$, then there is a projective map $g$ such that $g(B) = B$ and $g(p) = q$, and therefore $C_0(B - p) = C_0(B - q)$. In other words, the projective group acts transitively on any fixed ellipsoid $B$.

### 1. Preliminaries

We recall here some basic definitions in convexity used in our paper. More details can be found, for example, in the books by Gruber [3], Schneider [12], or Thompson [13].

A subset $K \subset V$ is convex if the line segment joining any two points $x, y \in K$ also lies in $K$.

A non-empty compact convex set $K \subset V$ is uniquely determined by its support function denoted here by $h_K : V^* \to \mathbb{R}$ and defined by

$$h_K(\xi) = \max_{x \in K} \langle \xi, x \rangle.$$  

Indeed, we have then

$$K = \{ x \ | \ \langle \xi, x \rangle \leq h_K(\xi) \ \text{for all} \ \xi \in V^* \}. \tag{2}$$

Note that $h_K$ is a positively homogeneous function of degree 1.

If $K$ contains the origin $o$ in its interior, we define its polar body $K^* \subset V^*$ with respect to the origin by

$$K^* := \{ \xi \ | \ \langle \xi, x \rangle \leq 1 \ \text{for all} \ x \in K \}. \tag{3}$$

One can also show that $K^* = \{ \xi \ | \ h_K(\xi) \leq 1 \}$. We can also define for each point $x \neq o$ in $K$ the positive number $\rho_K(x)$ such that $\rho_K(x) x \in \partial K$. The function $\rho_K$ is called the radial function and satisfies

$$\rho_K(x) = \frac{1}{h_K^*(x)}. \tag{4}$$

The antipodal function $a$ defined in the introduction is given by

$$a(p) = \rho_K(-p) \ \text{for all} \ p \in \partial K. \tag{5}$$

For convex sets $K$ and $L$ in $V$, the Minkowski sum $K + L$ is the convex set defined by

$$K + L := \{ x + y \ | \ x \in K \ \text{and} \ y \in L \}. \tag{6}$$

and, for $\alpha \in \mathbb{R}$, one can define the convex set $\alpha K$ by

$$\alpha K := \{ \alpha x \ | \ x \in K \}.$$  

Recall that $h_{K+L} = h_K + h_L$ and $h_{\alpha K} = \alpha h_K$ if $\alpha \geq 0$.

From now on, we will fix a Euclidean scalar product on $V$. We denote by $S^{n-1}$ the corresponding unit sphere in $V$ and by $dm$ the induced volume form on $V$ and $V^*$. We also will use the following notation: For an integrable homogeneous function of degree $-n$, $f : V \setminus \{ 0 \} \to \mathbb{R}$,

$$\int f \ dm := \int f(\theta) \ d\theta, \tag{7}$$

where $d\theta$ is the standard spherical measure on $S^{n-1}$. The value of this integral depends only on the volume measure $dm$ and is otherwise independent of the Euclidean scalar product chosen (see [14] for details).
2. Proof of Theorem 2

Let $K \in V$ be a bounded open convex body containing the origin and let $K^* \subset V^*$ be its polar with respect to the origin. Notice that for any point $p$ on the boundary $\partial K$ at which there exists a unique outer unit normal $v_K(p) \in V^*$ to $K$, the following holds:

$$h_K(v_K(p)) = \langle v_K(p), p \rangle.$$ \hfill (8)

Recall that the curvature function of $K$, $f_K: V \setminus \{0\} \to \mathbb{R}^+$ is defined as follows: for each $\theta \in S^{n-1}$ where $h_K$ is twice differentiable, the curvature function $f_K(\theta)$ is the sum of the determinants of the principal $(n-1)$-minors of the Hessian of $h_K$ (viewed as a function on $V^* \setminus \{0\}$). It is then extended as a function homogeneous of degree $-n - 1$. Recall that, for each $\theta \in S^{n-1}$ where the radial function $\rho_K$ is twice differentiable and the Gauss curvature $\kappa(p)$ is positive, where $p = \rho_K(\theta) \partial \in \partial K$,

$$f_K(v_K(p)) = \frac{1}{\kappa_K(p)}.$$

The volume of $K$ is given by

$$V(K) = \frac{1}{n} \int h_K^n \, dm = \frac{1}{n} \int \frac{1}{h_K^{n-1}} \, dm$$

and the affine surface area of $K$ is defined as

$$S(K) = \int f_K^{n-1} \, dm.$$

See Schneider’s book [12] for a more detailed discussion of affine surface area, which was defined by Blaschke for smooth convex bodies. The definition above, valid for all convex bodies, is due to Leichtweiss [6]. Lutwak [8] gave a different but equivalent definition. Also, see Santalo [11], Hug [4,5].

