C. R. Acad. Sci. Paris, Ser. I 357 (2019) 858–862

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Potential theory/Complex analysis

A class of maximal plurisubharmonic functions ☆

Une classe de fonctions pluri-sous-harmoniques maximales

Hoang-Son Do

Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Hanoi, Viet Nam

A R T I C L E I N F O

Article history:
Received 28 June 2019
Accepted after revision 5 November 2019
Available online 27 November 2019

Presented by the Editorial Board

A B S T R A C T

In this note, we introduce a class of maximal plurisubharmonic functions and use that class to prove some properties of maximal plurisubharmonics functions.

R É S U M É

Dans cette note, nous introduisons une classe de fonctions pluri-sous-harmoniques maximales et utilisons celle-ci pour prouver certaines propriétés des fonctions pluri-sous-harmoniques maximales.

0. Introduction

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain ($n \geq 2$). A function $u \in PSH(\Omega)$ is called maximal if, for every open set $G \Subset \Omega$, and for each upper semicontinuous function v on $\overline{G}$ such that $v \in PSH(G)$ and $v|_{\partial G} \leq u|_{\partial G}$, we have $v \leq u$. There are some equivalent descriptions of maximality that have been presented in [8] (see also [6]). The set of all maximal plurisubharmonic functions in Ω is denoted by $MPSH(\Omega)$.

By [8] and by the comparison principle [1], $u \in MPSH(\Omega)$ iff for every $U \Subset \Omega$, there exists a sequence of functions $PSH(U) \cap C(U) \ni u_j \searrow u$ such that $(dd^c u_j)^n$ is weakly convergent to 0 as $j \to \infty$. In the case where u belongs to the domain of definition of the Monge–Ampère operator $D(\Omega)$ (see [5,3]), this implies that maximality is a local notion. It has been conjectured by Blocki that maximality is also a local notion in the case where $u \notin D(\Omega)$.

In this note, we will introduce a class of plurisubharmonic functions and use it to study some properties of maximal plurisubharmonic functions. We say that a function $u \in PSH^-(\Omega)$ has property M_1 iff, for every open set $U \Subset \Omega$, there are $u_j \in PSH^-(U) \cap C(U)$ such that u_j is decreasing to u in U and

☆ The author was funded by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2017.306.
E-mail addresses: hoangson.do.vn@gmail.com, dhson@math.ac.vn.

https://doi.org/10.1016/j.crma.2019.11.003
1631-073X/

$$\lim_{j\to\infty}\left(\int\limits_{U\cap\{u_j>-t\}}(dd^c u_j)^n+\int\limits_{U\cap\{u_j>-t\}}du_j\wedge d^c u_j\wedge(dd^c u_j)^{n-1}\right)=0, \tag{1}$$

for any $t>0$. Denote by $M_1PSH(\Omega)$ the set of negative plurisubharmonic functions in Ω satisfying property M_1. We will show that $M_1PSH(\Omega)\subset MPSH(\Omega)$ and property M_1 is a local notion. Our main result is the following one.

Theorem 1. *Let Ω be a bounded domain in $\mathbb{C}^n$ and $u\in PSH^-(\Omega)$. Then the following conditions are equivalent:*

(i) $u\in M_1PSH(\Omega)$.
(ii) $\chi(u)\in MPSH(\Omega)$ *for any convex non-decreasing function* $\chi:\mathbb{R}\to\mathbb{R}$.
(iii) *For any open sets $U,\tilde{U}$ such that $U\Subset\tilde{U}\Subset\Omega$, for any $u_j\in PSH^-(\tilde{U})\cap C(\tilde{U})$ such that u_j is decreasing to u in $\tilde{U}$, we have:*

$$\lim_{j\to\infty}\left(\int\limits_U|u_j|^{-a}(dd^c u_j)^n+\int\limits_U|u_j|^{-a-1}du_j\wedge d^c u_j\wedge(dd^c u_j)^{n-1}\right)=0,$$

for all $a>n-1$.

In particular, property M_1 is a local notion and $M_1PSH(\Omega)\subset MPSH(\Omega)$.

In Section 2, by using Theorem 1, we will show [2] the following properties of maximal plurisubharmonic functions.

Corollary 2. *If $u,v\in MPSH_{\mathrm{loc}}(\Omega)$ and $\chi:\mathbb{R}\to\mathbb{R}$ is a convex non-decreasing function, then $(z,w)\mapsto\chi(u(z)+v(w))\in MPSH(\Omega\times\Omega)$.*

Corollary 3. *Let u be a negative maximal plurisubharmonic function in Ω and let $U,\tilde{U}$ be an open subset of Ω such that $U\Subset\tilde{U}\Subset\Omega$. Assume that $u_j\in PSH^-(\tilde{U})\cap C(\tilde{U})$ is decreasing to u in $\tilde{U}$. Then*

$$\int\limits_U|u_j|^{-a}(dd^c u_j)^n\overset{j\to\infty}{\longrightarrow}0,\ \forall a>n-1. \tag{2}$$

1. Proof of the main theorem

In this section, we prove Theorem 1.

