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Partial differential equations

Stability estimate in the inverse scattering for a single quantum particle in an external short-range potential

Estimation de stabilité en diffusion inverse pour une particule quantique en présence d'un potentiel à courte portée

Mourad Bellassoued [a], Luc Robbiano [b]

[a] *Université de Tunis El Manar, École nationale d'ingénieurs de Tunis, LAMSIN, B.P. 37, 1002 Tunis, Tunisia*
[b] *Laboratoire de mathématiques, Université de Versailles – Saint-Quentin-en-Yvelines, 78035 Versailles, France*

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ABSTRACT

In this paper we consider the inverse scattering problem for the Schrödinger operator with short-range electric potential. We prove in dimension $n \geq 2$ that the knowledge of the scattering operator determines the electric potential and we establish Hölder-type stability in determining the short range electric potential.

RÉSUMÉ

Dans cet article, nous considérons le problème de la diffusion inverse pour l'opérateur de Schrödinger avec un potentiel électrique à courte portée. Nous prouvons en dimension $n \geq 2$ que la connaissance de l'opérateur de diffusion détermine le potentiel électrique et nous établissons une estimation de stabilité de type Hölder pour la détermination du potentiel électrique à courte portée.

1. Introduction and main results

This paper concerns inverse scattering problems for a large class of Hamiltonian with short-range electric potential. A single quantum particle in an external potential is described by the Hilbert space $L^2(\mathbb{R}^n)$ and the family of Schrödinger Hamiltonians:

$$H = -\frac{1}{2}\Delta + V(x), \quad x \in \mathbb{R}^n. \tag{1.1}$$

We suppose that the electric potential $V \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$, with the short-range condition

E-mail addresses: mourad.bellassoued@enit.utm.tn (M. Bellassoued), luc.robbiano@uvsq.fr (L. Robbiano).

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$$|V(x)| \le C \langle x\rangle^{-\delta},$$

for some $\delta > 1$, where $\langle x\rangle = (1+|x|^2)^{1/2}$. Then we define

$$\mathcal{V}_\delta = \left\{ V \in \mathcal{C}^1(\mathbb{R}^n), \quad |V(x)| \le C\langle x\rangle^{-\delta}, \quad \delta > 1 \right\}.$$

Let $H_0 = \frac{1}{2}\Delta$ be the free Hamiltonian. We consider two strongly continuous unitary groups: e^{-itH_0} generate the free dynamic of the system and e^{-itH} a perturbation of this free dynamic. The state $u \in L^2(\mathbb{R}^n)$ is said asymptotically free as $t \to \pm\infty$ if there exists $\psi_\pm \in L^2(\mathbb{R}^n)$ such that

$$\lim_{t\to\pm\infty} \|e^{-itH}u - e^{-itH_0}\psi_\pm\| = 0. \tag{1.2}$$

Here ψ_+ is the outgoing (resp. incoming) asymptotic of the state u. The condition (1.2) is equivalent to the following two conditions:

$$\lim_{t\to\pm\infty} \|e^{itH_0}e^{-itH}u - \psi_\pm\| = 0, \quad \lim_{t\to\pm\infty} \|e^{itH}e^{-itH_0}\psi_\pm - u\| = 0.$$

The fundamental direct problems of scattering theory are: (a) to determine the set of asymptotically free states, i.e. the set of $u \in L^2(\mathbb{R}^n)$ such that

$$\lim_{t\to\pm\infty} e^{itH_0}e^{-itH}u = \psi_\pm$$

exist, (b) the condition of the scattering operator that maps the incoming ψ_- into the corresponding outgoing one ψ_+.

Let V be a short-range electric potential, by [10], Theorem 14.4.6, the wave operators, defined by

$$W_\pm(H, H_0)u = \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}u, \quad u \in L^2(\mathbb{R}^n)$$

exist as strong limits, are isometric operators, they intertwine the free and full Hamiltonian H and H_0:

$$W_\pm(H, H_0)H_0 = HW_\pm(H, H_0).$$

Their range is the projection of the space $L^2(\mathbb{R}^n)$ onto a continuous spectrum. Moreover, the wave operators $W_\pm(H_0, H)$ also exist and adjoint to $W_\pm(H, H_0)$. The scattering operator $S_V : \psi_- \mapsto \psi_+$ is defined as

$$S_V = W_+(H_0, H)W_-(H, H_0) = W_+(H, H_0)^* W_-(H, H_0).$$

It is well known that S_V is a unitary operator on $L^2(\mathbb{R}^n)$. We call $\mathcal{S}$ as a mapping from $\mathcal{V}_\delta$ into the set of bounded operators $\mathcal{L}(L^2(\mathbb{R}^n))$, $\mathcal{S}(V) = S_V$, the scattering map.

For $s > 0$, introducing the space $L^1_s(\mathbb{R}^n)$ be the weighted L^1 space in $\mathbb{R}^n$ with norm

$$\|u\|_{L^1_s(\mathbb{R}^n)} = \|\langle\cdot\rangle^s u\|_{L^1(\mathbb{R}^n)}.$$

The following is the main result of this paper.

Theorem 1.1. *Let $M > 0$, $\delta > n$ and $s \in (0, 1)$. There exist constants $C > 0$ and $\nu \in (0, 1)$ such that the following stability estimate holds*

$$\|V_1 - V_2\|_{H^{-1}(\mathbb{R}^n)} \le C\|S_{V_1} - S_{V_2}\|^\nu_{\mathcal{L}(L^2(\mathbb{R}^n))} \tag{1.3}$$

for every $V_1, V_2 \in \mathcal{V}_\delta$ such that $(V_1 - V_2) \in L^2(\mathbb{R}^n) \cap L^1_s(\mathbb{R}^n)$ and

$$\|V\|_{L^2(\mathbb{R}^n)} + \|V\|_{L^1_s(\mathbb{R}^n)} \le M. \tag{1.4}$$

In particular, the scattering map

$$\mathcal{S} : \mathcal{V}_\delta \longrightarrow \mathcal{L}(L^2(\mathbb{R}^n)), \quad V \longmapsto S_V,$$

is locally injective.

We describe now some previous results related with our problem. Let $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$. Define the unitary operator

$$\mathscr{F} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^+, L^2(\mathbb{S}^{n-1})), \quad \mathscr{F}(u)(\omega, \lambda) = 2^{-1/2}\lambda^{n-2/4}\hat{u}(\sqrt{\lambda}\omega),$$

where $L^2(\mathbb{R}^+, L^2(\mathbb{S}^{n-1}))$ denote the L^2-space of functions defined on $\mathbb{R}^+$ with value in $L^2(\mathbb{S}^{n-1})$. The spectral parameter λ plays the role of the energy of a quantum particle. Then,

$$\mathscr{F}(S_V u)(\lambda) = S_V(\lambda)\mathscr{F}(u)(\lambda).$$

The unitary operator $S_V(\lambda) : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{S}^{n-1})$ is called the scattering matrix at fixed energy λ with respect to the electric potential V.

The problem of identifying coefficients appearing in Schrödinger's equation was treated very well and there are many works that are relevant to this topic. In the case of a compactly supported electric potential and in dimension $n \geq 3$, uniqueness for the fixed energy scattering problem was given in [7,15,21,24]. In the earlier paper [23], this was done for small potentials. It is well known that, for compactly supported potentials, knowledge of the scattering amplitude (or the scattering matrix) at a fixed energy λ is equivalent to knowing the Dirichlet-to-Neumann map for the Schrödinger equation measured on the boundary of a large ball containing the support of the potential (see [29] for an account). Then the uniqueness result of Sylvester and Uhlmann [28] for the Dirichlet-to-Neumann map, based on special solutions called complex geometrical optics solutions, implies uniqueness at a fixed energy for compactly supported potentials. Melrose [14] proposed a related proof that uses the density of products of scattering solutions.

The uniqueness result with fixed energy was extended by Novikov to the case of exponentially decaying potentials [22]. Another proof applying arguments similar to the ones used for studying the Dirichlet-to-Neumann map was given in [30]. The fixed energy result for compactly supported potentials in the two-dimensional case follows from the corresponding uniqueness result for the Dirichlet-to-Neumann map of Bukhgeim [3], and this result was recently extended to potentials decaying faster than any Gaussian in [9].

We note that, in the absence of exponential decay for the potentials, there are counterexamples to uniqueness for inverse scattering at fixed energy. In two dimensions, Grinevich and Novikov [8] give a counterexample involving V in the Schwartz class, and in dimension three there are counterexamples with potentials decaying like $|x|^{-3/2}$ [16,26]. However, if the potentials have regular behavior at infinity (outside a ball they are given by convergent asymptotic sums of homogeneous functions in the radial variable), one still has uniqueness even in the magnetic case by the results of Weder and Yafaev [33,34] (see also Joshi and Sá Barreto [11,12]).

In the case of two-body Schrödinger Hamiltonians H with V short range, such a problem has been studied in [27] with high-frequency asymptotic methods. For short or long-range potentials, Enss and Weder [6] have used a geometrical method. They show that the potential is uniquely recovered by the high-velocity limit of the scattering operator. This method can be used to study Hamiltonians with electric and magnetic potentials on $L^2(\mathbb{R}^n)$, the Dirac equation, [8] and the N-body case [6]. In [19], Nicoleau used a stationary method to study Hamiltonians with smooth electric and magnetic potentials that have to be $\mathcal{C}^\infty$ functions with stronger decay assumption on higher derivatives, based on the construction of suitable modified wave operators (see also [17,18,20]). This approach gives the complete asymptotic expansion of the scattering operator at high energies. In [11], the author sees that the problem with obstacles can be treated in the same way by determining a class of test functions that have negligible interaction with the obstacle.