The following is straightforward if the boundary $\partial K$ is $C^2$ and has strictly positive Gauss curvature. The general case is due to Hug [5].

**Lemma 3.** For continuous function $\psi: \partial K \to \mathbb{R}$,

$$\int_{\partial K} \psi(p) \left( \frac{\kappa_K(p)}{\langle v_K(p), p \rangle^{n-1}} \right)^{1/2} \, dA(p) = \int_{S^{n-1}} \psi(\rho_K(\theta)) \left( \frac{f_K^*(\theta)}{h_K^{n-1}(\theta)} \right)^{1/2} \, d\theta.$$ \hfill (9)

**Proof.** By Theorem 3.2 and Eq. (1) in [5], Hug established that

$$\int_{\partial K} \left( \frac{\kappa_K(p)}{\langle v_K(p), p \rangle^{n-1}} \right)^{1/2} \, dA(p) = \int_{S^{n-1}} \left( \frac{f_K^*(\theta)}{h_K^{n-1}(\theta)} \right)^{1/2} \, d\theta.$$ \hfill (10)

However, in the proof of Theorem 3.2, Hug in fact proves that the two measures are equal via the bilipschitz map $\theta \mapsto \rho_K(\theta) \theta$. \hfill \square

Generalizations of Hug’s result can also be found in Ludwig [7]. In particular, using Theorem 4 and 5 in [7] applied with $\phi(t) = t^{1/2}$ one gets Eq. (10).

The new formula for the centro-projective area of a convex body $K$ is given by the following lemma.

**Lemma 4.** The centro-projective area of $K$ is equal to

$$C_o(K) = \int \left( \frac{2}{h_{K^*} + h_{-K^*}} \right)^{n-1} \rho_K^{1/2} \, dm.$$ \hfill (11)

**Proof.** If $a$ is the antipodal function defined by (5), then by Eqs. (4) and (5), we have

$$a(p) = \frac{1}{h_{K^*}(-p)},$$

and therefore, for each $\theta \in S^{n-1}$.
\[ a(\rho_K(\theta)\theta) = \frac{1}{h_{K^*}(\rho_K(\theta)\theta)} = \frac{h_{K^*}(\theta)}{h_{K^*}(\theta)} \]

Hence,
\[
\frac{2a(p(\theta))}{1 + a(p(\theta))} = \frac{2}{h_{K^*}(\theta)} - \frac{1}{h_{K^*}(\theta)} = \frac{2h_{K^*}(\theta)}{h_{K^*}(\theta) + h_{K^*}(\theta)}.
\]

The lemma now follows from Lemma 3 by setting
\[
\psi = \left( \frac{2a}{1 + a} \right)^{\frac{n+1}{2}}. \quad \Box
\]

To prove the theorem, we first apply the Hölder inequality to \( C_o(K) \):
\[
C_o(K) \leq \left( \int \left( \frac{2}{h_{K^*}(x) + h_{K^*}(-x)} \right)^n \ dm(x) \right)^{\frac{n-1}{n}} \cdot \left( \int \frac{n}{h_{K^*}^\infty} \ dm \right)^{\frac{n+1}{n}},
\]

where
\[
\pi(K) = \left[ \frac{1}{2}(K^* + (-K^*)) \right]^*.
\]

By the affine isoperimetric inequality (see, for example, [9] or [10]),
\[
S(K^*)^{n+1} \leq n^{(n+1)} V(K^*)^{n-1} V(B_n)^2,
\]
where equality holds if and only if \( K \) is an ellipsoid centered at the origin. Applying this to inequality (13) gives
\[
C_o(K) \leq n \cdot \left( V(\pi(K)) \cdot V(K^*) \right)^{\frac{n-1}{n}} V(B_n)\pi.
\]

