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Partial differential equations

On periodic subsolutions to steady second-order systems and applications

Sur des sous-solutions périodiques de systèmes d'équations du deuxième ordre stables et applications

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ABSTRACT

Bostan and Namah (Remarks on bounded solutions of steady Hamilton–Jacobi equations, C. R. Acad. Sci. Paris, Ser. I 347(15–16) (2009) 873–878) proved that constant functions are the only bounded solutions to $H(Du) = H(0)$ when H is superlinear and strictly convex. In this short note, we present a proof other than that of Bostan and Namah for equations that can be easily applied to some types of possibly degenerate parabolic systems. Our proof applies for periodic *sub*solutions instead of bounded solutions like that of Bostan and Namah; however, we need periodic *sub*solutions, which is quite restrictive. We do not consider Hopf–Lax's formula in our proof, so we can relax some restrictions on H. We also present an application to the large-time behavior of solutions to degenerate parabolic systems.

RÉSUMÉ

Bostan et Namah (Remarks on bounded solutions of steady Hamilton–Jacobi equations, C. R. Acad. Sci. Paris, Ser. I 347(15–16) (2009) 873–878) ont montré que les solutions bornées de l'équation $H(Du) = H(0)$, où H est superlinéaire et strictement convexe, sont les constantes. Dans cette note, on présente une autre preuve qui peut être appliquée facilement à des systèmes d'équations paraboliques dégénérées. Notre preuve s'applique à des *sous*-solutions périodiques au lieu des solutions bornées examinées par Bostan et Namah. Comme nous n'utilisons pas la formule d'Hopf–Lax dans la preuve, nous pouvons affaiblir un peu certaines régularités des hamiltoniens. Finalement, nous présentons une application au comportement asymptotique des solutions pour des systèmes d'équations paraboliques dégénérées.

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1. Introduction

We address in this paper the following problem studied in Bostan–Namah [2]. The authors consider the equation

$$H(Du) = H(0), \quad x \in \mathbb{R}^N, \tag{1.1}$$

and ask under which hypotheses the constant functions are the only bounded solutions to (1.1) (in viscosity sense).

H is said to be superlinear if $\frac{H(p)}{|p|}$ is coercive. Their result is stated as follows.

Theorem 1.1. *Assume that H is convex, superlinear, and such that its conjugate function is C^1 and strictly convex in a neighborhood of its minimum point. Then constants are the only bounded viscosity solutions to* (1.1)*.*

Our starting point is to understand what happens with the same question for the system:

$$H_i(Du_i) + \sum_{j=1}^{m} d_{ij} u_j = c_i, \quad x \in \mathbb{R}^N, \quad i = 1, \dots, m. \tag{1.2}$$

The system (1.2) does not always have a viscosity solution. Indeed, one can show that there exists a so-called ergodic constant $c \in \mathbb{R}$ such that the system (1.2), where on the right-hand side $c_i = c$ for any $i = 1, \dots, m$, admits a viscosity solution, see [9], [4], [10] for details.

To prove Theorem 1.1, the author uses Hopf–Lax's formula to extract information from the solutions. This formula in case of systems involving random jumps from one equation to another and hence is quite complicated to extract information. So we look for another proof as that of Bostan–Namah for equations that can be applied to systems. With this approach, we get the same result as in Bostan–Namah for periodic subsolutions. As we do not consider the conjugate of Hamiltonians in our proof, we do not need H to be superlinear as well as its conjugate is C^1. The new relaxed assumptions cannot be used with the approach using the Hopf–Lax formula (which only applies for solutions and superlinear Hamiltonians).

The proof of Bostan–Namah does not need the solutions to be periodic and this is the limit of our result. However, our result works for *sub*solutions, requires less regularities on H, and can be extended easily to second-order systems. We also present an application to the large-time behavior of solutions to second-order systems.

The notion of solution that we use in this article is the viscosity solution, see [6] and references therein.