All the above-mentioned papers are concerned only with uniqueness or reconstruction formula of the coefficients. Inspired by the works of Arian [1], Enss [4,5], Enss and Weder [6], Jung [13], Weder [31,32] and following the same strategy as in [6], we prove in this paper stability estimates in the recovery of the unknown coefficient V via the scattering map.

The paper is organized as follows. In Section 2, we examine the scattering problem associated with (1.1), by using the geometric time-dependent method developed by Enss and Weder. In Section 3, we prove some intermediate estimate of the X-ray transform of the potential V. In Section 4, we estimate the X-ray transform and the Fourier transform of the potential, in terms of the scattering map and we prove Theorem 1.1.

2. Scattering map

Here we recall some basic definitions of the scattering theory used throughout the paper. The Fourier transform on functions in $\mathbb{R}^n$ is defined by

$$\hat{f}(\xi) := \mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx,$$

and the inverse Fourier transform is

$$f(x) = \mathcal{F}^{-1}(\hat{f})(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi.$$

For $s \geq 0$, letting $H^s(\mathbb{R}^n)$ stand for the standard Sobolev space of those measurable functions f whose Fourier transform $\hat{f}$ satisfies

$$\|f\|_{H^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \langle\xi\rangle^{2s} |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2} < \infty, \quad \langle\cdot\rangle = (1 + |\cdot|^2)^{1/2}.$$

For $\delta > 0$, introducing the Hilbert space $L^2_\delta(\mathbb{R}^n)$ as the weighted $L^2(\mathbb{R}^n)$ space in $\mathbb{R}^n$ with norm

$$\|u\|_{L^2_\delta(\mathbb{R}^n)} = \|\langle\cdot\rangle^\delta u\|_{L^2(\mathbb{R}^n)},$$

we see that the Fourier transform $\mathcal{F}$ is a unitary transformation from $H^s(\mathbb{R}^n)$ onto $L^2_s(\mathbb{R}^n)$, that is,

$$\|u\|_{L^2_\delta(\mathbb{R}^n)} = \|\mathcal{F}(u)\|_{H^\delta(\mathbb{R}^n)}, \quad \forall u \in \mathcal{S}(\mathbb{R}^n). \tag{2.1}$$

Let $\mathrm{e}^{-\mathrm{i}tH_0}$ be the Schrödinger propagator, in term of the Fourier transform, this is given by

$$\mathrm{e}^{-\mathrm{i}tH_0}u = \mathcal{F}^{-1}(\mathrm{e}^{-\mathrm{i}t\frac{|\xi|^2}{2}}\mathcal{F}(u))(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{e}^{-\mathrm{i}t\frac{|\xi|^2}{2}}\hat{u}(\xi)\,\mathrm{d}\xi. \tag{2.2}$$

We also record the following properties of the wave operators $W_\pm$

$$W_\pm^* W_\pm = I, \quad \mathrm{e}^{-\mathrm{i}tH}W_\pm = W_\pm \mathrm{e}^{-\mathrm{i}tH_0}. \tag{2.3}$$

By Duhamel's formula, we have

$$W_\pm = I + \mathrm{i}\int_0^{\pm\infty} \mathrm{e}^{\mathrm{i}tH}V\mathrm{e}^{-\mathrm{i}tH_0}\,\mathrm{d}t. \tag{2.4}$$

The proof of (2.4) proceeds by differentiation and subsequent integration: For $u \in \mathscr{D}(H_0) = \mathscr{D}(H)$ one has the product rule

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}u\right) &= \mathrm{e}^{\mathrm{i}tH}iH\,\mathrm{e}^{-\mathrm{i}tH_0}u - \mathrm{e}^{\mathrm{i}tH}iH_0\,\mathrm{e}^{-\mathrm{i}tH_0}u \\ &= \mathrm{i}\,\mathrm{e}^{\mathrm{i}tH}V\,\mathrm{e}^{-\mathrm{i}tH_0}u.\end{aligned}$$

This is now integrated to yield

$$\mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}u - u = \mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}sH}V\mathrm{e}^{-\mathrm{i}sH_0}u\,\mathrm{d}s,$$

from which (2.4) follows after taking the limit $t \to \infty$.

Then from (2.4), we find out that

$$(W_+ - W_-)u = \mathrm{i}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tH}V\mathrm{e}^{-\mathrm{i}tH_0}u\,\mathrm{d}t, \tag{2.5}$$

for any state $u \in L^2(\mathbb{R}^n)$ for which the integral is well defined. We have a similar formula for $W_\pm^*$

$$W_\pm^* = I + i\int_{\pm\infty}^0 \mathrm{e}^{\mathrm{i}tH_0}V\mathrm{e}^{-\mathrm{i}tH}\,\mathrm{d}t.$$

It follows from the definition of the scattering operators that

$$S_V - I = (W_+ - W_-)^* W_-.$$

Then by Duhamel's formula and the intertwining relation (2.3), we have the following identity, giving a relation between the scattering operator S_V and the potential V

$$i(S_V - I)u = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}tH_0}VW_-\mathrm{e}^{-\mathrm{i}tH_0}u\,\mathrm{d}t, \quad u \in L^2(\mathbb{R}^n). \tag{2.6}$$

We need some elementary facts about pseudo-differential operators defined by the equality

$$a(D)u(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi}a(\xi)\hat{u}(\xi)\,\mathrm{d}\xi, \quad \forall u \in \mathcal{S}(\mathbb{R}^n),$$

where the symbol $a \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. It is then known that, for any $a \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $a(D)$ is bounded operator on $L^2(\mathbb{R}^n)$.

For $\varrho \in \mathbb{R}^n$, we define the conjugate pseudo-differential operator $a_\varrho(D)$ by

$$a_\varrho(D) = \mathrm{e}^{-\mathrm{i}x\cdot\varrho} a(D) \mathrm{e}^{\mathrm{i}x\cdot\varrho} := a(D+\varrho). \tag{2.7}$$

The symbol of the operator $a_\varrho(D)$ is given by $a_\varrho(\xi) = a(\xi+\varrho)$. Indeed, using the fact that $\mathcal{F}(\mathrm{e}^{\mathrm{i}x\cdot\varrho}u)(\xi) = \hat{u}(\xi-\varrho)$, we get

$$a_\varrho(D)u(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot(\xi-\varrho)} a(\xi)\hat{u}(\xi-\varrho)\,\mathrm{d}\xi, \quad \forall u \in \mathcal{S}(\mathbb{R}^n).$$

We define the linear unitary operator E_ϱ^t from $L^2(\mathbb{R}^n)$ into itself by the integral representation

$$E_\varrho^t u(x) = \mathrm{e}^{-\mathrm{i}t\varrho\cdot D} u(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi} \mathrm{e}^{-\mathrm{i}t\varrho\cdot\xi} \hat{u}(\xi)\,\mathrm{d}\xi = u(x-t\varrho), \quad \forall u \in \mathcal{S}(\mathbb{R}^n).$$

Hence, we obtain the following identity

$$E_{-\varrho}^t\, w\, E_\varrho^t u(x) = w(x+t\varrho)u(x), \quad \forall w,\, u \in L^2(\mathbb{R}^n). \tag{2.8}$$

By a simple calculation, it is easy to see that

$$\mathrm{e}^{-\mathrm{i}x\cdot\varrho} \mathrm{e}^{-\mathrm{i}tH_0} \mathrm{e}^{\mathrm{i}x\cdot\varrho} = \mathrm{e}^{-\mathrm{i}t|\varrho|^2} E_\varrho^t \mathrm{e}^{-\mathrm{i}tH_0} \quad \text{in}\, L^2(\mathbb{R}^n). \tag{2.9}$$

Let us recall the following result proved in Reed and Simon [25], XI, p. 39. The key of the proof is the application of the stationary phase method.

Lemma 2.1. *Let $g \in \mathcal{S}(\mathbb{R}^n)$ be a function such that $\hat{g}$ has a compact support. Let $\mathcal{O}$ be an open set containing the compact Supp$(\hat{g})$. Let*

$$\tilde{g}_t(x) = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi} \mathrm{e}^{-\mathrm{i}\frac{t}{2}|\xi|^2} \hat{g}(\xi)\,\mathrm{d}\xi. \tag{2.10}$$

Then, for any $m \in \mathbb{N}$, there exists $C > 0$ depending on m, g, and Supp$(\hat{g})$, such that

$$|\tilde{g}_t(x)| \le C(1+|x|+|t|)^{-m},$$

for all x, t with $xt^{-1} \notin \mathcal{O}$.

In the sequel, for $t \in \mathbb{R}^*$ and $\varrho \in \mathbb{R}^n$, we denote by A_1 and A_2 the following sets

$$A_1 = \left\{ |x-t\varrho| > \frac{1}{2}|t\varrho| \right\}, \quad A_2 = \left\{ |x| < \frac{1}{4}|t\varrho| \right\}. \tag{2.11}$$

For a measurable set $A \subset \mathbb{R}^n$, we denote by κ_A the characteristic function of A.

