C. R. Acad. Sci. Paris, Ser. I 357 (2019) 773–777

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Complex analysis

# Generalized directional Lelong number of a positive plurisubharmonic current

## *Nombre de Lelong directionnel généralisé d'un courant positif pluri-sous-harmonique*

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ARTICLE INFO

*Article history:*
Received 5 May 2019
Accepted after revision 1 August 2019
Available online 30 September 2019

Presented by the Editorial Board

ABSTRACT

Let $T$ be a positive plurisubharmonic (psh for short) current of bidegree $(k,k)$ on a neighborhood $\Omega$ of 0 in $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^m$ $(n = N - m \geqslant k)$, $B$ be a Borel subset of $L := \{0\} \times \mathbb{C}^m$ such that $B \Subset \Omega$. Taking $(z,t) \in \mathbb{C}^n \times \mathbb{C}^m$, we define a $C^2$ positive semi-exhaustive psh function on $\Omega$, $(z,t) \mapsto \varphi(z)$, such that $\log\varphi$ is also psh on the open set $\{\varphi > 0\}$ and consider $(z,t) \mapsto v(t)$ a continuous semi-exhaustive psh function on $\Omega$. This paper aims to prove that $T$ admits a generalized directional Lelong number along $L$ with respect to the functions $\varphi$ and $v$. Moreover, we give a theorem on the existence of a positive psh function $f$ on $L$, such that the Lelong number of $T$ is given by $f$. This theorem generalizes results studied by Alessandrini–Bassanelli and Toujani.



RÉSUMÉ

Soit $T$ un courant positif pluri-sous-harmonique $(psh)$ de bidegré $(k,k)$ dans un voisinage $\Omega$ de 0 dans $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^m$ $(n = N - m \geqslant k)$, et $B$ un borélien de $L := \{0\} \times \mathbb{C}^m$ tel que $B \Subset \Omega$. Étant donné $(z,t)$ dans $\mathbb{C}^n \times \mathbb{C}^m$, on définit une fonction de classe $C^2$ positive psh et semi-exhaustive sur $\Omega$, $(z,t) \mapsto \varphi(z)$, telle que $\log\varphi$ soit aussi psh sur l'ouvert $\{\varphi > 0\}$, et on considère une fonction $(z,t) \mapsto v(t)$ psh continue et semi-exhaustive sur $\Omega$. Dans cette note, on prouve que $T$ admet un nombre de Lelong directionnel généralisé relativement à $\varphi$ et $v$ le long de $L$ ; de plus, on prouve un théorème sur l'existence d'une fonction $f$ psh positive sur $L$ telle que le nombre de Lelong de $T$ soit donné par $f$. Ce théorème généralise des résultats étudiés par Alessandrini–Bassanelli et Toujani.



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https://doi.org/10.1016/j.crma.2019.07.009
1631-073X/

**Version française abrégée**

Il est bien connu qu'un courant pluri-subharmonique positif admet un nombre de Lelong (voir [6]) ; rappelons qu'un courant $T$ est dit pluri-subharmonique si $dd^cT \geqslant 0$. Demailly a défini dans [3] un nombre de Lelong généralisé lorsque $T$ est un courant positif d-fermé (un courant $T$ est d-fermé si $dT = 0$). De même, lorsque $T$ est un courant pluri-subharmonique positif, Alessandrini et Bassanelli ont prouvé dans [1] l'existence d'un nombre directionnel de Lelong, relativement aux fonctions $\varphi(z) = |z|^2$ et $v(t) = |t|^2$. Toujani a défini dans [7] le nombre de Lelong directionnel généralisé lorsque $\varphi$ et $v$ sont des fonctions psh semi-exhaustives positives de classe $C^2$ sur $\Omega$. Cette note traite le problème de la généralisation des résultats de Toujani [7] lorsque $v$ est une fonction psh supposée seulement continue et non plus de classe $C^2$.

