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Functional analysis

Joint spectra of spherical Aluthge transforms of commuting n-tuples of Hilbert space operators

Spectres joints des transformées d'Aluthge sphériques de n-uplets commutatifs d'opérateurs d'un espace de Hilbert

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ABSTRACT

Let $\mathbf{T} \equiv (T_1, \cdots, T_n)$ be a commuting n-tuple of operators on a Hilbert space $\mathcal{H}$, and let $T_i \equiv V_i P$ $(1 \le i \le n)$ be its canonical joint polar decomposition (i.e. $P := \sqrt{T_1^* T_1 + \cdots + T_n^* T_n}$, $(V_1, \cdots, V_n)$ a joint partial isometry, and $\bigcap_{i=1}^n \ker T_i = \bigcap_{i=1}^n \ker V_i = \ker P$). The spherical Aluthge transform of $\mathbf{T}$ is the (necessarily commuting) n-tuple $\hat{\mathbf{T}} := (\sqrt{P} V_1 \sqrt{P}, \cdots, \sqrt{P} V_n \sqrt{P})$. We prove that $\sigma_T(\hat{\mathbf{T}}) = \sigma_T(\mathbf{T})$, where σ_T denotes the Taylor spectrum. We do this in two stages: away from the origin, we use tools and techniques from criss-cross commutativity; at the origin, we show that the left invertibility of $\mathbf{T}$ or $\hat{\mathbf{T}}$ implies the invertibility of P. As a consequence, we can readily extend our main result to other spectral systems that rely on the Koszul complex for their definitions.

RÉSUMÉ

Soit $\mathbf{T} \equiv (T_1, \cdots, T_n)$ un n-uplet commutatif d'opérateurs sur un espace de Hilbert $\mathcal{H}$, et soient $T_i \equiv V_i P$ $(1 \le i \le n)$ sa décomposition polaire jointe canonique (i.e. $P := \sqrt{T_1^* T_1 + \cdots + T_n^* T_n}$, $(V_1, \cdots, V_n)$ une isométrie partielle jointe et $\bigcap_{i=1}^n \ker T_i = \bigcap_{i=1}^n \ker V_i = \ker P$). La transformée d'Aluthge sphérique de $\mathbf{T}$ est le n-uplet (nécessairement commutatif) $\hat{\mathbf{T}} := (\sqrt{P} V_1 \sqrt{P}, \cdots, \sqrt{P} V_n \sqrt{P})$. Nous démontrons que $\sigma_T(\hat{\mathbf{T}}) = \sigma_T(\mathbf{T})$, où σ_T désigne le spectre de Taylor. Nous procédons pour cela en deux étapes : en dehors de l'origine, nous utilisons les outils et les techniques de la commutativité criss-cross ; à l'origine, nous prouvons que l'inversibilité à gauche de $\mathbf{T}$ ou de $\hat{\mathbf{T}}$ implique l'inversibilité

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de P. Comme conséquence, nous pouvons étendre notre résultat à d'autres systèmes spectraux définis à partir des complexes de Koszul.

1. Introduction

Let $\mathcal{H}$ be a complex infinite-dimensional Hilbert space, let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$, and let $T \in \mathcal{B}(\mathcal{H})$. For $T \equiv V|T|$ the canonical polar decomposition of T, we let $\tilde{T} := |T|^{1/2}V|T|^{1/2}$ denote the Aluthge transform of T [1]. It is well known that T is invertible if and only if $\tilde{T}$ is invertible; moreover, the spectra of T and $\tilde{T}$ are equal. Over the last two decades, considerable attention has been given to the study of the Aluthge transform; cf. [2–4,35], [10], [13,14,17–28], [32], [39–42]). Moreover, the Aluthge transform has been generalized to the case of powers of $|T|$ different from $\frac{1}{2}$ ([5], [8], [9], [29]) and to the case of commuting pairs of operators ([13], [14]).