2. Preliminary and main results

For a square matrix $D = (d_{ij})_{1 \le i,j \le m}$, we define

$$\inf(D) = \inf_{i \in \{1,\cdots,m\}} \sum_{j=1}^{m} d_{ij}, \; \sup(D) = \sup_{i \in \{1,\cdots,m\}} \sum_{j=1}^{m} d_{ij}. \tag{2.1}$$

We introduce some common definitions of weakly coupled systems.

The coupling is called monotone if

$$d_{ii} \ge 0, \quad d_{ij} \le 0 \text{ for } i \ne j, \tag{2.2}$$

and degenerate if

$$d_{ii} \ge 0, \quad d_{ij} \le 0 \text{ for } i \ne j, \quad \text{and } \inf(D) = \sup(D) = 0. \tag{2.3}$$

Finally, we need the irreducibility of the coupling (which means essentially that the system is not separated into smaller ones).

$$\begin{array}{l} \text{We say that } D = (d_{ij})_{1 \le i,j \le m} \text{ is irreducible if, for any subset } \mathcal{I} \subsetneq \{1, \cdots, m\}, \\ \text{there exist } i \in \mathcal{I} \text{ and } j \notin \mathcal{I} \text{ such that } d_{ij} \ne 0. \end{array} \tag{2.4}$$

We recall some lemmata that will be needed in this paper. We say $(x_1, \dots, x_m) \in (\mathbb{R}^+)^m$ if $x_i > 0$ for all $i = 1, \dots, m$.

Lemma 2.1. ([4]) *Suppose that D satisfies* (2.3) *and* (2.4)*, then* $\ker(D) = \operatorname{span}\{(1, \cdots, 1)\}$*. Moreover, there exists a positive vector $\Lambda = (\Lambda_1, \dots, \Lambda_m) \in (\mathbb{R}^+)^m$ such that $D^\top \Lambda = 0$.*

and

Lemma 2.2. ([11]) *For any matrix $D \in M_m(\mathbb{R})$ satisfying* (2.4)*, the matrix $E \in M_{m-1}(\mathbb{R})$ obtaining from D after removing the ith row and ith column is invertible for any $i = 1, \dots, m$.*

Theorem 2.5 and Theorem 2.6 are particular cases of the following degenerate systems

$$-\mathrm{trace}(A_i D^2 u_i(x)) + H_i(Du_i) + \sum_{j=1}^{m} d_{ij} u_j = c_i \; x \in \mathbb{T}^N, i = 1, \dots, m, \tag{2.5}$$

and its parabolic version

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \mathrm{trace}(A_i D^2 u_i(x)) + H_i(Du_i) + \displaystyle\sum_{j=1}^{m} d_{ij} u_j = 0 & (x,t) \in \mathbb{T}^N \times (0, +\infty), \\ u_i(x, 0) = u_{0i}(x) & x \in \mathbb{T}^N, i = 1, \dots, m. \end{cases} \tag{2.6}$$

Before presenting the main results, we need an explicit calculation of the ergodic constant $(c, \dots, c)$. Recall that c is the unique constant such that the system (2.5), where on the right-hand side $c_i = c$ for any $i = 1, \dots, m$, admits a viscosity solution.

The proof of the following basic lemma adapts (Lemma 5.2 in [11]) to eikonal first-order systems.

Proposition 2.3. *Let*

$$c = \frac{\sum_{i=1}^{m} \Lambda_i H_i(0)}{\sum_{i=1}^{m} \Lambda_i}. \tag{2.7}$$

Then the system (2.5), where on the right-hand side $c_i = c$ for any $i = 1, \dots, m$, admits a constant solution.