Lemma 2.2. *Let $\varrho \in \mathbb{R}^n$ such that $|\varrho| > 4$, $t \in \mathbb{R}^*$, and let consider the two measurable sets A_1 and A_2 given by (2.11). Then, for any $a \in \mathcal{C}_0^\infty(B(0,1))$, and all $k \in \mathbb{N}$, there exists C such that*

$$\|\kappa_{A_1} \mathrm{e}^{-\mathrm{i}tH_0} a_{-\varrho}(D) \kappa_{A_2} u\|_{L^2(\mathbb{R}^n)} \le C \left\langle t\varrho \right\rangle^{-k} \|u\|_{L^2(\mathbb{R}^n)}, \tag{2.12}$$

for any $u \in L^2(\mathbb{R}^n)$. Here C depends only on k and n but does not depend on ϱ.

Proof. Let $a \in \mathcal{C}_0^\infty(B(0,1))$, A_1 and A_2 given by (2.11). For $u \in \mathcal{S}(\mathbb{R}^n)$, using (2.2) and (2.10), we easily see that

$$\begin{aligned}
\kappa_{A_1} \mathrm{e}^{-\mathrm{i}tH_0} a_{-\varrho}(D) \kappa_{A_2} u(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi} \mathrm{e}^{-\mathrm{i}\frac{t}{2}|\xi|^2} \kappa_{A_1}(x) a(\xi-\varrho) \mathcal{F}(\kappa_{A_2}u)(\xi)\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi} \mathrm{e}^{-\mathrm{i}\frac{t}{2}|\xi|^2} \kappa_{A_1}(x) a(\xi-\varrho) \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}y\cdot\xi} \kappa_{A_2}(y) u(y)\,\mathrm{d}y\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \kappa_{A_1}(x)\kappa_{A_2}(y) \tilde{a}_t^\varrho(x-y) u(y)\,\mathrm{d}y,
\end{aligned}$$

where the kernel $\tilde{a}_t^\varrho$ is given by

$$\tilde{a}_t^{\varrho}(z) = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}z\cdot\xi} \mathrm{e}^{-\mathrm{i}\frac{t}{2}|\xi|^2} a(\xi-\varrho)\,\mathrm{d}\xi.$$

Therefore, we have

$$\begin{aligned}
\|\kappa_{A_1}\mathrm{e}^{-\mathrm{i}tH_0}a_{-\varrho}(D)\kappa_{A_2}u\|^2_{L^2(\mathbb{R}^n)} &= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\kappa_{A_1}(x)\Big|\int_{\mathbb{R}^n}\kappa_{A_2}(y)\tilde{a}_t^{\varrho}(x-y)u(y)\,\mathrm{d}y\Big|^2\,\mathrm{d}x \\
&\le C\left(\int_{\mathbb{R}^n}\kappa_{A_1}(x)\left(\int_{\mathbb{R}^n}\kappa_{A_2}(y)|\tilde{a}_t^{\varrho}(x-y)|^2\,\mathrm{d}y\right)\mathrm{d}x\right)\|u\|^2_{L^2(\mathbb{R}^n)} \\
&\le \left(\int_{\mathbb{R}^n}|\tilde{a}_t^{\varrho}(z)|^2\left(\int_{\mathbb{R}^n}\kappa_{A_1}(x)\kappa_{A_2}(x-z)\,\mathrm{d}x\right)\mathrm{d}z\right)\|u\|^2_{L^2(\mathbb{R}^n)} \\
&\le \left(\int_{\mathbb{R}^n}|\tilde{a}_t^{\varrho}(z)|^2(\kappa_{A_1}*\check{\kappa}_{A_2})(z)\,\mathrm{d}z\right)\|u\|^2_{L^2(\mathbb{R}^n)}, \qquad (2.13)
\end{aligned}$$

where $\check{\kappa}_{A_2}(x) = \kappa_{A_2}(-x)$.

By a simple computation, we get

$$\begin{aligned}
\tilde{a}_t^{\varrho}(x) &= \mathrm{e}^{\mathrm{i}x\cdot\varrho}\mathrm{e}^{-\mathrm{i}\frac{t}{2}|\varrho|^2}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}(x-t\varrho)\cdot\xi}\mathrm{e}^{-\mathrm{i}\frac{t}{2}|\xi|^2}a(\xi)\,\mathrm{d}\xi \\
&:= \mathrm{e}^{\mathrm{i}x\cdot\varrho}\mathrm{e}^{-\mathrm{i}\frac{t}{2}|\varrho|^2}\tilde{g}_t(x-t\varrho), \qquad (2.14)
\end{aligned}$$

where

$$\tilde{g}_t(x) := \int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{e}^{-\mathrm{i}\frac{t}{2}|\xi|^2}a(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{e}^{-\mathrm{i}\frac{t}{2}|\xi|^2}\hat{g}(\xi)\,\mathrm{d}\xi$$

with $g = \mathcal{F}^{-1}(a)$. Thus, we arrive at

$$\begin{aligned}
\int_{\mathbb{R}^n}|\tilde{a}_t^{\varrho}(z)|^2(\kappa_{A_1}*\check{\kappa}_{A_2})(z)\,\mathrm{d}z &\le \int_{\mathbb{R}^n}|\tilde{g}_t(x)|^2(\kappa_{A_1}*\check{\kappa}_{A_2})(x+t\varrho)\,\mathrm{d}x \\
&\le \int_{\mathbb{R}^n}|\tilde{g}_t(x)|^2(\kappa_{A_1}*\check{\kappa}_{(A_2+t\varrho)})(x)\,\mathrm{d}x.
\end{aligned}$$

Since, $\mathrm{Supp}(\kappa_{A_1}*\check{\kappa}_{A_2+t\varrho}) \subset A_1-(A_2+t\varrho) \subset \left\{|x| \ge \frac{1}{4}|t\varrho|\right\}$, and

$$\|\kappa_{A_1}*\check{\kappa}_{A_2}\|_{L^\infty(\mathbb{R}^n)} \le \|\kappa_{A_2}\|_{L^1(\mathbb{R}^n)} \le C|t\varrho|^n,$$

the above, inserted in (2.13), yields the following inequality

$$\|\kappa_{A_1}\mathrm{e}^{-\mathrm{i}tH_0}a_{-\varrho}(D)\kappa_{A_2}u\|^2_{L^2(\mathbb{R}^n)} \le C|t\varrho|^n\left(\int_{\left\{|x|\ge\frac{1}{4}|t\varrho|\right\}}|\tilde{g}_t(x)|^2\,\mathrm{d}x\right)\|u\|^2_{L^2(\mathbb{R}^n)}. \qquad (2.15)$$

Let $r = \frac{1}{4}|t\varrho|$. In view of (2.14), we get from Lemma 2.1 for $m \in \mathbb{N}$, with $2m-k > 2n$, that

$$\begin{aligned}
|t\varrho|^n\int_{\left\{|x|\ge\frac{1}{4}|t\varrho|\right\}}|\tilde{g}_t(x)|^2\,\mathrm{d}x &\le Cr^n\int_{\{|x|\ge r\}}|\tilde{g}_t(x)|^2\,\mathrm{d}x \\
&\le C\,\langle r\rangle^{-k}\int_{\mathbb{R}^n}\langle x\rangle^{-2m+k+n}\,\mathrm{d}x, \qquad (2.16)
\end{aligned}$$

provided that $r > |t|$, which is satisfied if $|\varrho| > 4$.

Combining (2.16) and (2.15), we immediately deduce (2.12).

This completes the proof. □

Lemma 2.3. *Let $V \in \mathcal{V}_\delta$. Then, for any $a \in \mathcal{C}_0^\infty(B(0,1))$ and every $\varrho \in \mathbb{R}^n$, we have*

$$\|V\mathrm{e}^{-\mathrm{i}tH_0}a_{-\varrho}(D)u\|_{L^2(\mathbb{R}^n)} = \|V(x+t\varrho)\mathrm{e}^{-\mathrm{i}tH_0}a(D)\mathrm{e}^{-\mathrm{i}x\cdot\varrho}u\|_{L^2(\mathbb{R}^n)} \tag{2.17}$$

for any $u \in L^2(\mathbb{R}^n)$.