In this note, we focus on the spherical Aluthge transform [14]. Although our results hold for arbitrary $n > 2$, for the reader's convenience we will focus on the case $n = 2$, that is, the case of commuting pairs of Hilbert space operators. Let $\mathbf{T} \equiv (T_1, T_2)$ be a commuting pair of operators on $\mathcal{H}$. We now consider the canonical polar decomposition of the column operator $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$; that is, $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \equiv \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} P$, where $P := \sqrt{T_1^*T_1 + T_2^*T_2}$ and $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ is a (joint) partial isometry, and subject to the constraint $\bigcap_{i=1}^{2} \ker T_i = \bigcap_{i=1}^{2} \ker V_i = \ker P$.

The *spherical Aluthge transform* of $\mathbf{T}$ is the (necessarily commuting) n-tuple

$$\widehat{\mathbf{T}} := (\sqrt{P}V_1\sqrt{P}, \cdots, \sqrt{P}V_n\sqrt{P}) \quad ([13], [14]). \tag{1.1}$$

For a commuting pair $\mathbf{T} \equiv (T_1, T_2)$ of operators on $\mathcal{H}$, the Koszul complex associated with $\mathbf{T}$ is given as

$$K(\mathbf{T},\mathcal{H}) : 0 \xrightarrow{0} \mathcal{H} \xrightarrow{\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}} \mathcal{H} \oplus \mathcal{H} \xrightarrow{(-T_2\ T_1)} \mathcal{H} \xrightarrow{0} 0.$$

Definition 1.1. A commuting pair $\mathbf{T}$ is said to be *(Taylor) invertible* if its associated Koszul complex $K(\mathbf{T},\mathcal{H})$ is exact. The Taylor spectrum of $\mathbf{T}$ is

$$\sigma_T(\mathbf{T}) := \left\{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : K\left((T_1 - \lambda_1, T_2 - \lambda_2)\,,\ \mathcal{H}\right) \text{ is not invertible}\right\}.$$

The pair $\mathbf{T}$ is called *Fredholm* if each map in the Koszul complex $K(\mathbf{T}, \mathcal{H})$ has closed range and all the homology quotients are finite-dimensional. The Taylor essential spectrum is

$$\sigma_{Te}(\mathbf{T}) := \left\{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : (T_1 - \lambda_1, T_2 - \lambda_2) \text{ is not Fredholm}\right\}.$$

J.L. Taylor showed in [36] and [37] that, if $\mathcal{H} \neq \{0\}$, then $\sigma_T(\mathbf{T})$ is a nonempty, compact subset of the polydisc of multi-radius $r(\mathbf{T}) := (r(T_1), r(T_2))$, where $r(T_i)$ is the spectral radius of T_i $(i = 1, 2)$. (For additional facts about these joint spectra, the reader is referred to [11,12,15,16] and [38].)

As shown in [11] and [16], the Fredholmness of $\mathbf{T}$ can be detected in the Calkin algebra $\mathcal{Q}(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. (Here $\mathcal{K}$ denotes the closed two-sided ideal of compact operators; we also let $\pi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{Q}(\mathcal{H})$ denote the quotient map.) Concretely, $\mathbf{T}$ is Fredholm on $\mathcal{H}$ if and only if the pair of left multiplication operators $L_{\pi(\mathbf{T})} := (L_{\pi(T_1)}, L_{\pi(T_2)})$ is Taylor invertible when acting on $\mathcal{Q}(\mathcal{H})$. In particular, $\mathbf{T}$ is left Fredholm on $\mathcal{H}$ if and only if $L_{\pi(\mathbf{T})}$ is bounded below on $\mathcal{Q}(\mathcal{H})$.

Problem 1.2. *Let $\mathbf{T} \equiv (T_1, T_2)$ be a commuting pair of operators.*
(i) Assume that $\mathbf{T}$ is (Taylor) invertible (resp. Fredholm). Is $\widehat{\mathbf{T}}$ also (Taylor) invertible (resp. Fredholm)?
(ii) Is the Taylor spectrum (resp. Taylor essential spectrum) of $\widehat{\mathbf{T}}$ equal to that of $\mathbf{T}$?