Remark 2.4. To see why (2.7) appears naturally, assume a priori that we have a constant solution to

$$-\mathrm{trace}(A_i D^2 u_i(x)) + H_i(Du_i) + \sum_{j=1}^{m} d_{ij} u_j = c \; x \in \mathbb{T}^N, i = 1, \dots, m,$$

i.e.,

$$H_i(0) + \sum_{j=1}^{m} d_{ij} u_j = c \; x \in \mathbb{T}^N, i = 1, \dots, m.$$

Multiplying the ith equation with Λ_i, and summing all the equations to obtain

$$\sum_{i=1}^{m} \Lambda_i H_i(0) + \sum_{i=1}^{m} \Big(\sum_{j=1}^{m} \Lambda_i d_{ij} u_j\Big) = \Big(\sum_{i=1}^{m} \Lambda_i\Big) c.$$

From Lemma 2.1, $\sum_{i=1}^{m} (\sum_{j=1}^{m} \Lambda_i d_{ij} u_j) = 0$, and (2.7) appears.

Proof of Proposition 2.3. To prove this Lemma, it suffices to find an explicit solution associated with c.

From Lemma 2.2, we can find a constant vector $(c_1, ..., c_{m-1})$ satisfying

$$\sum_{j=1}^{m-1} d_{ij} c_j = c - H_i(0), \quad i = 1, \dots, m-1.$$

Choose $c_m = 0$, we have

$$\sum_{j=1}^{m} d_{ij} c_j = c - H_i(0), \quad i = 1, \dots, m-1. \tag{2.8}$$

We claim that (2.8) also holds for $i = m$. Let us multiply the ith equation in (2.8) by Λ_i, and sum all the equations to obtain

$$\sum_{i=1}^{m-1} \Big(\sum_{j=1}^{m} \Lambda_i d_{ij} c_j\Big) = \Big(\sum_{i=1}^{m-1} \Lambda_i\Big) c - \sum_{i=1}^{m-1} \Lambda_i H_i(0), \quad \text{i.e.,} \quad \sum_{j=1}^{m} -\Lambda_m d_{mj} u_j = -\Lambda_m (c - H_m(0)),$$

where the last inequality follows from Lemma 2.1. This yields

$$\sum_{j=1}^{m} d_{mj} u_j = c - H_m(0).$$

Finally, $(c_1, \dots, c_{m-1}, 0)$ is a solution to (2.5), where on the right-hand side 0 is replaced by c. □

We state our first result.

Theorem 2.5. *Assume that H_i is coercive and convex for $i = 1, \dots, m$, D satisfies (2.3) and (2.4). Assume that there exists $i \in \{1, \dots, m\}$ such that H_i is strictly convex at 0. We assume moreover that $A_i \geq 0$ are constant matrices, u_i is twice differentiable a.e. for $i = 1, \dots, m$. Then constants are the only bounded periodic subsolutions to (2.5), where on the right-hand side $c_i = c$, given in (2.7) for any $i = 1, \dots, m$.*

We present an important application of Theorem 2.5 for large-time behavior of solutions.

Theorem 2.6. *Assume that H_i is coercive and convex for all $i = 1, \dots, m$, D satisfies (2.3) and (2.4). Assume that there exists at least one H_i that is strictly convex at 0. We assume moreover that $A_i \geq 0$ are constant matrices, u_i is twice differentiable a.e. for $i = 1, \dots, m$. Let $(u_1, \dots, u_m)$ be a solution to (2.6), then there exists a unique so-called ergodic constant c as given in (2.7), a vector $(C_1, \dots, C_m) \in \mathbb{R}^m$ such that $u_i(x,t) + ct \to C_i$ uniformly for all $i = 1, \dots, m$ as $t \to \infty$.*

Remark 2.7. The strict convexity of just one Hamiltonian in Theorem 2.5 implies that all u_i's are constant, and Theorem 2.6 implies that all u_i's tend to constants.

Theorem 2.6 extends a result of [10] where the authors assumed that $H_i(p) = |p + c_i|^2 - |c_i|^2$, c_i's being constants and $A_i = 0$ for all $i = 1, \dots, m$. See [4], [10], [11], [8] and references therein for more discussions on the large-time behavior of solutions to systems of first and second order.