Proof. By a density argument, it is enough to consider (2.17) for $u \in \mathcal{S}(\mathbb{R}^n)$. By (2.7), we get

$$\|V\mathrm{e}^{-\mathrm{i}tH_0}a_{-\varrho}(D)u\|_{L^2(\mathbb{R}^n)} = \|V\mathrm{e}^{-\mathrm{i}tH_0}\mathrm{e}^{\mathrm{i}x\cdot\varrho}a(D)\mathrm{e}^{-\mathrm{i}x\cdot\varrho}u\|_{L^2(\mathbb{R}^n)}.$$

Using (2.8) and (2.9), we deduce that

$$\begin{aligned}\|V\mathrm{e}^{-\mathrm{i}tH_0}\mathrm{e}^{\mathrm{i}x\cdot\varrho}a(D)\mathrm{e}^{-\mathrm{i}x\cdot\varrho}u\|_{L^2(\mathbb{R}^n)} &= \|V\mathrm{e}^{\mathrm{i}x\cdot\varrho}E_\varrho^t\mathrm{e}^{-\mathrm{i}tH_0}a(D)\mathrm{e}^{-\mathrm{i}x\cdot\varrho}u\|_{L^2(\mathbb{R}^n)} \\ &= \|E_{-\varrho}^t V E_\varrho^t\mathrm{e}^{-\mathrm{i}tH_0}a(D)\mathrm{e}^{-\mathrm{i}x\cdot\varrho}u\|_{L^2(\mathbb{R}^n)} \\ &= \|V(x+t\varrho)\mathrm{e}^{-\mathrm{i}tH_0}a(D)\mathrm{e}^{-\mathrm{i}x\cdot\varrho}u\|_{L^2(\mathbb{R}^n)}.\end{aligned} \tag{2.18}$$

Thus we conclude the desired equality. □

Lemma 2.4. *Let $V \in \mathcal{V}_\delta$. Then for any $a \in \mathcal{C}_0^\infty(B(0,1))$, and every $\varrho \in \mathbb{R}^n$, $|\varrho| > 4$, we have*

$$\|V\mathrm{e}^{-\mathrm{i}tH_0}a_{-\varrho}(D)u\|_{L^2(\mathbb{R}^n)} = \|V(x+t\varrho)\mathrm{e}^{-\mathrm{i}tH_0}a(D)\mathrm{e}^{-\mathrm{i}x\cdot\varrho}u\|_{L^2(\mathbb{R}^n)} \leq C\left\langle t\varrho\right\rangle^{-\delta}\|u\|_{L^2_\delta(\mathbb{R}^n)}, \tag{2.19}$$

for any $u \in L^2_\delta(\mathbb{R}^n)$.

Proof. By a density argument, it is enough to consider (2.19) for $u \in \mathcal{S}(\mathbb{R}^n)$. Let A_1 and A_2 are given by (2.11). Then, we obtain

$$\begin{aligned}\|V\mathrm{e}^{-\mathrm{i}tH_0}a_{-\varrho}(D)u\|_{L^2(\mathbb{R}^n)} \leq &\|V\kappa_{A_1}\mathrm{e}^{-\mathrm{i}tH_0}a_{-\varrho}(D)u\|_{L^2(\mathbb{R}^n)} \\ &+ \|V\kappa_{A_1^c}\mathrm{e}^{-\mathrm{i}tH_0}a_{-\varrho}(D)u\|_{L^2(\mathbb{R}^n)} := I_1 + I_2.\end{aligned}$$

To estimate I_1, note that

$$I_1 \leq \|V\kappa_{A_1}\mathrm{e}^{-\mathrm{i}tH_0}a_{-\varrho}(D)\kappa_{A_2}u\|_{L^2(\mathbb{R}^n)} + \|V\kappa_{A_1}\mathrm{e}^{-\mathrm{i}tH_0}a_{-\varrho}(D)\kappa_{A_2^c}u\|_{L^2(\mathbb{R}^n)}.$$

Hence, by Lemma 2.2, we get

$$\begin{aligned}\|V\kappa_{A_1}\mathrm{e}^{-\mathrm{i}tH_0}a_{-\varrho}(D)\kappa_{A_2}u\|_{L^2(\mathbb{R}^n)} &\leq \|\kappa_{A_1}\mathrm{e}^{-\mathrm{i}tH_0}a_{-\varrho}(D)\kappa_{A_2}u\|_{L^2(\mathbb{R}^n)} \\ &\leq C\left\langle t\varrho\right\rangle^{-\delta}\|u\|_{L^2(\mathbb{R}^n)}.\end{aligned} \tag{2.20}$$

Furthermore, for any $u \in \mathcal{S}(\mathbb{R}^n)$, one has

$$\|V\kappa_{A_1}\mathrm{e}^{-\mathrm{i}tH_0}a_{-\varrho}(D)\kappa_{A_2^c}u\|_{L^2(\mathbb{R}^n)} \leq \|\kappa_{A_2^c}u\|_{L^2(\mathbb{R}^n)} \leq C\left\langle t\varrho\right\rangle^{-\delta}\|u\|_{L^2_\delta(\mathbb{R}^n)}. \tag{2.21}$$

Taking into account (2.20), (2.21), we see that

$$I_1 \leq C\left\langle t\varrho\right\rangle^{-\delta}\|u\|_{L^2_\delta(\mathbb{R}^n)}. \tag{2.22}$$

On the other hand, since $A_1^c \subset \left\{|x| \geq \frac{1}{2}|t\varrho|\right\}$ and $V \in \mathcal{V}_\delta$, we also have that

$$\begin{aligned}I_2 = \|V\kappa_{A_1^c}\mathrm{e}^{-\mathrm{i}tH_0}a_{-\varrho}(D)u\|_{L^2(\mathbb{R}^n)} &\leq C\left\langle t\varrho\right\rangle^{-\delta}\|\mathrm{e}^{-\mathrm{i}tH_0}a_{-\varrho}(D)u\|_{L^2(\mathbb{R}^n)} \\ &\leq C\left\langle t\varrho\right\rangle^{-\delta}\|u\|_{L^2(\mathbb{R}^n)}.\end{aligned} \tag{2.23}$$

Hence, by combining (2.23) and (2.22), we conclude the proof of the Lemma. □

Lemma 2.5. *Assume that $V \in \mathcal{V}_\delta$. Then, there exists $C > 0$ such that, for any $\Phi \in \mathcal{S}(\mathbb{R}^n)$ with $Supp(\hat{\Phi}) \subset B(0,1)$, we have*

$$\|(W_\pm - I)\mathrm{e}^{-\mathrm{i}tH_0}\Phi_\varrho\|_{L^2(\mathbb{R}^n)} \leq C|\varrho|^{-1}\|\Phi\|_{L^2_\delta(\mathbb{R}^n)},$$

for any $\varrho \in \mathbb{R}^n$, $|\varrho| > 4$, and uniformly for $t \in \mathbb{R}$. Here, $\Phi_\varrho = \mathrm{e}^{\mathrm{i}x\cdot\varrho}\Phi$.

Proof. It follows from Duhamel's formula (2.4) that

$$(W_+ - I)\mathrm{e}^{-\mathrm{i}tH_0}\Phi_\varrho = \mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}sH}V\mathrm{e}^{-\mathrm{i}sH_0}\mathrm{e}^{-\mathrm{i}tH_0}\Phi_\varrho\,\mathrm{d}s.$$

Take $a \in \mathcal{C}_0^\infty(B(0,1))$, such that $a(\xi-\varrho)\hat{\Phi}(\xi-\varrho) = \hat{\Phi}(\xi-\varrho)$, that is, $a_{-\varrho}(D)\Phi_\varrho = \Phi_\varrho$. Then, by Lemma 2.4, we get

$$\begin{aligned}\|(W_+ - I)\mathrm{e}^{-\mathrm{i}tH_0}\Phi_\varrho\|_{L^2(\mathbb{R}^n)} &\le \int_{-\infty}^{+\infty} \|V\mathrm{e}^{-\mathrm{i}sH_0}a_{-\varrho}(D)\Phi_\varrho\|_{L^2(\mathbb{R}^n)}\,\mathrm{d}s \\ &\le C\left(\int_{\mathbb{R}} \langle s\varrho\rangle^{-\delta}\,\mathrm{d}s\right)\|\Phi\|_{L^2_\delta(\mathbb{R}^n)} \\ &\le \frac{C}{|\varrho|}\left(\int_0^\infty \langle\tau\rangle^{-\delta}\,\mathrm{d}\tau\right)\|\Phi\|_{L^2_\delta(\mathbb{R}^n)},\end{aligned}$$

and the Lemma follows for W_+. The proof for W_- is similar. □

3. Stability of the X-ray transform of the potential

In this section, we prove some estimate for the X-ray transform of the electric potential V. We start with a preliminary property of the X-ray transform, which is needed to prove the main result.

Let $\omega \in \mathbb{S}^{n-1}$, and $\omega^\perp$ the hyperplane through the origin orthogonal to ω. We parametrize a line $\mathcal{L}(\omega, y)$ in $\mathbb{R}^n$ by specifying its direction $\omega \in \mathbb{S}^{n-1}$ and the point $y \in \omega^\perp$ where the line intersects the hyperplane $\omega^\perp$. The X-ray transform of function $f \in L^1(\mathbb{R}^n)$ is given by

$$X(f)(x,\omega) = X_\omega(f)(x) = \int_{\mathbb{R}} f(x+\tau\omega)\,\mathrm{d}\tau, \quad x \in \omega^\perp.$$

We see that $X(f)(x,\omega)$ is the integral of f over the line $\mathcal{L}(\omega, y)$ parallel to ω, which passes through $x \in \omega^\perp$. The following relation between the Fourier transform of $X_\omega(f)$ and f, called the Fourier slice theorem, will be useful: we denote by $\mathcal{F}_{\omega^\perp}$ the Fourier transform on the function in the hyperplane $\omega^\perp$. The Fourier slice theorem is summarized in the following identity (see [2]):

$$\begin{aligned}\mathcal{F}_{\omega^\perp}(X_\omega(f))(\eta) &= (2\pi)^{(1-n)/2}\int_{\omega^\perp} \mathrm{e}^{-\mathrm{i}x\cdot\eta}X_\omega(f)(x)\,\mathrm{d}x \\ &= (2\pi)^{(1-n)/2}\int_{\omega^\perp} \mathrm{e}^{-\mathrm{i}x\cdot\eta}\int_{\mathbb{R}} f(x+s\omega)\,\mathrm{d}s\,\mathrm{d}x \\ &= (2\pi)^{(1-n)/2}\int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}y\cdot\eta}f(y)\,\mathrm{d}y \\ &= \sqrt{2\pi}\mathcal{F}(f)(\eta), \quad \eta \in \omega^\perp. \end{aligned} \tag{3.1}$$

The main purpose here is to present a preliminary estimate, which relates the difference of the short-range potentials to the scattering map. As before, we let $V_1, V_2 \in \mathcal{V}_\delta$, $j = 1, 2$ be real valued potentials. We set:

$$V = V_1 - V_2,$$

such that

$$\|V\|_{L^2(\mathbb{R}^n)} \le M.$$

We start with the following lemma.