We first prove that $\sigma_{\mathrm{T}}(\widehat{\mathbf{T}}) = \sigma_{\mathrm{T}}(\mathbf{T})$. We do this in two stages: away from the origin, we use tools and techniques from criss-cross commutativity; at the origin we show that the left invertibility of $\mathbf{T}$ or $\widehat{\mathbf{T}}$ implies the invertibility of P; P then helps to establish an isomorphism between the relevant Koszul complexes. As a consequence, we can readily extend the above result to other spectral systems that rely on the Koszul complex for their definitions, including spectral systems on $\mathcal{Q}(\mathcal{H})$.

2. Main results

Recall the joint polar decomposition of $\mathbf{T}$ and the spherical Aluthge transform of $\mathbf{T}$; cf. (1.1). We now state our first main result.

Theorem 2.1. *Assume that $\mathbf{T}$ or $\widehat{\mathbf{T}}$ is left invertible; that is, the associated Koszul complex is exact at the left stage, and the range of the corresponding boundary map is closed. Then the operator P is invertible.*

Proof. Case 1. If $\mathbf{T}$ is left invertible, then $T_1^*T_1 + T_2^*T_2$ is invertible, and therefore P is invertible.

Case 2. If $\widehat{\mathbf{T}}$ is left invertible, then it is bounded below; that is, there exists a constant $c > 0$ such that

$$\left\| \sqrt{P} V_1 \sqrt{P} x \right\|^2 + \left\| \sqrt{P} V_2 \sqrt{P} x \right\|^2 \geq c^2 \left\| x \right\|^2 .$$

Since (V_1, V_2) is a joint partial isometry, it readily follows that

$$\left\| \sqrt{P} x \right\|^2 + \left\| \sqrt{P} x \right\|^2 \geq \frac{c^2}{\|P\|} \left\| x \right\|^2 .$$

As a result, $\sqrt{P}$ is bounded below, so P is invertible. □

We are now ready to state our second main result.

Theorem 2.2. *Let $\mathbf{T} = (T_1, T_2)$ be a commuting pair of operators on $\mathcal{H}$. Then,*

$$\mathbf{T} \text{ is (Taylor) invertible} \iff \widehat{\mathbf{T}} \text{ is (Taylor) invertible}.$$

We now recall the notion of criss-cross commutativity.

Definition 2.3. Let $\mathbf{A} \equiv (A_1, \cdots, A_n)$ and $\mathbf{B} \equiv (B_1, \cdots, B_n)$ be two n-tuples of operators on $\mathcal{H}$. We say that $\mathbf{A}$ and $\mathbf{B}$ criss-cross commute (or that $\mathbf{A}$ criss-cross commutes with $\mathbf{B}$) if $A_i B_j A_k = A_k B_j A_i$ and $B_i A_j B_k = B_k A_j B_i$ for all $i, j, k = 1, \cdots, n$. Observe that we do not assume that $\mathbf{A}$ or $\mathbf{B}$ is commuting.

Definition 2.4. Given two n-tuples $\mathbf{A}$ and $\mathbf{B}$ we define $\mathbf{AB} := (A_1 B_1, \cdots, A_n B_n)$ and $\mathbf{BA} := (B_1 A_1, \cdots, B_n A_n)$.

Remark 2.5. It is an easy consequence of Definition 2.3 that, if $\mathbf{A}$ and $\mathbf{B}$ criss-cross commute and $\mathbf{AB}$ is commuting, then $\mathbf{BA}$ is also commuting.

