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Some extension groups between exponential functors

Quelques groupes d'extensions entre foncteurs exponentiels

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ABSTRACT

Let $\mathcal{F}$ be the category of functors that send a finite-dimensional vector space over $\mathbb{F}_2$ to a vector space over $\mathbb{F}_2$. In this note, we describe the first extension groups between some exponential functors such as $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{S}^*, \Lambda^*)$, $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{S}^*_4, \Lambda^*)$, and $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{S}^*_4, \mathrm{S}^*_4)$, where S^*, Λ^*, S^*_4 are the symmetric power, the exterior power, and the truncated symmetric power at the power 4, respectively. Three main techniques are used: tri-graded Hopf algebra structure of the extension groups between two exponential functors, the polynomial filtration of these functors, and the hypercohomology spectral sequences.

RÉSUMÉ

Soit $\mathcal{F}$ la catégorie des foncteurs depuis la catégorie des $\mathbb{F}_2$-espaces vectoriels de dimension finie vers celle des $\mathbb{F}_2$-espaces vectoriels. Dans cette note, nous décrivons les premiers groupes d'extensions entre certains foncteurs exponentiels tels que $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{S}^*, \Lambda^*)$, $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{S}^*_4, \Lambda^*)$ et $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{S}^*_4, \mathrm{S}^*_4)$, où S^*, Λ^*, S^*_4 sont respectivement la puissance symétrique, la puissance extérieure et la puissance symétrique tronquée à la puissance 4. Trois techniques principales sont utilisées : la structure d'algèbre de Hopf tri-graduée des groupes d'extensions entre deux foncteurs exponentiels, la filtration polynomiale de ces foncteurs et les suites spectrales d'hypercohomologie.

1. Introduction

Let $\mathcal{F}$ be the category of functors that send a finite-dimensional vector space over $\mathbb{F}_2$ to a vector space over $\mathbb{F}_2$. Recall that a graded functor E^* is called “exponential” if there are two natural (graded) isomorphisms $E^*(V \oplus W) \cong E^*(V) \otimes E^*(W)$ and $E^0(V) \cong \mathbb{F}_2$ (see [2]). This isomorphism equips E^* with a canonical Hopf algebra structure. Let A^* and B^* be two exponential functors. Then $\mathrm{Ext}^*_{\mathcal{F}}(A^*, B^*)$ becomes a tri-graded Hopf algebra and $\mathrm{Hom}_{\mathcal{F}}(A^*, B^*)$ a sub-Hopf algebra (see [3]). We note that the product structure of $\mathrm{Ext}^*_{\mathcal{F}}(A^*, B^*)$ (and $\mathrm{Hom}_{\mathcal{F}}(A^*, B^*)$) is not the Yoneda product. In Section 2, we

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describe the bi-graded Hopf algebra structure of $\mathrm{Hom}_{\mathcal{F}}(A^*, B^*)$, which will be used when we apply the theorem of Franjou, Friedlander, Scorichenko and Suslin [3, Theorem 1.7] to find a decomposition of $\mathrm{Ext}^1_{\mathcal{F}}(A^*, B^*)$.

It is deduced from a result of Milnor and Moore ([6, Theorem 4.4]) that

$$\mathrm{Ext}^*_{\mathcal{F}}(A^*, B^*) \cong \mathrm{Hom}_{\mathcal{F}}(A^*, B^*) \otimes \left[\mathbb{F}_2 \otimes_{\mathrm{Hom}_{\mathcal{F}}(A^*,B^*)} \mathrm{Ext}^*_{\mathcal{F}}(A^*, B^*)\right].$$

This means that $\mathrm{Ext}^*_{\mathcal{F}}(A^*, B^*)$ is a free module over $\mathrm{Hom}_{\mathcal{F}}(A^*, B^*)$. It follows that $\mathrm{Ext}^1_{\mathcal{F}}(A^*, B^*)$ is also free over $\mathrm{Hom}_{\mathcal{F}}(A^*, B^*)$. In this note, we describe a basis of $\mathrm{Ext}^1_{\mathcal{F}}(A^*, B^*)$ over $\mathrm{Hom}_{\mathcal{F}}(A^*, B^*)$ where A^* and B^* are chosen among S^*, Λ^*, S^*_4.