$(iii\Rightarrow i)$: Obvious.

$(i\Rightarrow ii)$: Assume that $U\Subset\tilde{U}\Subset\Omega$. Let $u_j\in PSH^-(U)\cap C(U)$ such that u_j is decreasing to u in U and condition (1) is satisfied.

If χ is smooth and χ is constant in some interval $(-\infty,-m)$, then

$$\begin{aligned}(dd^c\chi(u_j))^n&=(\chi'(u_j))^n(dd^c u_j)^n+n\chi''(u_j)(\chi'(u_j))^{n-1}du_j\wedge d^c u_j\wedge(dd^c u_j)^{n-1}\\&\le C\mathbf{1}_{\{u_j>-t\}}(dd^c u_j)^n+C\mathbf{1}_{\{u_j>-t\}}du_j\wedge d^c u_j\wedge(dd^c u_j)^{n-1},\end{aligned}$$

where $C,t>0$ depend only on χ. Hence,

$$\int\limits_U(dd^c\chi(u_j))^n\overset{j\to\infty}{\longrightarrow}0.$$

Then, $\chi(u)$ is maximal on U for any open set $U\Subset\Omega$. Thus, $\chi(u)\in MPSH(\Omega)$.

In the general case, for any convex non-decreasing function χ, we can find $\chi_l\searrow\chi$ such that χ_l is smooth, convex and $\chi|_{(-\infty,-m)}=const$ for some m. By the argument above, $\chi_l(u)\in MPSH(\Omega)$ for any $l\in\mathbb{N}$. Hence $\chi(u)\in MPSH(\Omega)$.

$(ii\Rightarrow iii)$: For any $0<\alpha<\frac{1}{n}$, the function

$$\Phi_\alpha(t)=-(-t)^\alpha$$

is convex and non-decreasing in $\mathbb{R}^-$. Assume that u satisfies (ii), we have $\Phi_\alpha(u)\in MPSH(\Omega)$.

By [2] (see also [4]), for any $0<\alpha<\frac{1}{n}$, we have $\Phi_\alpha(u)\in D(\Omega)$. Then, for any $u_j\in PSH^-(\tilde{U})\cap C(\tilde{U})$ such that u_j is decreasing to u in $\tilde{U}$, we have

$$\int\limits_U (dd^c \Phi_\alpha(u_j))^n \overset{j\to\infty}{\longrightarrow} 0, \forall 0 < \alpha < \frac{1}{n},$$

which implies (iii).

By $(i \Leftrightarrow ii)$, we conclude that $M_1 PSH(\Omega) \subset MPSH(\Omega)$. Finally, we need to show that property M_1 is a local notion. Assume that u has local property M_1. Let $U \Subset \tilde{U} \Subset \Omega$ be open sets. By the compactness of $\overline{U}$ and by the local property M_1 of u, there are open sets $U_1, ..., U_m, \tilde{U}_1, ..., \tilde{U}_m$ such that

$$\forall k = 1, ..., m : U_k \Subset \tilde{U}_k \Subset \tilde{U} \text{ and } u \in M_1 PSH(\tilde{U}_k),$$

and

$$\overline{U} \subset \bigcup_{k=1}^{m} U_k.$$

Let $u_j \in PSH^-(\tilde{U}) \cap C(\tilde{U})$ such that u_j is decreasing to u in $\tilde{U}$, we have, by $(i \Leftrightarrow iii)$,

$$\lim_{j\to\infty}\left(\int\limits_{U_k} |u_j|^{-a}(dd^c u_j)^n + \int\limits_{U_k} |u_j|^{-a-1} du_j \wedge d^c u_j \wedge (dd^c u_j)^{n-1}\right) = 0,$$

for all $a > n-1$ and $k = 1, ..., m$. Hence,

$$\lim_{j\to\infty}\left(\int\limits_{U} |u_j|^{-a}(dd^c u_j)^n + \int\limits_{U} |u_j|^{-a-1} du_j \wedge d^c u_j \wedge (dd^c u_j)^{n-1}\right) = 0, \ \forall a > n-1.$$

Then u satisfies (iii). Thus u has property M_1.

2. Proof of Corollary 2 and Corollary 3

2.1. Proof of Corollary 2

Without loss of generality, we can assume that $u, v \in PSH^-(\Omega)$.