Nous utilisons maintenant les notations suivantes : on note $(z,t) \in \mathbb{C}^n \times \mathbb{C}^m$ et on considère une fonction positive de classe $C^2$ psh sur $\Omega$, $(z,t) \mapsto \varphi(z)$, telle que $\log\varphi$ soit aussi psh sur l'ouvert $\{\varphi > 0\}$. Nous supposons $\varphi : \Omega \cap (\mathbb{C}^n \times \{0\}) \to [-\infty, +\infty[$, semi-exhaustive, c'est-à-dire qu'il existe un nombre réel $R$ tel que, pour tout $c \in ]-\infty, R[$, on ait $\{z,\ \varphi(z) < c\} \Subset \Omega \cap (\mathbb{C}^n \times \{0\})$ ; de la même manière, soit $(z,t) \mapsto v(t)$ une fonction semi-exhaustive continue psh sur l'ouvert $\Omega$. Pour $r < R$ et $r_1 < r_2 < R$, nous notons :

$$B(r) = \{z \in \Omega;\ \varphi(z) < r\}, \quad S(r) = \{z \in \Omega;\ \varphi(z) = r\}, \quad B(r_1, r_2) = \{z \in \Omega;\ r_1 \leqslant \varphi(z) < r_2\},$$

Pour simplifier, nous utiliserons aussi les notations suivantes :

$$\beta_z = dd^c(\varphi(z)),\ \ \gamma_t = dd^c(v(t)),\ \ \alpha_z = dd^c(\log(\varphi(z))),\ \ \omega_t = dd^c(|t|^2),\ \ \omega_z = dd^c(|z|^2),$$

avec

$$(z = (z_1, \dots, z_n), t = (t_1, \dots, t_m)) \in \mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^N;\ |z|^2 = \sum_{j=1}^{n} z_j \overline{z_j};\ |t|^2 = \sum_{j=1}^{m} t_j \overline{t_j}.$$

On note $\omega = \omega_t + \omega_z$ la forme euclidienne sur $\mathbb{C}^n \times \mathbb{C}^m$ et, si $z_j = x_j + iy_j \in \mathbb{C}$, on note :

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right),\ \frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right),\ d = \partial + \overline{\partial}\ ,\ d^c = i(\overline{\partial} - \partial),\ dd^c = 2i\partial\overline{\partial}.$$

On donne, dans la proposition suivante, une formule importante de type Lelong–Jensen. Cette formule a été utilisée par Demailly [4], Alessandrini–Bassanelli [1] et Toujani [7].

**Proposition.** *Pour tous $0 < r_1 < r_2$ tels que $B(r_2) \times B \Subset \Omega$, et pour tous $1 \leqslant p \leqslant n-k,\ 0 \leqslant q < p$, on a*

$$\begin{aligned}
\int_{B(r_1,r_2)\times B} T \wedge (\frac{\alpha_z}{\pi})^p \wedge \beta_z^{n-k-p} \wedge \gamma_t^m &= \frac{1}{(\pi r_2)^{q+1}} \int_{B(r_2)\times B} T \wedge (\frac{\alpha_z}{\pi})^{p-q-1} \wedge \beta_z^{n-k-p+q+1} \wedge \gamma_t^m \\
&- \frac{1}{(\pi r_1)^{q+1}} \int_{B(r_1)\times B} T \wedge (\frac{\alpha_z}{\pi})^{p-q-1} \wedge \beta_z^{n-k-p+q+1} \wedge \gamma_t^m \\
&- \int_{r_1}^{r_2} \left(\frac{1}{(\pi s)^{q+1}} - \frac{1}{(\pi r_2)^{q+1}}\right) ds \int_{B(s)\times B} dd^cT \wedge (\frac{\alpha_z}{\pi})^{p-q-1} \wedge \beta_z^{n-k-p+q} \wedge \gamma_t^m \\
&- \int_{0}^{r_1} \left(\frac{1}{(\pi r_1)^{q+1}} - \frac{1}{(\pi r_2)^{q+1}}\right) ds \int_{B(s)\times B} dd^cT \wedge (\frac{\alpha_z}{\pi})^{p-q-1} \wedge \beta_z^{n-k-p+q} \wedge \gamma_t^m.
\end{aligned}$$

Notre résultat principal consiste dans le théorème suivant.