The following computations are the main results of this note.

Theorem 1.1. *As a module over* $\mathrm{Hom}_{\mathcal{F}}(\mathrm{S}^*, \Lambda^*)$, $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{S}^*, \Lambda^*)$ *is freely generated by two classes, which are the generators of the* 1*-dimensional* $\mathbb{F}_2$*-vector spaces* $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{S}^1, \Lambda^2)$ *and* $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{S}^2, \Lambda^2)$.

Theorem 1.2. *As a module over* $\mathrm{Hom}_{\mathcal{F}}(\mathrm{S}^*_4, \Lambda^*)$, $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{S}^*_4, \Lambda^*)$ *is freely generated by three classes, which are the generators of the* 1*-dimensional* $\mathbb{F}_2$*-vector spaces* $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{S}^1_4, \Lambda^2)$, $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{S}^4_4, \Lambda^1)$, *and* $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{S}^2_4, \Lambda^2)$.

Theorem 1.3. *As a module over* $\mathrm{Hom}_{\mathcal{F}}(\mathrm{S}^*_4, \mathrm{S}^*_4)$, $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{S}^*_4, \mathrm{S}^*_4)$ *is freely generated by four classes, which are the generators of the* 1*-dimensional* $\mathbb{F}_2$*-vector spaces* $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{S}^1_4, \mathrm{S}^4_4)$, $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{S}^4_4, \mathrm{S}^1_4)$, $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{S}^2_4, \mathrm{S}^4_4)$, *and* $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{S}^4_4, \mathrm{S}^2_4)$.

A motivation of this work comes from the decreasing filtration of the functor $V \mapsto K(2)^*(BV^\sharp)$, where $K(2)^*(-)$ is the second Morava K-theory at $p = 2$, V a finite-dimensional vector space over $\mathbb{F}_2$ and $BV^\sharp$ the classifying space of the dual vector space of V (see [7]). Each successive quotient of this filtration is isomorphic to S^k_4. To understand the functor $V \mapsto K(2)^*(BV^\sharp)$ in relation to its sub-objects and sub-quotients, it is necessary to study the extension group $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{S}^*_4, \mathrm{S}^*_4)$. The group $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{S}^*_4, \Lambda^*)$ appears when we use the hypercohomology spectral sequences to compute $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{S}^*_4, \mathrm{S}^*_4)$. In a similar way, we obtain the result about $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{S}^*, \Lambda^*)$. This is the case that was not considered in [3]. These computations should be useful to decide whether or not there exists, up to isomorphism, a unique indecomposable (in each degree) filtered functor having the same subquotients as $V \mapsto \widetilde{K}(2)^*(BV^\sharp)$, where $\widetilde{K}(2)^*(-)$ is the reduced Morava K-theory. If one has an exponential structure on the functor, work of A. Touzé shows that this is true (see [8]). This result should be analogous to the well-known fact that the functor $V \mapsto \widetilde{K}(1)^*(BV^\sharp)$ is uniserial.

In order to prove the above theorems, let us recall that $\mathrm{S}^{2^{k_1}+\cdots+2^{k_s}}$ is a direct factor of $\mathrm{S}^{2^{k_1}} \otimes \cdots \otimes \mathrm{S}^{2^{k_s}}$ if $k_1, \ldots, k_s$ are distinct (the same thing happens to Λ^* and S^*_4). Because of this fact, if $m \neq 2^k$ or $n \neq 2^h$, all elements of $\mathrm{Ext}^1_{\mathcal{F}}(A^m, B^n)$ are decomposable. Hence, to understand $\mathrm{Ext}^1_{\mathcal{F}}(A^*, B^*)$, it suffices to compute the groups $\mathrm{Ext}^1_{\mathcal{F}}(A^{2^k}, B^{2^h})$. To do this, we use the hypercohomology spectral sequences associated with a certain complex that begins with B^{2^h}. This technique was first used by Franjou, Lannes, Schwartz (see [4]), then by Franjou and coworkers (see [2], [3]).