Remark 2.8. The question of when we have $u_i \in W^{1,\infty}(\mathbb{T}^N)$ for (2.5) and $u_i \in W^{1,\infty}(\mathbb{T}^N \times [0,\infty))$ for (2.6) was systematically studied in the literature. It holds under much more general assumptions than the ones in our context; we refer the readers to [1], [8] and the references therein for more results.

The assumption that u_i is twice differentiable a.e. is of course not necessary in first-order systems (i.e. when $A_i = 0$ for all $i = 1, \dots, m$). The twice differentiability a.e. of the solution is discussed in Proposition 2.10.

Proof of Theorems 2.5 and 2.6. *Step 1. Lipschitz regularity of solutions and boundedness of the solution to (2.6) where 0 is replaced by c on the right-hand side of (2.6).*

An application of [[8], Theorem 3.1] yields that any solution to (2.5) lies in $(W^{1,\infty}(\mathbb{T}^N))^m$. Now, [[8], Corollary 3.8] claims that, for any initial data lying in $(W^{1,\infty}(\mathbb{T}^N))^m$, the solution to (2.6) lies in $(W^{1,\infty}(\mathbb{T}^N \times [0,\infty)))^m$. Once we have the result obtained at Step 4, the general case when the initial data is purely continuous can be handled easily by approximating continuous functions by $W^{1,\infty}(\mathbb{T}^N)$ functions.

Suppose now that the existence of the ergodic constant for (2.5) has been established. Given $(u_1(x,t), \dots, u_m(x,t))$, the solution to (2.6), then the new function $(u_1(x,t) + ct, \dots, u_m(x,t) + ct)$ still satisfies (2.6), where on the right-hand side of (2.6), 0 is replaced by c given in (2.7). It is proved quite easily using the comparison principle, see for instance in [[4], Proposition 5.2], that the new function is bounded. We still denote the new function $(u_1(x,t), \dots, u_m(x,t))$.

Step 2. Proof of Theorem 2.5; we claim that any continuous subsolution $(u_1, \dots, u_m)$ to (2.5) (on the right-hand side 0 is replaced by c given in (2.7)) is a constant.

Using Lemma 2.1, we have the existence of $\Lambda_i > 0$ such that

$$\sum_{i=1}^{m} -\Lambda_i \operatorname{trace}(A_i D^2 u_i(x)) + \sum_{i=1}^{m} \Lambda_i H_i(Du_i) \leq (\sum_{i=1}^{m} \Lambda_i)c = \sum_{i=1}^{m} \Lambda_i H_i(0).$$

It follows that

$$-\sum_{i=1}^{m} \Lambda_i \int_{\mathbb{T}^N} \operatorname{trace}(A_i D^2 u_i(x))\,\mathrm{d}x + \sum_{i=1}^{m} \Lambda_i \int_{\mathbb{T}^N} H_i(Du_i)\,\mathrm{d}x \leq \sum_{i=1}^{m} \Lambda_i H_i(0). \tag{2.9}$$

It follows from the periodicity of u_i that $\int_{\mathbb{T}^N} \operatorname{trace}(A_i D^2 u_i(x))\,\mathrm{d}x = 0$, and hence Jensen's inequality yields $\int_{\mathbb{T}^N} H_i(Du_i)\,\mathrm{d}x \geq H_i[\int_{\mathbb{T}^N} Du_i\,\mathrm{d}x] = H_i(0)$ for $i = 1, \dots, m$.

We derive

$$\sum_{i=1}^{m} \Lambda_i H_i(0) \geq \int_{\mathbb{T}^N} \sum_{i=1}^{m} \Lambda_i H_i(Du_i)\,\mathrm{d}x \geq \int_{\mathbb{T}^N} \Lambda_1 H_1(Du_1)\,\mathrm{d}x + \sum_{i=2}^{m} \Lambda_i H_i(0);$$

this implies

$$H_1(0) \geq \int_{\mathbb{T}^N} H_1(Du_1)\,\mathrm{d}x.$$

Since H_1 is strictly convex at 0, we have a better estimate for this strictly convex Hamiltonian.