**Théorème.** *Il existe un voisinage ouvert $X$ de $0$ dans $L$, $X \subset \Omega$ et une fonction pluri-subharmonique $f : X \to \mathbb{R}_+$ telles que*

$$\nu(T, L, B) = \lim_{r\to 0} \frac{1}{(\pi r)^{n-k}} \int_{B(r)\times B} T \wedge \beta_z^{n-k} \wedge \gamma_t^m = \int_B f(t)\gamma_t^m$$

*pour chaque borélien $B$ dans $L$ tel que $B \Subset X$.*

## 1. Introduction and preliminaries

It is well known that a positive plurisubharmonic current admits Lelong numbers (see [6]); let us recall that a current $T$ is said to be plurisubharmonic if $dd^cT \geqslant 0$. Demailly has defined in [3] generalized Lelong numbers whenever $T$ is d-closed positive current (a current $T$ is d-closed if $dT = 0$). Similarly, when $T$ is a positive plurisubharmonic current, Alessandrini and Bassanelli proved in [1] the existence of a directional Lelong number, with respect to the functions $\varphi(z) = |z|^2$ and $v(t) = |t|^2$. Toujani defined in [7] the generalized directional Lelong number, with $\varphi$ and $v$ are $C^2$ positive semi-exhaustive psh functions on $\Omega$.

This paper treats the problem by generalizing the results section of Toujani [7] considering a function $v$ psh continuous instead of a function psh of class $C^2$.

Now, we shall use the following notations: we take $(z,t) \in \mathbb{C}^n \times \mathbb{C}^m$ and consider a $C^2$ positive psh function on $\Omega$, $(z,t) \mapsto \varphi(z)$, such that $\log \varphi$ is also psh on the open set $\{\varphi > 0\}$. We suppose that $\varphi : \Omega \cap (\mathbb{C}^n \times \{0\}) \to [-\infty, +\infty[$ is semi-exhaustive, i.e. that there exists a real number $R$ such that, for all $c \in ]-\infty, R[$, we have $\{z,\ \varphi(z) < c\} \Subset \Omega \cap (\mathbb{C}^n \times \{0\})$, and we let $(z,t) \mapsto v(t)$ be a semi-exhaustive continuous psh function on the open set $\Omega$. For $r < R$ and $r_1 < r_2 < R$, we denote

$$B(r) = \{z \in \Omega;\ \varphi(z) < r\}, \quad S(r) = \{z \in \Omega;\ \varphi(z) = r\}, \quad B(r_1, r_2) = \{z \in \Omega;\ r_1 \leqslant \varphi(z) < r_2\}.$$

For simplicity, we shall also use the following notation:

$$\beta_z = dd^c(\varphi(z)),\ \ \gamma_t = dd^c(v(t)),\ \ \alpha_z = dd^c(\log(\varphi(z))),\ \ \omega_t = dd^c(|t|^2),\ \ \omega_z = dd^c(|z|^2),$$

with

$$(z = (z_1, \ldots, z_n), t = (t_1, \ldots, t_m)) \in \mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^N;\ |z|^2 = \sum_{j=1}^{n} z_j \overline{z_j};\ |t|^2 = \sum_{j=1}^{m} t_j \overline{t_j}.$$

We denote by $\omega = \omega_t + \omega_z$ the Euclidean form on $\mathbb{C}^n \times \mathbb{C}^m$. If $z_j = x_j + iy_j \in \mathbb{C}$, then we set as usual:

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right),\ \ \frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right),\ \ d = \partial + \overline{\partial},\ \ d^c = i(\overline{\partial} - \partial),\ \ dd^c = 2i\partial\overline{\partial}.$$

We state the following Lelong–Jensen type formula, which is an important identity. This formula was used by Demailly [4], Alessandrini–Bassanelli [1], and Toujani [7].

**Proposition.** *For every $0 < r_1 < r_2$ such that $B(r_2) \times B \Subset \Omega$, and for every $1 \leqslant p \leqslant n-k,\ 0 \leqslant q < p$, we have*