Next, we use the Blaschke–Santaló inequality, which states that, for any convex body \( C \subset V \) that is symmetric with respect to the origin,
\[
V(C) \times V(C^*) \leq V(B_n)^2.
\]

Again, equality holds if and only if \( C \) is an ellipsoid. Setting
\[
C = \frac{1}{2} K^* + \frac{1}{2} (-K^*) \text{ and } C^* = \pi(K)
\]
the Blaschke–Santaló inequality and (15) imply
\[
C_o(K) \leq nV(B_n) \cdot \left( \frac{V(K^*)}{V(\frac{1}{2}K^* + \frac{1}{2}(-K^*))} \right)^{\frac{n-1}{n}}.
\]

The theorem now follows by
\[
V\left( \frac{1}{2} K^* + \frac{1}{2} (-K^*) \right)^{1/n} \geq \frac{1}{2} V(K^*)^{1/n} + \frac{1}{2} V(-K^*)^{1/n} = V(K^*)^{1/n},
\]
which follows from the Brunn–Minkowski inequality, and the identity \( C_o(B_n) = nV(B_n) \).

Let us stress out that the equality conditions of the Brunn–Minkowski inequality, the Blaschke–Santaló inequality, and the affine isoperimetric inequality imply that equality holds in Theorem 2 if and only if \( K \) is an ellipsoid that contains the origin in its interior, but is not necessarily centered at the origin.

3. Centro-projective invariance

We remark that the invariance of (11) under centro-projective transformations of \( K \) is easy to show. It suffices to show that it is invariant under linear transformations of \( K \) and translations of \( K^* \). The invariance of (11) under linear transformations of \( K \) is established in [14]. The invariance of \( f_{K^*} \) and \( h_{K^*} \) under translations of \( K^* \) follows directly from their definitions.
4. Application in Hilbert geometries

A Hilbert geometry is a metric space structure defined as follows on a proper open convex domain of a finite-dimensional affine space. By proper, we mean that the domain does not contain any line. The distance between two points in the domain is defined using cross-ratios in the same way one constructs the projective model of the hyperbolic space on a Euclidean ball (see, for example, [2]). Such a metric is called a Hilbert metric. The Hausdorff measure associated with that metric is called a Busemann measure.

Given an open bounded convex domain \( K \subset V \) and a point \( p \in K \), let \( V_{K,p}(r) \) denote the Busemann measure of the metric ball of radius \( r \) centered at \( p \). This defines, for each pointed convex domain \( (K, p) \) of \( V \), a function \( V_{K,p} : \mathbb{R}^+ \to \mathbb{R}^+ \).

Since the Busemann measure is defined in terms of the Hilbert metric, which in turns is defined using the cross-ratios, the function \( (K, p) \to V_{K,p} \) is a projective invariant of \( K \).

One can therefore ask two questions:

- Is it true that for any pointed convex domain \( (K, p) \) and \( r > 0 \) one has
  \[
  V_{K,p}(r) \leq V_{B_n,0}(r) \tag{18}
  \]

- Is the map \( (K, p) \to V_{K,p} \) injective? That is, if \( (K, p) \) and \( (K', p') \) are pointed convex sets such that \( V_{K,p} = V_{K',p'} \), does there exist a projective transformation \( g \) such that
  \[
  (g(K), g(p)) = (K', p') \tag{18}
  \]

A partial answer can be given if we assume the domain \( K \) to have regularity \( C^{1,1} \). Geometrically, this means that there exists a ball of some fixed radius that can roll inside \( K \) and touch every point on the boundary. It was proved in [2] that, for any convex domain \( K \),

\[
\lim_{r \to +\infty} \frac{V_{K,p}(r)}{V_{B_n,0}(r)} = \frac{C_0(K-p)}{C_0(B_n)}.
\]

Theorem 2 shows that this limit is strictly smaller that 1, when \( K \) is not an ellipsoid. In particular, for any \( p \in K \), there exists \( r_{K,p} > 0 \) such that, for all \( r > r_{K,p} \), the inequality (18) holds and is strict.

References