Proposition 2.9. *Let H be a convex function satisfying $H(0)=0$, and that H is strictly convex at 0. Set*

$$A=\{f\in W^{1,\infty}(\mathbb{T}^N)\bigcap C(\mathbb{T}^N) \text{ such that } ||f||_{W^{1,\infty}(\mathbb{T}^N)}\le C,\ \max_{\mathbb{T}^N} f-\min_{\mathbb{T}^N} f\ge 1\}.$$

Then we have, for all $f\in A$,

$$\int_{\mathbb{T}^N} H(Df)\,\mathrm{d}x\ge\beta>0,\ \ \textit{where } \beta \textit{ is independent of } f.$$

We apply Proposition 2.9 to the strictly convex function $p\mapsto H_1(p)-H_1(0)$. If u_1 is not a constant, we get a contradiction, since

$$H_1(0)\ge\int_{\mathbb{T}^N} H_1(Du_1)\,\mathrm{d}x\ge H_1(0)+\beta\ \ ,\beta>0.$$

It implies that u_1 is a constant, let us say C_1.

Step 3. Showing that $u_i=C_i$ for any $i=2,\dots,m$.
Set $M_i=\max_{x\in\mathbb{T}^N} u_i(x)$. By taking 0 as a test function, we have

$$H_i(0)+\sum_{j=1}^m d_{ij}M_j\le c, i=1,\dots,m.$$

Multiplying the ith equation with Λ_i, and summing all equations taking into account Lemma (2.1) and (2.7), we deduce that

$$H_i(0)+\sum_{j=1}^m d_{ij}M_j=c, i=1,\dots,m. \tag{2.10}$$

Integrating each equation of (2.5) as done in (2.9), we get

$$H_i(0)+\sum_{j=1}^m d_{ij}\int_{\mathbb{T}^N} u_j\,\mathrm{d}x\le c, i=1,\dots,m. \tag{2.11}$$

Repeating the arguments used above to get (2.10), we have

$$H_i(0)+\sum_{j=1}^m d_{ij}\int_{\mathbb{T}^N} u_j\,\mathrm{d}x=c, i=1,\dots,m. \tag{2.12}$$

From (2.10) and (2.12), we have

$$\sum_{j=1}^m d_{ij}(M_j-\int_{\mathbb{T}^N} u_j\,\mathrm{d}x)=0, i=1,\dots,m.$$

This latter equality implies $(M_1-\int_{\mathbb{T}^N} u_1\,\mathrm{d}x,\dots,M_m-\int_{\mathbb{T}^N} u_m\,\mathrm{d}x)\in\ker(D)$. Lemma 2.1 claims that ker(D)=span$\{(1,\cdots,1)\}$; as a consequence,

$$M_i-\int_{\mathbb{T}^N} u_i\,\mathrm{d}x=M_1-\int_{\mathbb{T}^N} u_1\,\mathrm{d}x=0\ \ \text{for any } i=2,\dots,m.$$

Thanks to the continuities of u_i, we conclude that

$$u_i=M_i\ \ \text{for any } i=1,\dots,m.$$

Step 4. Proof of Theorem 2.6. We show that any solution $(u_1,\dots,u_m)$ of (2.6) satisfies $u_i\to C_i$ uniformly for all $i=1,\dots,m$.