2. Bi-graded Hopf algebra structure over the hom-sets between exponential functors

Let A^* and B^* two exponential functors chosen among S^*, Λ^*, S^*_4. First, using Kuhn's techniques about the characteristic of a natural transformation from S^m to S^n (see [5, Lemma 6.15]), we can easily calculate the linear structure of $\mathrm{Hom}_{\mathcal{F}}(A^*, B^*)$. Then, using the definition of Hopf product, we obtain some algebraic relations on this basis. For example, the vector space $\mathrm{Hom}_{\mathcal{F}}(\mathrm{S}^*, \Lambda^*)$ is freely generated by $b_m : \mathrm{S}^m \to \Lambda^m$ for all $m \in \mathbb{N}$, where $b_m(x_1 \cdots x_m) = x_1 \wedge \cdots \wedge x_m$. Furthermore, when we consider the Hopf product on $\mathrm{Hom}_{\mathcal{F}}(\mathrm{S}^*, \Lambda^*)$, we always have two facts: $f^2 = 0$ for all $f \in \mathrm{Hom}_{\mathcal{F}}(\mathrm{S}^*, \Lambda^*)$, and $b_m = b_{2^{k_1}} \cdots b_{2^{k_s}}$ where $m = 2^{k_1} + \cdots + 2^{k_s}$ $(k_1 < \cdots < k_s)$. We deduce the following result:

Proposition 2.1. *As bi-graded Hopf algebras,* $\mathrm{Hom}_{\mathcal{F}}(\mathrm{S}^*, \Lambda^*) \cong \bigotimes_{k\in\mathbb{N}} \Lambda(b_{2^k})$*. Moreover, the coproduct is determined by* $\delta(b_m) = \sum_{i=0}^m b_i \otimes b_{m-i}$.

In the same way, we get the following result for $\mathrm{Hom}_{\mathcal{F}}(\mathrm{S}^*_4, \Lambda^*)$.

Proposition 2.2. *As bi-graded Hopf algebras,* $\mathrm{Hom}_{\mathcal{F}}(\mathrm{S}^*_4, \Lambda^*) \cong \bigotimes_{k\in\mathbb{N}} \Lambda(\tilde{b}_{2^k})$*, where* $\tilde{b}_{2^m} : \mathrm{S}^*_4 \to \Lambda^m$ *is induced by* b_m*. The coproduct is characterized by the Verschiebung morphism, which is determined by* $V(\tilde{b}_1) = 0$, $V(\tilde{b}_{2^k}) = \tilde{b}_{2^{k-1}}$ *for* $k \geq 1$.

For the case of $\mathrm{Hom}_{\mathcal{F}}(\mathrm{S}^*_4, \mathrm{S}^*_4)$, let us remark that $f : \mathrm{S}^m_4 \to \mathrm{S}^n_4$ is non-zero if and only if $m \leq n \leq 2m$. In this case, consider the morphism $b_{[m,n]} : \mathrm{S}^m_4 \to \mathrm{S}^n_4$ defined by $b_{[m,n]}(x_1 \cdots x_m) = \sum_{|I|=2m-n} x_I x^2_{\{1,\ldots,m\}\setminus I}$, where $x_L := x_{l_1} \cdots x_{l_s}$ and $x^2_L := x^2_{l_1} \cdots x^2_{l_s}$ for $L = \{l_1, \ldots, l_s\}$ (we also agree that $x_L = x^2_L = 1 \in \mathbb{F}_2$ if $L = \varnothing$).

Proposition 2.3. *As bi-graded Hopf algebras,*

$$\mathrm{Hom}_{\mathcal{F}}(S_4^*, S_4^*) \cong \bigotimes_{k\in\mathbb{N}} \Lambda(b_{[2^k,2^k]}) \otimes \bigotimes_{l\in\mathbb{N}} \Lambda(b_{[2^h,2^{h+1}]}).$$

Moreover, the coproduct is characterized by the Verschiebung morphism given by $V(b_{[1,1]}) = V(b_{[1,2]}) = 0$, $V(b_{[2^k,2^k]}) = b_{[2^{k-1},2^{k-1}]}$ *and* $V(b_{[2^k,2^{k+1}]}) = b_{[2^{k-1},2^k]}$ *for* $k \geq 1$.