Lemma 3.1. *Let $V_j \in \mathcal{V}_\delta$, $j = 1, 2$. Then there exist $C > 0$, $\lambda_0 > 0$ and $\gamma \in (0,1)$ such that, for any $\omega \in \mathbb{S}^{n-1}$ and $\Phi, \Psi \in \mathcal{S}(\mathbb{R}^n)$ with Supp$(\hat{\Phi})$, Supp$(\hat{\Psi}) \subset B(0,1)$, the following estimate holds true*

$$|\int_{\mathbb{R}^n} X(V)(x,\omega)\Phi(x)\overline{\Psi}(x)\,dx| \le C\sqrt{\lambda}\|S_{V_1}-S_{V_2}\|\|\Phi\|_{L^2(\mathbb{R}^n)}\|\Psi\|_{L^2(\mathbb{R}^n)} + \lambda^{-\gamma/2}\left(\|\Phi\|_{H^2(\mathbb{R}^n)}+\|\Phi\|_{L^2_\delta(\mathbb{R}^n)}\right)\left(\|\Psi\|_{H^2(\mathbb{R}^n)}+\|\Psi\|_{L^2_\delta(\mathbb{R}^n)}\right) \tag{3.2}$$

for any $\lambda > \lambda_0$. *Here* $V = V_1 - V_2$.

Proof. Let $\varrho = \sqrt{\lambda}\omega$ with $\lambda > 0$ and $\omega \in \mathbb{S}^{n-1}$. In what follows for Φ, $\Psi \in \mathcal{S}(\mathbb{R}^n)$ with $\mathrm{Supp}(\hat{\Phi})$, and $\mathrm{Supp}(\hat{\Psi}) \subset B(0,1)$, we denote

$$\Phi_\varrho = e^{ix\cdot\varrho}\Phi, \quad \Psi_\varrho = e^{ix\cdot\varrho}\Psi, \quad \varrho = \sqrt{\lambda}\omega.$$

From the identity (2.6) of the wave and scattering operators, it is easily seen that

$$\sqrt{\lambda}\left(i(S_{V_j}-I)\Phi_\varrho,\Psi_\varrho\right) = \int_{\mathbb{R}} \ell_j(\tau,\lambda,\omega)\,d\tau + R_j(\lambda,\omega), \quad j=1,2, \tag{3.3}$$

where the leading ℓ_j is given by

$$\ell_j(\tau,\lambda,\omega) := \left(V_j e^{-i\tau\lambda^{-1/2}H_0}\Phi_\varrho, e^{-i\tau\lambda^{-1/2}H_0}\Psi_\varrho\right),$$

and the remaining term R_j is given by

$$R_j(\lambda,\omega) = \int_{\mathbb{R}} \left((W_-^j - I)e^{-i\tau\lambda^{-1/2}H_0}\Phi_\varrho, V_j e^{-i\tau\lambda^{-1/2}H_0}\Psi_\varrho\right)d\tau.$$

At first, we estimate the remaining term R_j. Let $a \in C_0^\infty(B(0,1))$ such that $a(\xi)\hat{\Phi}(\xi) = \hat{\Phi}(\xi)$ and $a(\xi)\hat{\Psi}(\xi) = \hat{\Psi}(\xi)$. Lemma 2.4 gives uniformly in λ the integral bound

$$\|V_j e^{-i\tau\lambda^{-1/2}H_0}\Phi_\varrho\| = \|V_j e^{-i\tau\lambda^{-1/2}H_0}a_{-\varrho}(D)\Phi_\varrho\| = \|V_j(x+\tau\omega)e^{-i\tau\lambda^{-1/2}H_0}a(D)\Phi\| \le C\langle\tau\rangle^{-\delta}\|\Phi\|_{L^2_\delta(\mathbb{R}^n)}. \tag{3.4}$$

Similarly, we get

$$\|V_j e^{-i\tau\lambda^{-1/2}H_0}\Psi_\varrho\| \le C\langle\tau\rangle^{-\delta}\|\Psi\|_{L^2_\delta(\mathbb{R}^n)}. \tag{3.5}$$

By Lemma 2.5 and (3.5), we obtain

$$|R_j(\lambda,\omega)| \le C\int_{\mathbb{R}} \|(W_-^j - I)e^{-i\tau\lambda^{-1/2}H_0}\Phi_\varrho\|\|V_j e^{-i\tau\lambda^{-1/2}H_0}\Psi_\varrho\|\,d\tau \le \frac{C}{\sqrt{\lambda}}\|\Phi\|_{L^2_\delta(\mathbb{R}^n)}\|\Psi\|_{L^2_\delta(\mathbb{R}^n)}.$$

Thus $R_j(\lambda,\omega)$ satisfies the remaining estimate in (3.2).

We consider now the leading term ℓ_j. Taking into account (2.8), (2.9) a simple calculation gives

$$\ell_j(\tau,\lambda,\omega) = \left(V_j(x+\tau\omega)e^{-i\tau\lambda^{-1/2}H_0}\Phi, e^{-i\tau\lambda^{-1/2}H_0}\Psi\right),$$

and therefore

$$\ell_j(\tau,\lambda,\omega) - \left(V_j(x+\tau\omega)\Phi,\Psi\right) = \ell_j^{(1)}(\tau,\lambda,\omega) + \ell_j^{(2)}(\tau,\lambda,\omega)$$

where

$$\ell_j^{(1)}(\tau,\lambda,\omega) = \left(V_j(x+\tau\omega)e^{-i\tau\lambda^{-1/2}H_0}\Phi, (e^{-i\tau\lambda^{-1/2}H_0} - I)\Psi\right),$$

and

$$\ell_j^{(2)}(\tau,\lambda,\omega) = \left((e^{-i\tau\lambda^{-1/2}H_0} - I)\Phi, V_j(x+\tau\omega)\Psi\right).$$

Since $\hat{\Psi}$ has compact support, we obtain:

$$\left(e^{-i\tau\lambda^{-1/2}H_0} - I\right)\Psi = \int_0^{\tau/\sqrt{\lambda}} \frac{d}{ds}(e^{-isH_0}\Psi)\,ds = -i\int_0^{\tau/\sqrt{\lambda}} e^{-isH_0}H_0\Psi\,ds.$$

Therefore, we have

$$\|(e^{-i\tau\lambda^{-1/2}H_0}-I)\Psi\|_{L^2(\mathbb{R}^n)} \le \frac{|\tau|}{\sqrt{\lambda}}\|H_0\Psi\|_{L^2(\mathbb{R}^n)} \le \frac{|\tau|}{\sqrt{\lambda}}\|\Psi\|_{H^2(\mathbb{R}^n)},$$

and using the fact that

$$\|(e^{-i\tau\lambda^{-1/2}H_0}-I)\Psi\|_{L^2(\mathbb{R}^n)} \le 2\|\Psi\|_{L^2(\mathbb{R}^n)},$$

we deduce the following estimation

$$\|(e^{-i\tau\lambda^{-1/2}H_0}-I)\Psi\|_{L^2(\mathbb{R}^n)} \le C\left(\frac{|\tau|}{\sqrt{\lambda}}\right)^{\gamma}\|\Psi\|_{H^2(\mathbb{R}^n)}, \tag{3.6}$$

for all $\gamma\in(0,1)$. Then by (3.6) and (3.4), we find

$$|\ell_j^{(1)}(\tau,\lambda,\omega)| \le \frac{C}{\lambda^{\gamma/2}}\langle\tau\rangle^{-(\delta-\gamma)}\|\Psi\|_{H^2(\mathbb{R}^n)}\|\Phi\|_{L^2_\delta(\mathbb{R}^n)}.$$

Hence, by selecting γ small such that $\delta-\gamma>1$, it follows that

$$\int_{\mathbb{R}}|\ell_j^{(1)}(\tau,\lambda,\omega)|\,d\tau \le \frac{C}{\lambda^{\gamma/2}}\|\Psi\|_{H^2(\mathbb{R}^n)}\|\Phi\|_{L^2_\delta(\mathbb{R}^n)}. \tag{3.7}$$

Moreover, we have

$$\begin{aligned}\int_{\mathbb{R}^n}|V_j(x+t\omega)\Psi(x)|^2\,dx &\le C\int_{\left\{|x+\tau\omega|>\frac12|\tau|\right\}}\langle x+\tau\omega\rangle^{-2\delta}|\Psi(x)|^2\,dx\\ &\quad+\int_{\left\{|x+\tau\omega|\le\frac12|\tau|\right\}}\langle x+\tau\omega\rangle^{-2\delta}|\Psi(x)|^2\,dx\\ &\le C\left(\langle\tau\rangle^{-2\delta}\int_{\mathbb{R}^n}|\Psi(x)|^2\,dx+\langle\tau\rangle^{-2\delta}\int_{\mathbb{R}^n}\langle x\rangle^{2\delta}|\Psi(x)|^2\,dx\right)\le C\langle\tau\rangle^{-2\delta}\|\Psi\|^2_{L^2_\delta(\mathbb{R}^n)}.\end{aligned}$$