We reuse many arguments made at Step 3. Set $M_i(t) = \max_{x\in\mathbb{T}^N} u_i(x,t)$, $P_i(t) = \min_{x\in\mathbb{T}^N} u_i(x,t)$ and $M(t) = \max_{i\in\{1,\ldots,m\}} M_i(t)$, by the comparison principle, M is decreasing and hence $M(t) \to L$ as $t \to \infty$. By taking 0 as a test function, we have

$$\frac{\partial M_i}{\partial t} + H_i(0) + \sum_{j=1}^{m} d_{ij} M_j \leq c, t \in (0,+\infty), 1 \leq i \leq m,$$

and

$$\frac{\partial P_i}{\partial t} + H_i(0) + \sum_{j=1}^{m} d_{ij} P_j \geq c, t \in (0,+\infty), 1 \leq i \leq m.$$

We find that

$$\frac{\partial (M_i - P_i)}{\partial t} + \sum_{j=1}^{m} d_{ij}(M_j - P_j) \leq 0, t \in (0,+\infty), 1 \leq i \leq m.$$

By applying [[11], Lemma 3.4], we find that $M_i(t) - P_i(t) \to l$ as $t \to \infty$ where l is independent of i.

If $l = 0$, it is straightforward to see that $u_i(x,t) \to L_i$ as $t \to \infty$ for all i. We now assume that $l > 0$ and wlog $l > 2$. Hence there exist t_0 such that $M_i(t) - P_i(t) \geq 1$, for any $t \geq t_0$. Reusing the arguments from Step 2, we come to

$$\frac{\partial}{\partial t} \int_{\mathbb{T}^N} \sum_{i=1}^{m} \Lambda_i u_i \,\mathrm{d}x \leq -\beta < 0, t \geq t_0.$$

This yields $\sum_{i=1}^{m} \Lambda_i u_i(x,t) \to -\infty$, which is a contradiction. □

Now, we turn to the proof of Proposition 2.9.

Proof of Proposition 2.9. Note that, by periodicity of f, $\int_{\mathbb{T}^N} Df = 0$. Therefore, by using Jensen's inequality and the periodicity of f, we always have that $\int_{\mathbb{T}^N} H(Df)\,\mathrm{d}x \geq 0$.

We now assume by contradiction that such a β does not exist; therefore, we can find a sequence $(f_n) \in A$ such that

$$\int_{\mathbb{T}^N} H(Df_n)\,\mathrm{d}x < \frac{1}{n}.$$

Ascoli's theorem claims, by passing to a subsequence if necessary, the existence of $f_0 \in W^{1,\infty}(\mathbb{T}^N)$ such that

$$f_n \to f_0 \text{ in } C(\mathbb{T}^N). \tag{2.13}$$

In particular, f_0 is not a constant because $\max_{\mathbb{T}^N} f_0 - \min_{\mathbb{T}^N} f_0 \geq 1$.

Moreover, since $W^{1,2}(\mathbb{T}^N)$ is a reflexive Banach space, by passing to a subsequence if necessary, we have

$$f_n \rightharpoonup g_0 \text{ in } W^{1,2}(\mathbb{T}^N). \tag{2.14}$$

From (2.13) and (2.14), we obtain

$$g_0 = f_0 \text{ a.e.}$$

It then follows

$$0 \leq \int_{\mathbb{T}^N} H(Df_0) \leq \liminf_n \int_{\mathbb{T}^N} H(Df_n) \leq 0.$$

It yields

$$\int_{\mathbb{T}^N} H(Df_0) = 0.$$

We apply Jensen's inequality to deduce that $Df_0(x) = 0$ a.e., and therefore f_0 is constant by continuity. It leads to a contradiction. □

Recall that the semi-concavity yields the twice differentiability a.e. of a function defined on a subset of finite-dimensional spaces. We say that g is semi-concave with constant $M > 0$ if for any $y, h \in \mathbb{R}^N$, we have

$$g(y+h) - 2g(y) + g(y-h) \le M|h|^2. \tag{2.15}$$

We turn to the semi-concavity of solutions for systems of type (2.5). More precisely, we consider the following system

$$\begin{cases} -\mathrm{trace}(A_i D^2 u_i(x)) + H_i(Du_i) + \sum_{j=1}^m d_{ij} u_j = f_i(x) & (x,t) \in \mathbb{T}^N \times (0,+\infty), \\ u_i(x,0) = u_{0i}(x) & x \in \mathbb{T}^N. \end{cases} \tag{2.16}$$

For more results about the semiconcavity of solutions, we refer the reader to [7], [5], [3] and the references therein.