If $u, v \in MPSH_{loc}(\Omega)$ then for any $z_0, w_0 \in \Omega$, there are open balls $U, \tilde{U}, V, \tilde{V}$ such that $z_0 \in U \Subset \tilde{U} \Subset \Omega$, $w_0 \in V \Subset \tilde{V} \Subset \Omega$, $u \in MPSH(\tilde{U})$ and $v \in MPSH(\tilde{V})$. We need to show that $u(z) + v(w)$ has property M_1 in $U \times V$.

Let $u_j \in PSH^-(\tilde{U}) \cap C(\tilde{U})$ and $v_j \in PSH^-(\tilde{V}) \cap C(\tilde{V})$ such that u_j is decreasing to u in $\tilde{U}$ and v_j is decreasing to v in $\tilde{V}$. By [9], there are $\tilde{u}_j \in PSH^-(\tilde{U}) \cap C(\tilde{U})$, $\tilde{v}_j \in PSH^-(\tilde{V}) \cap C(\tilde{V})$ such that

$$\begin{cases} \tilde{u_j} = u_j & \text{in } \tilde{U} \setminus U, \\ \tilde{v_j} = v_j & \text{in } \tilde{V} \setminus V, \\ (dd^c \tilde{u}_j)^n = 0 & \text{in } U, \\ (dd^c \tilde{v}_j)^n = 0 & \text{in } V. \end{cases}$$

By the maximality of u and v, we conclude that $\tilde{u}_j$ is decreasing to u in $\tilde{U}$ and $\tilde{v}_j$ is decreasing to v in $\tilde{V}$. In $U \times V$, we have:

$$\begin{aligned} &(dd^c(\tilde{u}_j(z) + \tilde{v}_j(w)))^{2n} = C_{2n}^n (dd^c \tilde{u}_j)_z^n \wedge (dd^c \tilde{v}_j)_w^n = 0 \\ &d(\tilde{u}_j(z) + \tilde{v}_j(w)) \wedge d^c(\tilde{u}_j(z) + \tilde{v}_j(w)) \wedge (dd^c(\tilde{u}_j(z) + \tilde{v}_j(w)))^{2n-1} \\ &= C_{2n-1}^{n-1} d_z \tilde{u}_j \wedge d_z^c \tilde{u}_j \wedge (dd^c \tilde{u}_j)_z^{n-1} \wedge (dd^c \tilde{v}_j)_w^n + C_{2n-1}^{n-1} d_w \tilde{v}_j \wedge d_w^c \tilde{v}_j \wedge (dd^c \tilde{v}_j)_w^{n-1} \wedge (dd^c \tilde{u}_j)_z^n \\ &= 0. \end{aligned}$$

Then $u(z) + v(w)$ has property M_1 in $U \times V$. By Theorem 1, property M_1 is a local notion. Hence, $u(z) + v(w) \in M_1 PSH(\Omega \times \Omega)$. This implies that $\chi(u(z) + v(w)) \in MPSH(\Omega \times \Omega)$ for any convex non-decreasing function χ.

2.2. Proof of Corollary 3

Let $v=|z_1|^2+\ldots+|z_{n-1}|^2+x_n+y_n-M$, where $M=\sup_\Omega(|z|^2+|x_n|+|y_n|)$. Then $v\in MPSH(\Omega)$. By Corollary 2, $\chi(u(z)+v(w))\in MPSH(\Omega\times\Omega)$ for any convex non-decreasing function χ.

By [2,4], for any $0<\alpha<\frac{1}{2n}$, we have $\Phi_\alpha(u(z)+v(w))\in D(\Omega\times\Omega)$, where Φ_α is defined as in the proof of Theorem 1. Then

$$\int_{U\times U}(dd^c\Phi(u_j(z)+v(w)))^{2n}\overset{j\to\infty}{\longrightarrow}0,$$

for any $0<\alpha<\frac{1}{2n}$. Hence,

$$\int_U |u_j|^{-2n-1+2n\alpha}(dd^cu_j)^n\overset{j\to\infty}{\longrightarrow}0,\ \forall 0<\alpha<\frac{1}{2n}. \tag{3}$$

Moreover, $\Phi_\beta(u)\in D(\Omega)$ for any $0<\beta<\frac{1}{n}$. Then, for any $0<\beta<\frac{1}{n}$, there is $C_\beta>0$ such that

$$\int_U(dd^c\Phi_\beta(u_j))^n\le C_\beta,\ \forall j>0.$$

Hence,

$$\int_U |u_j|^{-n+n\beta}(dd^cu_j)^n\le C_\beta,\ \forall j>0, \forall 0<\beta<\frac{1}{n}. \tag{4}$$

Combining (3), (4) and using Hölder's inequality, we obtain (2).

3. Further remarks on the class $M_1PSH(\Omega)$

In this section, we introduce some additional properties of the class $M_1PSH(\Omega)$. By Theorem 1, we have the following proposition.