Then we show as the proof of (3.7) that

$$\begin{aligned}\int_{\mathbb{R}}|\ell_j^{(2)}(\tau,\lambda,\omega)|\,d\tau &\le \int_{\mathbb{R}}\|(e^{-i\tau\lambda^{-1/2}H_0}-I)\Phi\|_{L^2(\mathbb{R}^n)}\|V_j(x+\tau\omega)\Psi\|_{L^2(\mathbb{R}^n)}\,d\tau\\ &\le\frac{C}{\lambda^{\gamma/2}}\left(\int_{\mathbb{R}}\langle\tau\rangle^{-(\delta-\gamma)}\,d\tau\right)\|\Phi\|_{H^2(\mathbb{R}^n)}\|\Psi\|_{L^2_\delta(\mathbb{R}^n)}.\end{aligned}$$

Then, we easily see that

$$\int_{\mathbb{R}}|\ell_j^{(1)}(\tau,\lambda,\omega)|\,d\tau+\int_{\mathbb{R}}|\ell_j^{(2)}(\tau,\lambda,\omega)|\,d\tau \le \frac{C}{\lambda^{\gamma/2}}\left(\|\Psi\|_{H^2(\mathbb{R}^n)}\|\Phi\|_{L^2_\delta(\mathbb{R}^n)}+\|\Phi\|_{H^2(\mathbb{R}^n)}\|\Psi\|_{L^2_\delta(\mathbb{R}^n)}\right). \tag{3.8}$$

From (3.8) and (3.3), we deduce that

$$i\sqrt{\lambda}\left((S_{V_1}-S_{V_2})\Phi_\varrho,\Psi_\varrho\right)=\int_{\mathbb{R}}(\ell_1-\ell_2)(\tau,\lambda,\omega)\,d\tau+(R_1-R_2)(\lambda,\omega):=\int_{\mathbb{R}}\ell(\tau,\lambda,\omega)\,d\tau+R(\lambda,\omega),$$

where the leading and remaining terms, respectively, satisfy

$$\begin{aligned}\left|\int_{\mathbb{R}}(\ell(\tau,\lambda,\omega)-(V(x+\tau\omega)\Phi,\Psi))\,d\tau\right| &\le\sum_{j=1}^{2}\int_{\mathbb{R}}\left(|\ell_j^{(1)}(\tau,\lambda,\omega)|+|\ell_j^{(2)}(\tau,\lambda,\omega)|\right)d\tau\\ &\le\frac{C}{\lambda^{\gamma/2}}\left(\|\Psi\|_{H^2(\mathbb{R}^n)}\|\Phi\|_{L^2_\delta(\mathbb{R}^n)}+\|\Phi\|_{H^2(\mathbb{R}^n)}\|\Psi\|_{L^2_\delta(\mathbb{R}^n)}\right),\end{aligned}$$

and

$$|R(\lambda,\omega)| \le |R_1(\lambda,\omega)| + |R_2(\lambda,\omega)| \le \frac{C}{\sqrt{\lambda}} \|\Phi\|_{L^2_\delta(\mathbb{R}^n)} \|\Psi\|_{L^2_\delta(\mathbb{R}^n)}.$$

This completes the proof of Lemma 3.1. □

4. Proof of the stability estimate

In this section, we complete the proof of Theorem 1.1. We are going to use the estimate proved in the previous section; this will provide information on the X-ray transform of the difference of short-range electric potentials.

Let $\omega \in \mathbb{S}^{n-1}$ and $V \in \mathcal{V}_\delta$. We denote

$$f(x) = X(V)(x,\omega) = \int_{\mathbb{R}} V(x+t\omega)\,\mathrm{d}t.$$

Then f satisfies the following estimate

$$\begin{aligned}|f(x)| = |f(x-(\omega\cdot x)\omega)| &\le C\int_{\mathbb{R}} \langle x-(\omega\cdot x)\omega + t\omega\rangle^{-\delta}\,\mathrm{d}t \\ &\le \frac{C}{\langle x-(x\cdot\omega)\omega\rangle^{\delta-1}} \int_{\mathbb{R}} \langle t\rangle^{-\delta}\,\mathrm{d}t, \quad \forall x\in\mathbb{R}^n.\end{aligned}$$

In particular, we have $f \in L^1(\omega^\perp)$, since $\delta > n$.

For any Φ, Ψ with $\mathrm{Supp}(\hat{\Phi}) \subset B(0,1)$ and $\mathrm{Supp}(\hat{\Psi}) \subset B(0,1)$, we have, by (3.2):

$$\begin{aligned}|\int_{\mathbb{R}^n} f(x)\Phi(x)\overline{\Psi}(x)\,\mathrm{d}x| &\le \sqrt{\lambda}\|S_{V_1}-S_{V_2}\| \|\Phi\|_{L^2(\mathbb{R}^n)} \|\Psi\|_{L^2(\mathbb{R}^n)} \\ &\quad + C\lambda^{-\gamma/2}\left(\|\Phi\|_{H^2(\mathbb{R}^n)} + \|\Phi\|_{L^2_\delta(\mathbb{R}^n)}\right)\left(\|\Psi\|_{H^2(\mathbb{R}^n)} + \|\Psi\|_{L^2_\delta(\mathbb{R}^n)}\right).\end{aligned}$$

Let $\eta \in \omega^\perp$ be fixed and let $\Psi \in L^2(\mathbb{R}^n)$ such that $\mathrm{Supp}(\hat{\Psi}) \subset B(\eta/2,1)$. Denote by $\Psi_{\eta/2} = \mathrm{e}^{\mathrm{i}x\cdot\eta/2}\Psi$, then $\mathrm{Supp}(\hat{\Psi}_{\eta/2}) \subset B(0,1)$. Applying the last inequality with $\Psi = \Psi_{\eta/2}$, we find:

$$\begin{aligned}|\int_{\mathbb{R}^n} f_{-\eta/2}(x)\Phi(x)\overline{\Psi}(x)\,\mathrm{d}x| &\le \sqrt{\lambda}\|S_{V_1}-S_{V_2}\| \|\Phi\|_{L^2(\mathbb{R}^n)} \|\Psi\|_{L^2(\mathbb{R}^n)} \\ &\quad + C\langle\eta\rangle^2\lambda^{-\gamma/2}\left(\|\Phi\|_{H^2(\mathbb{R}^n)} + \|\Phi\|_{L^2_\delta(\mathbb{R}^n)}\right)\left(\|\Psi\|_{H^2(\mathbb{R}^n)} + \|\Psi\|_{L^2_\delta(\mathbb{R}^n)}\right),\end{aligned} \tag{4.1}$$

where $f_{-\eta/2} = \mathrm{e}^{-\mathrm{i}x\cdot\eta/2} f$.

Lemma 4.1. *Let $V_j \in \mathcal{V}_\delta$, $j=1,2$. Then there exist $C>0$, $\lambda_0>0$, $\gamma\in(0,1)$ and $\sigma > n/2+\gamma$ such that, for any $\omega \in \mathbb{S}^{n-1}$ and $\Phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\mathrm{Supp}(\hat{\Phi}) \subset B(0,1)$, the following estimate holds true*

$$\begin{aligned}|\mathcal{F}(f\Phi)(\eta)| &\le \varepsilon^{-n/2}\sqrt{\lambda}\|S_{V_1}-S_{V_2}\| \|\Phi\|_{L^2(\mathbb{R}^n)} \\ &\quad + C\langle\eta\rangle^4\lambda^{-\gamma/2}\varepsilon^{-n/2-\delta}\left(\|\Phi\|_{H^2(\mathbb{R}^n)} + \|\Phi\|_{L^2_\delta(\mathbb{R}^n)}\right) + C\varepsilon^\gamma\|\Phi\|_{L^2_\sigma(\mathbb{R}^n)}\end{aligned}$$

for any $\lambda > \lambda_0$, $\eta \in \omega^\perp$ and $\varepsilon \in (0,1)$.