Proposition 2.10. *Let $A_i \ge 0$ be constant matrices for $i \in \{1,\dots,m\}$, D satisfies (2.2) with $\inf(D) > 0$. Assume that $(u_1,\dots,u_m)$ is a bounded continuous solution to (2.16). If f_i is semi-concave with constant M_i, then u is semi-concave with constant $M = \dfrac{\max\{M_1,\dots,M_m\}}{\inf(D)}$.*

Proof of Proposition (2.10)**.** *Step 1. Test function*

The goal is to show that, for any $y, h \in \mathbb{R}^N$ and $i = 1,\dots,m$, we have

$$u_i(y+h) - 2u_i(y) + u_i(y-h) - M|h|^2 \le 0.$$

Assume that this is not the case, hence there exist for instance $y_{\max}, h_{\max} \in \mathbb{R}^N$ such that

$$u_1(y_{\max} + h_{\max}) - 2u_1(y_{\max}) + u_1(y_{\max} - h_{\max}) - M|h_{\max}|^2 > 0. \tag{2.17}$$

Consider

$$\Phi(i,x,y,z) = u_i(x) - 2u_i(y) + u_i(z) - \frac{\alpha|x-2y+z|^2}{2} - \frac{M|x-y|^2}{2} - \frac{M|y-z|^2}{2}.$$

Set $M(x,y,z) = \dfrac{\alpha|x-2y+z|^2}{2} + \dfrac{M|x-y|^2}{2} + \dfrac{M|y-z|^2}{2}$ and let $(i, x_\alpha, y_\alpha, z_\alpha) \in \{1,\dots,m\} \times (\mathbb{T}^N)^3$ be the point such that $\Phi(i,x_\alpha,y_\alpha,z_\alpha) = \max_{(j,x,y,z)\in\{1,\dots,m\}\times(\mathbb{R}^N)^3} \Phi(j,x,y,z)$. Now, for any sequence $\alpha \to \infty$, since $\Phi(1, y+h, y, y-h) \le \Phi(i, x_\alpha, y_\alpha, z_\alpha)$ for any $y, h \in \mathbb{R}^N$, we then have

$$\begin{aligned} 0 &< u_1(y_{\max}+h_{\max}) - 2u_1(y_{\max}) + u_1(y_{\max}-h_{\max}) - M|h_{\max}|^2 \le \Phi(i,x_\alpha,y_\alpha,z_\alpha) \\ &\le u_i(x_\alpha) - 2u_i(y_\alpha) + u_i(z_\alpha) - \frac{M|x_\alpha - y_\alpha|^2}{2} - \frac{M|y_\alpha - z_\alpha|^2}{2}. \end{aligned} \tag{2.18}$$

Step 2. Writing viscosity inequalities. Since $(u_1,\dots,u_m)$ is a viscosity solution to (2.16), [[6], Theorem 3.2] claims that, for every $\alpha > 1$, we have the symmetric matrices X, Y, Z such that

$$\begin{aligned} &(D_x M, X) \in \overline{J}^{2,+} u_i(x_\alpha), \quad (-D_y M/2, -Y/2) \in \overline{J}^{2,-} u_i(y_\alpha), \qquad (2.19) \\ &(D_z M, Z) \in \overline{J}^{2,+} u_i(z_\alpha), \\ &-(\alpha^2 + |\mathcal{A}|)I \le \begin{pmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{pmatrix} \le \mathcal{A} + \frac{1}{\alpha^2}\mathcal{A}^2, \quad \mathcal{A} = D^2 M(x_\alpha, y_\alpha, z_\alpha). \qquad (2.20) \end{aligned}$$