3. Proof of Theorem 1.1

We first recall some notations about the hypercohomology spectral sequences (see [1, Chapter XVII] for more information). Let $\mathcal{C}$ be an abelian category that has enough injectives. Let C^* be a complex in $\mathcal{C}$ and $I^{*,*}$ a Cartan–Eilenberg injective resolution of C^*. Consider the bi-complex formed by applying the functor $\mathrm{Hom}_{\mathcal{C}}(A, -)$ to $I^{*,*}$, where $A \in \mathcal{C}$. Then the initial pages of the associated hypercohomology spectral sequences are given by $\mathbf{I}_1^{s,t} \cong \mathrm{Ext}^t_{\mathcal{C}}(A, C^s)$ and $\mathbf{II}_2^{s,t} \cong \mathrm{Ext}^s_{\mathcal{C}}(A, H^t(C^*))$. The differentials d_r of the r^{th} pages are of bi-degree $(r, 1-r)$.

We now describe the linear structure of $\mathrm{Ext}^1_{\mathcal{F}}(S^{2^k}, \Lambda^{2^h})$ for $k, h \in \mathbb{N}$.

Among the cases where $k = 0$ or $h = 0$ or $h > k \geq 1$, the only one that gives a non-vanishing result is the case $k = 0$ and $h = 1$ where $\mathrm{Ext}^1_{\mathcal{F}}(\mathrm{Id}, \Lambda^2) \cong \mathbb{F}_2$. We can prove this by using the polynomial filtration of the functor S^n, which was studied carefully in the work of A. Troesch [9]. In fact, among the successive quotients of the polynomial filtration of S^{2^k}, the one that has the highest degree is the cosocle Λ^{2^k}. The result is deduced from the fact that $\mathrm{Ext}^1_{\mathcal{F}}(\Lambda^i, \Lambda^j) = 0$ if $|i-j| \neq 1$.

For the case of $k > h \geq 1$, we can also prove that $\mathrm{Ext}^1_{\mathcal{F}}(S^{2^k}, \Lambda^{2^h})$ is zero. Consider the complex

$$\Lambda^*_{2^h} : 0 \to \Lambda^{2^h} \to \Lambda^{2^h-1} \otimes \Lambda^1 \to \cdots \to \Lambda^{2^{h-1}} \otimes \Lambda^{2^{h-1}} \to \cdots \to \Lambda^1 \otimes \Lambda^{2^h-1} \to \Lambda^{2^h} \to 0$$

where the differential from $\Lambda^i \otimes \Lambda^j$ to $\Lambda^{i-1} \otimes \Lambda^{j+1}$ is induced from the diagonal $\Lambda^i \to \Lambda^{i-1} \otimes \Lambda^1$ and the product $\Lambda^1 \otimes \Lambda^j \to \Lambda^{j+1}$. This complex is exact at all positions except the middle one, whose homology is $\Lambda^{2^{h-1}}$. We now study the hypercohomology spectral sequences where the initial pages are given by $\mathbf{I}_1^{s,t} = \mathrm{Ext}^t_{\mathcal{F}}(S^{2^k}, \Lambda^{2^h-s} \otimes \Lambda^s)$ and $\mathbf{II}_2^{s,t} = \mathrm{Ext}^s_{\mathcal{F}}(S^{2^k}, H^t(\Lambda^*_{2^h}))$. Using Proposition 2.1, it is clear that $\mathbf{II}_2^{0,*}$, $\mathbf{II}_2^{*,0}$ and $\mathbf{I}_1^{*,0}$ are null. It follows that the differential from $\mathbf{I}_1^{0,1} = \mathrm{Ext}^1_{\mathcal{F}}(S^{2^k}, \Lambda^{2^h})$ to $\mathbf{I}_1^{1,1} = \mathrm{Ext}^1_{\mathcal{F}}(S^{2^k}, \Lambda^{2^h-1} \otimes \Lambda^1)$ is injective. It is also easy to prove that $\mathbf{I}_1^{1,1}$ is isomorphic to $\mathrm{Ext}^1_{\mathcal{F}}(S^{2^k-1}, \Lambda^{2^h-1})$, which is included in $\mathrm{Ext}^1_{\mathcal{F}}(S^1 \otimes S^2 \otimes \cdots \otimes S^{2^{k-1}}, \Lambda^{2^h-1})$. Moreover, it follows from [3, Theorem 1.7] that $\mathrm{Ext}^1_{\mathcal{F}}(S^1 \otimes S^2 \otimes \cdots \otimes S^{2^{k-1}}, \Lambda^{2^h-1})$ is isomorphic to $\bigoplus_{i=0}^{k-1} \mathrm{Ext}^1_{\mathcal{F}}(S^{2^i}, \Lambda^{2^h+2^i-2^k})$, which is trivial because $2^h + 2^i - 2^k \leq 0$ for $0 \leq i \leq k-1$. So, $\mathbf{I}_1^{0,1}$ is null.