Proof. Let $\psi_0 \in \mathcal{C}_0^\infty(B(0,1))$, with $\|\psi_0\|_{L^1(\mathbb{R}^n)} = 1$, we define

$$\psi_\varepsilon(\xi) = \varepsilon^{-n}\psi_0(\varepsilon^{-1}(\xi-\eta/2)), \quad \mathrm{Supp}(\psi_\varepsilon) \subset B(\eta/2,\varepsilon) \subset B(\eta/2,1),$$

and let $\Psi_\varepsilon = \mathcal{F}^{-1}(\psi_\varepsilon)$. By Plancherel's formula, we get:

$$\int_{\mathbb{R}^n} f_{-\eta/2}(x)\Phi(x)\overline{\Psi}_\varepsilon(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \mathcal{F}(f_{-\eta/2}\Phi)(\xi)\psi_\varepsilon(\xi)\,\mathrm{d}\xi. \tag{4.2}$$

Taking into account (4.2) and applying (4.1) with $\Psi = \Psi_\varepsilon$, we obtain:

$$|\int_{\mathbb{R}^n} \mathcal{F}(f_{-\eta/2}\Phi)(\xi)\psi_\varepsilon(\xi)\,\mathrm{d}\xi| \le \sqrt{\lambda}\|S_{V_1} - S_{V_2}\|\|\Phi\|_{L^2(\mathbb{R}^n)}\|\psi_\varepsilon\|_{L^2(\mathbb{R}^n)}$$
$$+ C\langle\eta\rangle^2\lambda^{-\gamma/2}\left(\|\Phi\|_{H^2(\mathbb{R}^n)} + \|\Phi\|_{L^2_\delta(\mathbb{R}^n)}\right)\left(\|\Psi_\varepsilon\|_{H^2(\mathbb{R}^n)} + \|\Psi_\varepsilon\|_{L^2_\delta(\mathbb{R}^n)}\right). \tag{4.3}$$

Furthermore, there exists $C > 0$ such that

$$\|\psi_\varepsilon\|^2_{L^2(\mathbb{R}^n)} = \varepsilon^{-n}\int_{\mathbb{R}^n} |\psi_0(\xi)|^2\,\mathrm{d}\xi \le C\varepsilon^{-n}, \tag{4.4}$$

and

$$\|\Psi_\varepsilon\|^2_{H^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \langle\xi\rangle^4|\psi_\varepsilon(\xi)|^2\,\mathrm{d}\xi = \varepsilon^{-2n}\int_{\mathbb{R}^n}\langle\xi\rangle^4|\psi_0(\varepsilon^{-1}(\xi-\eta/2))|^2\,\mathrm{d}\xi \le C\varepsilon^{-n}\langle\eta\rangle^4. \tag{4.5}$$

Using the fact that

$$\mathcal{F}(\psi_\varepsilon)(y) = \mathrm{e}^{-\mathrm{i}y\cdot\eta/2}\hat{\psi}_0(\varepsilon y)$$

and (2.1), we get:

$$\|\Psi_\varepsilon\|_{L^2_\delta(\mathbb{R}^n)} = \|\psi_\varepsilon\|_{H^\delta(\mathbb{R}^n)} \le C\varepsilon^{-n/2-\delta}. \tag{4.6}$$

Then, by (4.3), (4.4), (4.5) and (4.6), one gets

$$|\int_{\mathbb{R}^n} \mathcal{F}(f_{-\eta/2}\Phi)(\xi)\psi_\varepsilon(\xi)\,\mathrm{d}\xi| \le \sqrt{\lambda}\varepsilon^{-n/2}\|S_{V_1} - S_{V_2}\|\|\Phi\|_{L^2(\mathbb{R}^n)}$$
$$+ C\langle\eta\rangle^4\lambda^{-\gamma/2}\varepsilon^{-n/2-\delta}\left(\|\Phi\|_{H^2(\mathbb{R}^n)} + \|\Phi\|_{L^2_\delta(\mathbb{R}^n)}\right).$$

Moreover, we have:

$$\mathcal{F}(f_{-\eta/2}\Phi)(\eta/2) = \int_{\mathbb{R}^n}\mathcal{F}(f_{-\eta/2}\Phi)(\xi)\psi_\varepsilon(\xi)\,\mathrm{d}\xi$$
$$+ \int_{\mathbb{R}^n}\left(\mathcal{F}(f_{-\eta/2}\Phi)(\eta/2) - \mathcal{F}(f_{-\eta/2}\Phi)(\xi)\right)\psi_\varepsilon(\xi)\,\mathrm{d}\xi.$$

Furthermore, for any $\gamma \in (0,1)$, there exists $C = C(\gamma) > 0$ such that

$$|\mathcal{F}(f_{-\eta/2}\Phi)(\eta/2) - \mathcal{F}(f_{-\eta/2}\Phi)(\xi)| \le C|\xi-\eta/2|^\gamma\int_{\mathbb{R}^n}\langle x\rangle^\gamma|\Phi(x)|\,\mathrm{d}x$$
$$\le C|\xi-\eta/2|^\gamma\left(\int_{\mathbb{R}^n}\langle x\rangle^{-2\sigma+2\gamma}\,\mathrm{d}x\right)^{1/2}\|\Phi\|_{L^2_\sigma(\mathbb{R}^n)},$$

for some $\sigma > \gamma + n/2$. We deduce that

$$|\int_{\mathbb{R}^n}\left(\mathcal{F}(f_{-\eta/2}\Phi)(\eta/2) - \mathcal{F}(f_{-\eta/2}\Phi)(\xi)\right)\psi_\varepsilon(\xi)\,\mathrm{d}\xi|$$
$$\le C\|\Phi\|_{L^2_\sigma(\mathbb{R}^n)}\int_{\mathbb{R}^n}|\eta/2-\xi|^\gamma|\psi_\varepsilon(\xi)|\,\mathrm{d}\xi \le C\varepsilon^\gamma\|\Phi\|_{L^2_\sigma(\mathbb{R}^n)},$$

which imply

$$|\mathcal{F}(f\Phi)(\eta)| = |\mathcal{F}(f_{-\eta/2}\Phi)(\eta/2)| \le \varepsilon^{-n/2}\sqrt{\lambda}\|S_{V_1} - S_{V_2}\|\|\Phi\|_{L^2(\mathbb{R}^n)}$$
$$+ \langle\eta\rangle^4\lambda^{-\gamma/2}\varepsilon^{-n/2-\delta}\left(\|\Phi\|_{H^2(\mathbb{R}^n)} + \|\Phi\|_{L^2_\delta(\mathbb{R}^n)}\right) + C\varepsilon^\gamma\|\Phi\|_{L^2_\sigma(\mathbb{R}^n)}.$$

This completes the proof of the lemma. □

We give now the following lemma to be used later.

Lemma 4.2. *Let $\theta \in \mathcal{C}_0^\infty((-\frac{1}{2}, \frac{1}{2}))$ and $\varphi \in \mathcal{C}_0^\infty(\omega^\perp \cap B(0, \frac{1}{2}))$. Putting*

$$\Phi(y) = \mathcal{F}_0^{-1}(\theta)(y \cdot \omega)\mathcal{F}_{\omega^\perp}^{-1}(\varphi)(y - (y \cdot \omega)\omega), \quad y \in \mathbb{R}^n,$$

where $\mathcal{F}_0$ denote the Fourier transform on the function in $\mathbb{R}$. Then we have Supp$(\hat{\Phi}) \subset B(0, 1)$ and

$$\hat{\Phi}(\xi) = \theta(\omega \cdot \xi)\varphi(\xi - (\omega \cdot \xi)\omega), \quad \forall \xi \in \mathbb{R}^n.$$

Moreover, for all $s \geq 0$, we have

$$\|\Phi\|_{H^s(\mathbb{R}^n)} \leq \|\theta\|_{L^2_s(\mathbb{R})}\|\varphi\|_{L^2_s(\omega^\perp)}.$$

Finally, for any $\delta \geq 0$, there exists $C > 0$ such that

$$\|\Phi\|_{L^2_\delta(\mathbb{R}^n)} \leq C\|\varphi\|_{H^\delta(\omega^\perp)}.$$

Here C depends on norms of θ.

The next step in the proof is to deduce an estimate that links the Fourier transform of the unknown coefficient to the measurement $S_{V_1} - S_{V_2}$.

Lemma 4.3. *Let $V_j \in \mathcal{V}_\delta$, $j = 1, 2$. Then there exist $C > 0$, $\lambda_0 > 0$, $\gamma \in (0, 1)$ and $\alpha_j > 0$, $j = 1, 2, 3$, such that, for any $\omega \in \mathbb{S}^{n-1}$, the following estimate holds true:*

$$|\mathcal{F}_{\omega^\perp}(f)(\eta)| \leq C\varepsilon^{-\alpha_1}\sqrt{\lambda}\|S_{V_1} - S_{V_2}\| + \lambda^{-\gamma/2}\varepsilon^{-\alpha_2}\langle\eta\rangle^4 + C\varepsilon^{\alpha_3}, \tag{4.7}$$

for any $\lambda > \lambda_0$, $\eta \in \omega^\perp$ and $\varepsilon \in (0, 1)$.