We have

$$\begin{cases} -\mathrm{trace}(A_i X) + H_i(\alpha(x_\alpha - 2y_\alpha + z_\alpha) + M(x_\alpha - y_\alpha)) + \sum_{j=1}^m d_{ij} u_j(x_\alpha) \le f_i(x_\alpha), \\ -\mathrm{trace}(A_i Z) + H_i(\alpha(x_\alpha - 2y_\alpha + z_\alpha) + M(z_\alpha - y_\alpha)) + \sum_{j=1}^m d_{ij} u_j(z_\alpha) \le f_i(z_\alpha), \\ -\mathrm{trace}[A_i \frac{-Y}{2}] + H_i(\alpha(x_\alpha - 2y_\alpha + z_\alpha) + \frac{M}{2}(x_\alpha - y_\alpha) + \frac{M}{2}(z_\alpha - y_\alpha)) + \sum_{j=1}^m d_{ij} u_j(y_\alpha) \ge f_i(y_\alpha). \end{cases}$$

Adding the three above inequalities and noting that H is convex, we have

$$\sum_{j=1}^m d_{ij}[u_j(x_\alpha) + u_j(z_\alpha) - 2u_j(y_\alpha)] - \mathrm{trace}[A_i X + A_i Z + A_i Y] \le f_i(x_\alpha) + f_i(z_\alpha) - 2f_i(y_\alpha).$$

Since $u_i(x_\alpha) + u_i(z_\alpha) - 2u_i(y_\alpha) \ge u_j(x_\alpha) + u_j(z_\alpha) - 2u_j(y_\alpha)$ for any $j = 1,\dots,m$, it follows

$$(\sum_{j=1}^{m} d_{ij})[u_i(x_\alpha)+u_i(z_\alpha)-2u_i(y_\alpha)] - \operatorname{trace}[A_iX + A_iZ + A_iY] \le f_i(x_\alpha)+f_i(z_\alpha)-2f_i(y_\alpha).$$

Since X, Y, Z satisfy (2.20), we have $\operatorname{trace}(AX+AY+AZ)\le 0$. Hence we get

$$\inf(D)(u_i(x_\alpha)+u_i(z_\alpha)-2u_i(y_\alpha)) \le f_i(x_\alpha)+f_i(z_\alpha)-2f_i(y_\alpha). \tag{2.21}$$

Now since $\Phi(i,x_\alpha,y_\alpha,z_\alpha)\ge \Phi(i,0,0,0)=0$, we have

$$\frac{\alpha|x_\alpha-2y_\alpha+z_\alpha|^2}{2} + \frac{M|x_\alpha-y_\alpha|^2}{2} + \frac{M|y_\alpha-z_\alpha|^2}{2} \le C.$$

By passing to a subsequence if necessary, we can assume that as $\alpha\to\infty$

$$y_\alpha \to y_0, x_\alpha - y_\alpha \to h_0, z_\alpha - y_\alpha \to h_0.$$

Hence,

$$\limsup_{\alpha\to\infty} \Phi(i,x_\alpha,y_\alpha,z_\alpha) \le u_i(y_0+h_0)-2u_i(y_0)+u_i(y_0-h_0)-c|h_0|^2 = \Phi(i,y_0+h_0,y_0,y_0-h_0).$$

This implies

$$\frac{\alpha|x_\alpha-2y_\alpha+z_\alpha|^2}{2} \to 0 \text{ as } \alpha\to\infty.$$

By subtracting $\dfrac{M \quad \inf(D)|x_\alpha-y_\alpha|^2}{2} + \dfrac{M \quad \inf(D)|y_\alpha-z_\alpha|^2}{2}$ to both sides of (2.21), we let $\alpha\to\infty$ to have

$$\limsup_{\alpha\to\infty}(u_i(x_\alpha)+u_i(z_\alpha)-2u_i(y_\alpha) - \frac{M|x_\alpha-y_\alpha|^2}{2} - \frac{M|y_\alpha-z_\alpha|^2}{2}) \le 0,$$

which is in clear contradiction with (2.18). □

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