Lemma 2.6. *Let $\mathbf{T} \equiv (T_1, T_2)$ be a commuting pair of operators on $\mathcal{H}$, let $P := \sqrt{T_1^*T_1 + T_2^*T_2}$, and let $\widehat{\mathbf{T}}$ be its spherical Aluthge transform. Then $\mathbf{A} \equiv (A_1, A_2) := (\sqrt{P}, \sqrt{P})$ and $\mathbf{B} \equiv (B_1, B_2) := (V_1\sqrt{P}, V_2\sqrt{P})$ criss-cross commute. As a consequence, $\widehat{\mathbf{T}} (= \mathbf{BA})$ is commuting.*

Lemma 2.7. *(cf. [6] and [7]) Let $\mathbf{A}$ criss-cross commute with $\mathbf{B}$ on $\mathcal{H}$, and assume that $\mathbf{AB}$ is commuting. Then $\sigma_{\mathrm{T}}(\mathbf{BA}) \setminus \{\mathbf{0}\} = \sigma_{\mathrm{T}}(\mathbf{AB}) \setminus \{\mathbf{0}\}$.*

We now prove our third main result.

Theorem 2.8. *Let $\mathbf{T} = (T_1, T_2)$ be a commuting pair of operators on $\mathcal{H}$. Then*

$$\sigma_{\mathrm{T}}(\mathbf{T}) = \sigma_{\mathrm{T}}(\widehat{\mathbf{T}}).$$

Proof. Let $\lambda \in \mathbb{C}^2$. If $\lambda = (0, 0)$, use Theorem 2.2; if $\lambda \neq (0, 0)$, use Lemma 2.7. □

Remark 2.9. (i) Theorems 2.1, 2.2 and 2.8 can be easily extended to other spectral systems whose definition is given in terms of the Koszul complex; e.g., the left k-spectral systems $\sigma_{\pi,k}$ defined by W. Słodkowski and W. Żelazko ([33], [34]). For, the Proof of Theorem 2.1 (which uses only left invertibility of the relevant Koszul complex) works well in case $\mathbf{T}$ or $\widehat{\mathbf{T}}$. Once we know that $\sqrt{P}$ is invertible, the Koszul complexes of $\mathbf{T}$ and $\widehat{\mathbf{T}}$ are isomorphic, so $0 \notin \sigma_{\pi,k}(\mathbf{T})$ if and only if $0 \notin \sigma_{\pi,k}(\widehat{\mathbf{T}})$.
(ii) Similarly, Theorem 2.7 admits an easy extension to Słodkowski's left k-spectra (cf. [6], [7]), since the Proof of Theorem 2.7 relies on the isomorphism of the Koszul complexes for $\mathbf{T}$ and $\widehat{\mathbf{T}}$, implemented by $\sqrt{P}$.
(iii) On the other hand, the above results cannot be extended to Słodkowski's right k-spectra; for, consider the adjoint U_+^* of the (unweighted) unilateral shift U_+. It is easy to see that U_+^* is onto while $\widehat{U_+^*}$ is not.

Our final main result deals with Fredholmness.

Theorem 2.10. *Let* $\mathbf{T} = (T_1, T_2)$ *be a commuting pair of operators on* $\mathcal{H}$*. Then*

$$\sigma_{Te}(\mathbf{T}) = \sigma_{Te}(\widehat{\mathbf{T}}).$$

Moreover, for each $\lambda \notin \sigma_{Te}(\mathbf{T})$*, we have*

$$ind\ (\mathbf{T} - \lambda) = ind\ (\widehat{\mathbf{T}} - \lambda),$$

where ind denotes the Fredholm index.

Sketch of proof. In Theorem 2.1, one can replace "left invertible" for the Koszul complex with "left Fredholm" and "invertible" for P with "Fredholm." A similar adjustment works for Theorems 2.2 and 2.8. In the analog of Theorem 2.2, one first proves that $\sqrt{P}$ is bounded below in the orthogonal complement of $\ker T_1 \cap \ker T_2$; since this kernel is finite dimensional, it follows that $\sqrt{P}$ is Fredholm. In Theorem 2.8, one needs to replace Lemma 2.7 with the results for Fredholmness proved in [6], [7], [30], and [31]. While Li's results only guarantee that ind $(\mathbf{T} - \lambda) =$ ind $(\widehat{\mathbf{T}} - \lambda)$ whenever $\lambda \neq (0, 0)$, the continuity of the Fredholm index (cf. [16]) does the rest. □

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