Proof. Let $\theta \in \mathcal{C}_0^\infty(-1/4, 1/4)$ and $\varphi \in \mathcal{C}_0^\infty(\omega^\perp \cap B(0, 1/2))$. Putting

$$\Phi(y) = \mathcal{F}_0^{-1}(\theta)(y \cdot \omega)\mathcal{F}_{\omega^\perp}^{-1}(\varphi)(y - (y \cdot \omega)\omega), \quad y \in \mathbb{R}^n.$$

We assume further $\theta(0) = 1$ and $\|\theta\|_{L^2(\mathbb{R})} = 1$. Then we have, by Lemma 4.2, Supp$(\hat{\Phi}) \subset B(0, 1)$. The change of variable $x = y + t\omega \in \omega^\perp \oplus \mathbb{R}\omega$, $\mathrm{d}x = \mathrm{d}y\,\mathrm{d}t$ yields, after noting that $\eta \in \omega^\perp$

$$\begin{aligned}\mathcal{F}(f\Phi)(\eta) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\cdot\eta}\mathcal{F}_0^{-1}\theta(x \cdot \omega)\mathcal{F}_{\omega^\perp}^{-1}(\varphi)(x - (x \cdot \omega)\omega)f(x)\,\mathrm{d}x \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}}\int_{\omega^\perp} \mathrm{e}^{-\mathrm{i}y\cdot\eta}\mathcal{F}_0^{-1}(\theta)(t)\mathcal{F}_{\omega^\perp}^{-1}(\varphi)(y)f(y)\,\mathrm{d}y\,\mathrm{d}t \\ &= (2\pi)^{-(n-1)/2}\int_{\omega^\perp} \mathrm{e}^{-\mathrm{i}y\cdot\eta}\mathcal{F}_{\omega^\perp}^{-1}(\varphi)(y)f(y)\,\mathrm{d}y \\ &= \mathcal{F}_{\omega^\perp}(f\mathcal{F}_{\omega^\perp}^{-1}(\varphi))(\eta) = \mathcal{F}_{\omega^\perp}(f) * \varphi(\eta),\end{aligned} \tag{4.8}$$

where we have used $f(y - t\omega) = f(y)$ for any $t \in \mathbb{R}$. Taking into account (4.8) and applying Lemma 4.1 and Lemma 4.2, one gets

$$\begin{aligned}|\int_{\omega^\perp} \mathcal{F}_{\omega^\perp}(f)(\xi)\varphi(\eta - \xi)\,\mathrm{d}\xi| &\leq \varepsilon^{-n/2}\sqrt{\lambda}\|S_{V_1} - S_{V_2}\|\|\varphi\|_{L^2(\omega^\perp)} \\ &\quad + C\langle\eta\rangle^4\lambda^{-\gamma/2}\varepsilon^{-n/2-\delta}\left(\|\varphi\|_{L^2_2(\omega^\perp)} + \|\varphi\|_{H^\delta(\omega^\perp)}\right) + C\varepsilon^\gamma\|\varphi\|_{H^\sigma(\omega^\perp)}.\end{aligned} \tag{4.9}$$

Now, we specify the choice of the function φ. Let $\varphi_0 \in \mathcal{C}_0^\infty(\omega^\perp \cap B(0, 1/2))$ with $\|\varphi_0\|_{L^1(\omega^\perp)} = 1$, we define, for h small

$$\varphi_h(\xi) = h^{-n+1}\varphi_0(h^{-1}\xi), \quad \xi \in \omega^\perp.$$

Applying (4.9) with $\varphi = \varphi_h$, we get

$$\begin{aligned}|\int_{\omega^\perp} \mathcal{F}_{\omega^\perp}(f)(\xi)\varphi_h(\eta - \xi)\,\mathrm{d}\xi| &\leq \varepsilon^{-n/2}\sqrt{\lambda}\|S_{V_1} - S_{V_2}\|\|\varphi_h\|_{L^2(\omega^\perp)} \\ &\quad + C\langle\eta\rangle^4\lambda^{-\gamma/2}\varepsilon^{-n/2-\delta}\left(\|\varphi_h\|_{L^2_2(\omega^\perp)} + \|\varphi_h\|_{H^\delta(\omega^\perp)}\right) + C\varepsilon^\gamma\|\varphi_h\|_{H^\sigma(\omega^\perp)}.\end{aligned}$$

Since

$$\|\varphi_h\|_{L^2(\omega^\perp)} = h^{(1-n)/2}\|\varphi_0\|_{L^2(\omega^\perp)}, \quad \|\varphi_h\|_{L^2_2(\omega^\perp)} \le Ch^{(1-n)/2}\|\varphi_0\|_{L^2_2(\omega^\perp)}$$

and,

$$\|\varphi_h\|_{H^\sigma(\omega^\perp)} \le Ch^{-\sigma+(1-n)/2}\|\varphi_0\|_{H^\sigma(\omega^\perp)},$$

we obtain

$$\Big|\int_{\omega^\perp} \mathcal{F}_{\omega^\perp}(f)(\xi)\varphi_h(\eta-\xi)\,\mathrm{d}\xi\Big| \le \varepsilon^{-n/2}\sqrt{\lambda}h^{(1-n)/2}\|S_{V_1}-S_{V_2}\| + C\,\langle\eta\rangle^4\,\lambda^{-\gamma/2}\varepsilon^{-n/2-\delta}h^{-\delta+(1-n)/2} + C\varepsilon^\gamma h^{-\sigma+(1-n)/2}.$$

Moreover

$$\mathcal{F}_{\omega^\perp}(f)(\eta) = \int_{\omega^\perp}\mathcal{F}_{\omega^\perp}(f)(\xi)\varphi_h(\eta-\xi)\,\mathrm{d}\xi - \int_{\omega^\perp}(\mathcal{F}_{\omega^\perp}(f)(\xi)-\mathcal{F}_{\omega^\perp}(f)(\eta))\varphi_h(\eta-\xi)\,\mathrm{d}\xi.$$

Using the fact that,

$$\begin{aligned}|\mathcal{F}_{\omega^\perp}(f)(\xi)-\mathcal{F}_{\omega^\perp}(f)(\eta)| &\le C\int_{\omega^\perp}|\mathrm{e}^{-\mathrm{i}x\cdot\xi}-\mathrm{e}^{-\mathrm{i}x\cdot\eta}||f(x)|\,\mathrm{d}x\\ &\le C|\xi-\eta|^{\gamma'}\int_{\omega^\perp}\langle x\rangle^{\gamma'}|f(x)|\,\mathrm{d}x\\ &\le C|\xi-\eta|^{\gamma'}\|V\|_{L^1_{\gamma'}(\mathbb{R}^n)},\end{aligned}$$

with $\gamma'>0$ sufficiently small. We deduce that

$$\begin{aligned}\Big|\int_{\omega^\perp}(\mathcal{F}_{\omega^\perp}(f)(\xi)-\mathcal{F}_{\omega^\perp}(f)(\eta))\varphi_h(\eta-\xi)\,\mathrm{d}\xi\Big| &\le CM\int_{\omega^\perp}|\xi-\eta|^{\gamma'}|\varphi_h(\eta-\xi)|\,\mathrm{d}\xi\\ &\le CMh^{\gamma'}.\end{aligned}$$

We obtain, for any $\eta\in\omega^\perp$

$$\begin{aligned}|\mathcal{F}_{\omega^\perp}(f)(\eta)| \le{}& \varepsilon^{-n/2}\sqrt{\lambda}h^{(1-n)/2}\|S_{V_1}-S_{V_2}\| + \lambda^{-\gamma/2}\varepsilon^{-n/2-\delta}\,\langle\eta\rangle^4\,h^{-\delta+(1-n)/2}\\ &+C\varepsilon^\gamma h^{-\sigma+(1-n)/2} + Ch^{\gamma'}.\end{aligned}$$

Selecting h such that $\varepsilon^\gamma h^{-\sigma+(1-n)/2} = h^{\gamma'}$, we obtain:

$$|\mathcal{F}_{\omega^\perp}(f)(\eta)| \le \varepsilon^{-\alpha_1}\sqrt{\lambda}\|S_{V_1}-S_{V_2}\| + \lambda^{-\gamma/2}\varepsilon^{-\alpha_2}\,\langle\eta\rangle^4 + C\varepsilon^{\alpha_3}.$$

This completes the proof of the lemma. □

We return now to the proof of Theorem 1.1. Since ω is arbitrary, we deduce, from (4.7) and (3.1),

$$|\mathcal{F}(V)(\eta)| \le \varepsilon^{-\alpha_1}\sqrt{\lambda}\|S_{V_1}-S_{V_2}\| + \lambda^{-\gamma/2}\varepsilon^{-\alpha_2}\,\langle\eta\rangle^4 + C\varepsilon^{\alpha_3}, \quad \forall\eta\in\mathbb{R}^n. \tag{4.10}$$

In light of the above reasoning and decomposing the $H^{-1}(\mathbb{R}^n)$ norm of V as

$$\|V\|^2_{H^{-1}(\mathbb{R}^n)} = \int_{|\eta|\le R}\langle\eta\rangle^{-2}\,|\mathcal{F}(V)(\eta)|^2\,\mathrm{d}\eta + \int_{|\eta|>R}\langle\eta\rangle^{-2}\,|\mathcal{F}(V)(\eta)|^2\,\mathrm{d}\eta$$

then, by (4.10), Plancherel's Theorem and (1.4), we get

$$\|V\|^2_{H^{-1}(\mathbb{R}^n)} \le C\left(R^n(\varepsilon^{-\alpha_1}\sqrt{\lambda}\|S_{V_1}-S_{V_2}\| + \lambda^{-\gamma/2}\varepsilon^{-\alpha_2}R^2 + C\varepsilon^{\alpha_3}) + \frac{M^2}{R^2}\right).$$

The next step is to choose in such a way $\varepsilon^{\alpha_3}R^n = R^{-2}$. In this case, we get

$$\|V\|^2_{H^{-1}(\mathbb{R}^n)} \leq C\left(R^{\beta_1}\sqrt{\lambda}\|S_{V_1} - S_{V_2}\| + \lambda^{-\gamma/2}R^{\beta_2} + \frac{1}{R^2}\right).$$

Now we choose $R > 0$ in such that a way $\lambda^{-\gamma/2}R^{\beta_2} = R^{-2}$. In this case, we get

$$\|V\|^2_{H^{-1}(\mathbb{R}^n)} \leq C\left(\lambda^{\mu_1}\|S_{V_1} - S_{V_2}\| + \lambda^{-\mu_2}\right),$$

for some positive constants μ_1, μ_2. Finally, minimizing the right-hand side with respect to λ, we obtain the desired estimate of Theorem 1.1.

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