$$\begin{aligned}
\int_{B(r_1,r_2)\times B} T \wedge (\frac{\alpha_z}{\pi})^p \wedge \beta_z^{n-k-p} \wedge \gamma_t^m &= \frac{1}{(\pi r_2)^{q+1}} \int_{B(r_2)\times B} T \wedge (\frac{\alpha_z}{\pi})^{p-q-1} \wedge \beta_z^{n-k-p+q+1} \wedge \gamma_t^m \\
&- \frac{1}{(\pi r_1)^{q+1}} \int_{B(r_1)\times B} T \wedge (\frac{\alpha_z}{\pi})^{p-q-1} \wedge \beta_z^{n-k-p+q+1} \wedge \gamma_t^m \\
&- \int_{r_1}^{r_2} \left(\frac{1}{(\pi s)^{q+1}} - \frac{1}{(\pi r_2)^{q+1}}\right) ds \int_{B(s)\times B} dd^cT \wedge (\frac{\alpha_z}{\pi})^{p-q-1} \wedge \beta_z^{n-k-p+q} \wedge \gamma_t^m \\
&- \int_{0}^{r_1} \left(\frac{1}{(\pi r_1)^{q+1}} - \frac{1}{(\pi r_2)^{q+1}}\right) ds \int_{B(s)\times B} dd^cT \wedge (\frac{\alpha_z}{\pi})^{p-q-1} \wedge \beta_z^{n-k-p+q} \wedge \gamma_t^m.
\end{aligned}$$

**Consequence.** For $p = n-k$, $q = p-1$, the quantity

$$A = \frac{1}{(\pi r_2)^{n-k}} \int_{B(r_2)\times B} T \wedge \beta_z^{n-k} \wedge \gamma_t^m - \frac{1}{(\pi r_1)^{n-k}} \int_{B(r_1)\times B} T \wedge \beta_z^{n-k} \wedge \gamma_t^m$$

is positive, since $T \wedge \beta_z^{n-k} \wedge \gamma_t^m$ and $dd^cT \wedge \beta_z^{n-k-1} \wedge \gamma_t^m$ are positive measures, and since we have $0 \leqslant \int_{B(r_1,r_2)} T \wedge (\frac{\alpha_z}{\pi})^{n-k} \wedge \gamma_t^m \leqslant A$. We infer that the function: $r \to \frac{1}{(\pi r)^{n-k}} \int_{B(r)\times B} T \wedge \beta_z^{n-k} \wedge \gamma_t^m$ increases with $r$. As a consequence, the limit of $\frac{1}{(\pi r)^{n-k}} \int_{B(r)\times B} T \wedge \beta_z^{n-k} \wedge \gamma_t^m$ as $r \to 0$ exists; it will be denoted by $\nu(T, L, B)$.

**Definition.** The number $\nu(T, L, B)$, which is finite, is called the generalized directional Lelong number of $T$ along the direction $L$, with respect to the functions $\varphi(z)$ and $v(t)$.

**Remarks.** If $m = 0$ and $\varphi(z) = |z|^2$ with $z \in \mathbb{C}^n$, we get the classical Lelong number of $T$ at 0, see [6].

If $m = 0$ and $T$ is a d-closed positive current, we get the number introduced by Demailly [3].

If $\varphi(z) = |z|^2$, $v(t) = |t|^2$ and $B$ is an open ball in $L$, with $z \in \mathbb{C}^n$, $t \in \mathbb{C}^m$, we get the number introduced by Alessandrini–Bassanelli [1].

The main result of this paper is given by the following theorem.

**Theorem.** *There exists an open neighborhood $X$ of 0 in $L$, $X \subset \Omega$ and a plurisubharmonic function $f : X \to \mathbb{R}_+$, such that*

$$\nu(T, L, B) = \lim_{r\to 0} \frac{1}{(\pi r)^{n-k}} \int\limits_{B(r)\times B} T \wedge \beta_z^{n-k} \wedge \gamma_t^m = \int\limits_B f(t)\gamma_t^m$$