Proposition 4. *Let Ω be a bounded domain in $\mathbb{C}^n$.*

(i) *If $u\in M_1PSH(\Omega)$ then $\chi(u)\in M_1PSH(\Omega)$ for any convex non-decreasing function $\chi:\mathbb{R}^-\to\mathbb{R}^-$.*
(ii) *If $u_j\in M_1PSH(\Omega)$ and u_j is decreasing to u, then $u\in M_1PSH(\Omega)$.*
(iii) *Let $u\in PSH^-(\Omega)\cap C^2(\Omega\setminus F)$, where $F=\{z:u(z)=-\infty\}$ is closed. If*

$$(dd^cu)^n=du\wedge d^cu\wedge(dd^cu)^{n-1}=0$$

in $\Omega\setminus F$, then $u\in M_1PSH(\Omega)$.

In some special cases, we can easily check property M_1 using the following criteria.

Proposition 5. *Let Ω be a bounded domain in $\mathbb{C}^n$. Let $\chi:\mathbb{R}\to\mathbb{R}$ be a smooth convex increasing function such that $\chi''(t)>0$ for any $t\in\mathbb{R}$. Assume also that χ is lower bounded. If $u\in PSH^-(\Omega)$ and $\chi(u)\in MPSH(\Omega)$, then $u\in M_1PSH(\Omega)$.*

Proof. Let $U\Subset\tilde{U}\Subset\Omega$ and $u_j\in PSH(\tilde{U})\cap C(\tilde{U})$ such that u_j is decreasing to u. Then

$$dd^c(\chi(u_j))=\chi'(u_j)dd^cu_j+\chi''(u_j)du_j\wedge d^cu_j$$

and

$$(dd^c\chi(u_j))^n=(\chi'(u_j))^n(dd^cu_j)^n+n\chi''(u_j)(\chi'(u_j))^{n-1}du_j\wedge d^cu_j\wedge(dd^cu_j)^{n-1}.$$

For any $t>0$, there exists $C>0$ depending only on t and χ such that

$$(dd^c\chi(u_j))^n\ge C\mathbf{1}_{\{u_j>-t\}}(dd^cu_j)^n+C\mathbf{1}_{\{u_j>-t\}}du_j\wedge d^cu_j\wedge(dd^cu_j)^{n-1}. \tag{5}$$

Note that $\chi(u)\in D(\Omega)\cap MPSH(\Omega)$. Hence,

$$\lim_{j\to\infty}\int_U (dd^c\chi(u_j))^n = 0. \tag{6}$$

Combining (5) and (6), we have

$$\lim_{j\to\infty}\left(\int_{U\cap\{u_j>-t\}} (dd^c u_j)^n + \int_{U\cap\{u_j>-t\}} du_j\wedge d^c u_j\wedge (dd^c u_j)^{n-1}\right) = 0.$$

Thus, $u \in M_1PSH(\Omega)$. □

Example 6. (i) If u is a negative plurisubharmonic function in $\Omega\subset\mathbb{C}^n$ depending only on $n-1$ variables, then u has M_1 property.
(ii) If $f:\Omega\to\mathbb{C}^n$ is a holomorphic mapping of rank $<n$ then $(dd^c|f|^2)^n = 0$ (see, for example, in [7]). By Proposition 5, $\log|f| \in M_1PSH(\Omega)$ if it is negative in Ω.

References

[1] E. Bedford, B.A. Taylor, A new capacity for plurisubharmonic functions, Acta Math. 149 (1–2) (1982) 1–40.
[2] E. Bedford, Survey of pluri-potential theory, in: Several Complex Variables, Stockholm, 1987/1988, in: Math. Notes, vol. 38, Princeton University Press, Princeton, NJ, USA, 1993, pp. 48–97.
[3] Z. Blocki, The domain of definition of the complex Monge–Ampère operator, Amer. J. Math. 128 (2) (2006) 519–530.
[4] Z. Blocki, Remark on the definition of the complex Monge–Ampère operator, in: Functional Analysis and Complex Analysis, in: Contemp. Math., vol. 481, Amer. Math. Soc., Providence, RI, USA, 2009, pp. 17–21.
[5] U. Cegrell, The general definition of the complex Monge–Ampère operator, Ann. Inst. Fourier (Grenoble) 54 (1) (2004) 159–179 (in English, with French summary).
[6] M. Klimek, Pluripotential Theory, Oxford University Press, Oxford, UK, 1991.
[7] A. Rashkovskii, Maximal plurisubharmonic functions associated with holomorphic mappings, Indiana Univ. Math. J. 47 (1) (1998) 297–309.
[8] A. Sadullaev, Plurisubharmonic measures and capacities on complex manifolds, Russ. Math. Surv. 36 (1981) 61–119.
[9] J.B. Walsh, Continuity of envelopes of plurisubharmonic functions, J. Math. Mech. 18 (1968) 143–148.