If $k = l \geq 1$, we prove by induction that $\mathrm{Ext}^1_{\mathcal{F}}(S^{2^k}, \Lambda^{2^k})$ is a 1-dimensional vector space. The first step can be easily checked. In order to compute $\mathrm{Ext}^1_{\mathcal{F}}(S^{2^k}, \Lambda^{2^k})$, we use the hypercohomology spectral sequences associated with the complex $\Lambda^*_{2^k}$ and we get an inclusion from $\mathrm{Ext}^1_{\mathcal{F}}(S^{2^k}, \Lambda^{2^k})$ into $\mathrm{Ext}^1_{\mathcal{F}}(S^{2^k}, \Lambda^{2^k-1} \otimes \Lambda^1) \cong \mathrm{Ext}^1_{\mathcal{F}}(S^{2^k-1}, \Lambda^{2^k-1})$. From the inductive hypothesis, we deduce that the dimension of last one is at most 1. Hence, $\mathrm{Ext}^1_{\mathcal{F}}(S^{2^k}, \Lambda^{2^k})$ is of dimension 1 because it is generated by the Hopf product $b_{2^k-2}\epsilon_{[2,2]}$, where $\epsilon_{[2,2]}$ is the generator of $\mathrm{Ext}^1_{\mathcal{F}}(S^2, \Lambda^2)$.

4. Proof of the Theorem 1.3

We first show that $\mathrm{Ext}^1_{\mathcal{F}}(S_4^{2^k}, S_4^{2^h})$ is null for positive numbers k, h such that $|k-h| \geq 2$. We use the polynomial filtration of $S_4^{2^h}$ which is induced by that of S^{2^h} (see [9, §1.5.3] or [7]).

Lemma 4.1. *The functor* $S_4^{2^h}$ *admits the polynomial filtration*

$$0 \subset F_0^h \subset F_1^h \subset \cdots \subset F^h_{2^{h-1}+1} = S_4^{2^h},$$

whose successive quotient F_i^h / F_{i-1}^h *is isomorphic to* $\Lambda^{2i-2} \otimes \Lambda^{2^{h-1}-i+1}$.

The vanishing of $\mathrm{Ext}^1_{\mathcal{F}}(S_4^{2^k}, S_4^{2^h})$ in the case under consideration follows from the fact that $\mathrm{Ext}^1_{\mathcal{F}}(\Lambda^i, \Lambda^j)$ is null if $|i-j| \geq 2$. Using this result, the remaining cases that we need to compute are $\mathrm{Ext}^1_{\mathcal{F}}(S_4^{2^{h+1}}, S_4^{2^h})$, $\mathrm{Ext}^1_{\mathcal{F}}(S_4^{2^h}, S_4^{2^h})$ and $\mathrm{Ext}^1_{\mathcal{F}}(S_4^{2^h}, S_4^{2^{h+1}})$.