*for every Borel subset $B$ in $L$ with $B \Subset X$.*

## 2. Proof of the main theorem

### 2.1. Proof of theorem for $k = n$

For $k = n$, we have $\nu(T, L, B) = \lim_{r\to 0} \int_{B(r)\times B} T \wedge \gamma_t^m$. We let $R = 1_{\Omega\setminus Y} T$, where $Y = L \cap \Omega$ and $1_{\Omega\setminus Y}$ is the characteristic function of $\Omega \setminus Y$. Then $Y$ is an analytic subset of $\Omega$ and has codimension $k = n$. As $T$ is of order zero, $||T||_K$ is finite for all compact subsets $K \subset \Omega$. Therefore, $R$ has a locally finite mass across $Y = L \cap \Omega$ and $T$ is a positive psh current. By the results of [2], we know that $T$ and $R$ are $\mathbb{C}$-flat. Hence, thanks again to [2], we find $T = 1_Y T + \widetilde{R}$, where $\widetilde{R}$ is the trivial extension of $R$ across $Y$. As $k = n$, we have $1 \leqslant k < N$, and by [2], there exists a psh function $f \geqslant 0 \in L^1_{\mathrm{loc}(Y)}$ such that $1_Y T = f[Y]$, with $[Y]$ the current integration on $Y$. Thus, we have $T = f[Y] + \widetilde{R}$. So, $\int_{B(r)\times B} T \wedge \gamma_t^m = \int_B f(t)\gamma_t^m + \int_{B(r)\times B} \widetilde{R} \wedge \gamma_t^m$, for $r \ll 1$ and $B \subset L$, $B \Subset \Omega$.

We will show that $\lim_{r\to 0} \int_{B(r)\times B} \widetilde{R} \wedge \gamma_t^m = 0$. We can here rely on the existence of a sequence of functions $v_p$ of class $C^\infty$ psh that decreases and converges simply towards $v(t)$, see [5]. We also know that the sequence $\int_{B(r)\times B} \widetilde{R} \wedge (\mathrm{dd^c} v_p)^m$ converges weakly to $\int_{B(r)\times B} \widetilde{R} \wedge \gamma_t^m$ when $p$ tends to $+\infty$ see [8]. According to Toujani (see [8]), we have

$$\int\limits_{B(r)\times B} \widetilde{R} \wedge (\mathrm{dd^c} v_p)^m \leqslant \int\limits_{\overline{B(r)}\times\overline{B}} \widetilde{R} \wedge (\mathrm{dd^c} v_p)^m \leqslant C\Big(||\widetilde{R}||_{\overline{B(2r)}\times\overline{B}} + ||\widetilde{\mathrm{dd^c} R}||_{\overline{B(2r)}\times\overline{B}}\Big)||v_p||^m_{\infty(\overline{B(2r)}\times\overline{B})}$$

with $C$ is a positive constant independent of $\widetilde{R}$ and $v_p$. Since, $||v_p||_\infty$ is bounded on the compact $\overline{B(2r)} \times \overline{B}$, we can then replace $|v_p|$ by $\frac{|v_p|}{||v_p||_\infty}$ and reduce to the case $|v_p| \leqslant 1$. Therefore, we infer

$$0 \leqslant \int\limits_{B(r)\times B} \widetilde{R} \wedge (\mathrm{dd^c} v_p)^m \leqslant C\Big(||\widetilde{R}||_{\overline{B(2r)}\times\overline{B}} + ||\widetilde{\mathrm{dd^c} R}||_{\overline{B(2r)}\times\overline{B}}\Big).$$

We pass to the limit when $p$ tends to $+\infty$ and get

$$0 \leqslant \int\limits_{B(r)\times B} \widetilde{R} \wedge \gamma_t^m \leqslant C\Big(||\widetilde{R}||_{\overline{B(2r)}\times\overline{B}} + ||\widetilde{\mathrm{dd^c} R}||_{\overline{B(2r)}\times\overline{B}}\Big). \tag{1}$$

In a previous article of Toujani (see [7]), the function $v$ was assumed to be of class $C^2$, and we took advantage of the regularity (in effect, the continuity) of the coefficients of the form $\mathrm{dd^c} v$ to obtain the theorem. In the present situation where the function $v$ is merely assumed to be continuous, we made use of the inequality (1), which involves the masses $||\widetilde{R}||$ and $||\widetilde{\mathrm{dd^c} R}||$. Note that the case where $k = n$ is the most important one for the proof of the theorem. However, the latter also deals with the case $k < n$. Notice that

$$\lim_{r\to 0} ||\widetilde{R}||_{\overline{B(2r)}\times\overline{B}} = ||\widetilde{R}||_{\overline{S(0)}\times\overline{B}} = \int\limits_{\overline{S(0)}\times\overline{B}} \widetilde{R} \wedge (\mathrm{dd^c}\omega)^m = 0,$$