In order to study the first two cases, we make use of the hypercohomology spectral sequences associated with the complex

$$(S_4)^*_{2^h} : 0 \to S_4^{2^h} \to S_4^{2^h-1} \otimes S_4^1 \to S_4^{2^h-2} \otimes S_4^2 \to S_4^{2^h-3} \otimes S_4^3 \to \cdots \to S_4^1 \otimes S_4^{2^h-1} \to S_4^{2^h} \to 0$$

where $H_1((S_4)^*_{2^h}) = H_2((S_4)^*_{2^h}) = 0$, $H_0((S_4)^*_{2^h}) \cong \Lambda^{2^{h-1}}$ and $H_3((S_4)^*_{2^h}) \cong \Lambda^{2^{h-1}-2} \otimes \Lambda^1$. It follows that part of the second hypercohomology spectral sequence can be deduced from $\mathrm{Ext}^1_{\mathcal{F}}(S^*_4, \Lambda^*)$. This group can be completely determined by the same method as in the previous section, and thus we obtain Theorem 1.2.

Using this result, we can easily show that $\mathrm{Ext}^1_{\mathcal{F}}(S_4^{2^{h+1}}, S_4^{2^h}) = 0$ if $h > 1$. The case $h = 1$ is reduced to the computation of $\mathrm{Ext}^1_{\mathcal{F}}(S^4, S^2)$ by using the short exact sequence $0 \to S^1 \to S^4 \to S^4_4 \to 0$.

Similarly, we can compute $\mathrm{Ext}^1_{\mathcal{F}}(S_4^{2^h}, S_4^{2^h})$ for $h \le 2$. The case where $h > 2$ is more complicated. We can find two independent generators of $\mathbf{I}_1^{0,1}$ and we have an inclusion from $\mathbf{I}_1^{0,1} = \mathrm{Ext}^1_{\mathcal{F}}(S_4^{2^h}, S_4^{2^h})$ into $\mathbf{I}_1^{1,1} = \mathrm{Ext}^1_{\mathcal{F}}(S_4^{2^h}, S_4^{2^h-1} \otimes S^1_4)$. The difficulty is that we want to prove that $\mathbf{I}_1^{0,1}$ is of dimension 2, but the dimension of $\mathbf{I}_1^{1,1}$ is 3. To solve this difficulty, we have to analyze the differential from $\mathbf{I}_1^{1,1}$ to $\mathbf{I}_1^{2,1}$, which is induced by the Hopf algebra structure of $\mathrm{Ext}^*_{\mathcal{F}}(S^*_4, S^*_4)$. It is non-trivial by an *ad hoc* argument. We then get the result.

For the case of $\mathrm{Ext}^1_{\mathcal{F}}(S_4^{2^h}, S_4^{2^{h+1}})$, we use the polynomial filtration of $S_4^{2^{h+1}}$. In detail, using the long exact sequences

$$\cdots \to \mathrm{Hom}_{\mathcal{F}}(S_4^{2^h}, \Lambda^{2i} \otimes \Lambda^{2^h-i}) \to \mathrm{Ext}^1_{\mathcal{F}}(S_4^{2^h}, F_i^{h+1}) \to \mathrm{Ext}^1_{\mathcal{F}}(S_4^{2^h}, F_{i+1}^{h+1}) \to \mathrm{Ext}^1_{\mathcal{F}}(S_4^{2^h}, \Lambda^{2i} \otimes \Lambda^{2^h-i}) \to \cdots$$

for $1 \le i \le 2^h$, we get an isomorphism of groups $\mathrm{Ext}^1_{\mathcal{F}}(S_4^{2^h}, S_4^{2^{h+1}}) \cong \mathrm{Ext}^1_{\mathcal{F}}(S_4^{2^h}, F_2^{h+1})$, where the latter can be computed using the Loewy structure of F_2^{h+1}.

5. Perspective

By considering an appropriate complex, we can reduce the problem of computing $\mathrm{Ext}^1_{\mathcal{F}}(S^*_{2^m}, S^*_{2^n})$ to $\mathrm{Ext}^1_{\mathcal{F}}(S^*_{2^m}, S^*_{2^{n-1}})$. So, if we are interested in this type of extension group, the first one that we have to study is $\mathrm{Ext}^1_{\mathcal{F}}(S^*_{2^m}, \Lambda^*)$. This can be computed by the same method as that described in the proof of Theorem 1.1 or 1.2.

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