because the set $\overline{S(0)} = S(0) = \{z \in \Omega \ ; \ \varphi(z) = 0\} = \{z \in \Omega \ ; \ \log\varphi(z) = -\infty\}$ is compact and complete pluripolar in $\Omega \cap (\mathbb{C}^n \times \{0\})$, as the function $\varphi$ is $C^2$ psh semi-exhaustive and $\log\varphi$ is psh on $\{\varphi > 0\}$. Similarly, the set $\overline{B}$ is a compact and it is a complete pluripolar in $\Omega \cap (\{0\} \times \mathbb{C}^m)$; therefore, $\lambda(\overline{S(0)} \times \overline{B}) = 0$, where $\lambda$ is the Lebesgue measure in $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^m$. Similarly, $\lim_{r\to 0} ||\widetilde{\mathrm{dd^c} R}||_{\overline{B(2r)}\times\overline{B}} = 0$, and we finally get $\lim_{r\to 0} \int_{B(r)\times B} \widetilde{R} \wedge \gamma_t^m = 0$. Therefore, we have:

$$\nu(T, L, B) = \lim_{r\to 0} \int\limits_{B(r)\times B} T \wedge \gamma_t^m = \int\limits_B f(t)\gamma_t^m. \qquad \square$$

In the sequel, we suppose that $0 \leqslant k < n$.

### 2.2. Proof of the theorem when $T$ is pluriharmonic

To show the theorem when $T$ is positive pluriharmonic (i.e. $\mathrm{dd^c} T = 0$), we need the following lemma.

**Lemma 2.1.** *There exist currents $T^{(0)}, T^{(1)}, \ldots, T^{(n-k)}$ on $U$, such that, for a suitable subsequence $\{T_{\nu_\mu}\}_{\mu\in\mathbb{N}}$, we have*

*i)* $\lim_{\mu\to+\infty} \left(\widetilde{T_{\nu_\mu} \wedge (\frac{\alpha_z}{\pi})^p}\right) = T^{(p)}$ *weakly on $U$* $(0 \leqslant p \leqslant n-k)$,

*ii) let $X = U \cap L$, there exists a positive function $f \in L^1_{\mathrm{loc}}(X)$ such that $1_X T^{(n-k)} = f[X]$.*

Lemma 2.1 is of a great importance in the proof of the main theorem; for its proof, see [7]. To prove the theorem when $T$ is pluriharmonic, we show that the current $T^{(n-k)}$ indicated in Lemma 2.1 is pluriharmonic of bidegree $(n, n)$; in this way, we are reduced to the case $k = n$ to complete the proof, see [7].

### 2.3. Proof of the theorem when $T$ is positive plurisubharmonic

To prove the main theorem when $T$ is positive psh, we need the following lemma (see [1]).

**Lemma 2.2.** *There exists an open neighborhood $X$ of 0 in $L$ with $X \Subset \Omega$, such that, for every Borel subset $B$ in $L$ with $B \Subset X$, we have*

$$\nu(T, L, B) = \nu(T^{(1)} + S^{(0)}, L, B),$$

*with* $T^{(1)} = \lim_{\mu\to+\infty} \widetilde{R^1_\mu}$, *where* $R^1_\mu = T_{\nu_\mu} \wedge \frac{\alpha_z}{\pi}$, $S^{(0)} = \lim_{\mu\to+\infty} \widetilde{R'^0_\mu}$ *and* $R'^0_\mu = \frac{-\log(\varphi)}{\pi} \mathrm{dd^c} T_{\nu_\mu}$, *for a subsequence $(T_{\nu_\mu})$ of $(T_\nu)$.*

In order to prove the main theorem for $T$ plurisubharmonic, we use Lemma 2.2. This shows that

$$\nu(T^{(1)} + S^{(0)}, L, B) = \nu(T, L, B),$$

and as $T^{(1)} + S^{(0)}$ is a pluriharmonic current, there exists a psh function $f : X \to \mathbb{R}$, $f \geqslant 0$ such that $\nu(T^{(1)} + S^{(0)}, L, B) = \int_B f(t)\gamma_t^m$. Therefore, $\nu(T, L, B) = \int_B f(t)\gamma_t^m$.

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