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Combinatorics/Number theory

On two congruence conjectures

Sur deux conjectures de congruence

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we mainly prove a congruence conjecture of M. Apagodu [3] and a supercongruence conjecture of Z.-W. Sun [25].

R É S U M É

Nous montrons dans cette Note une congruence conjecturée par M. Apagodu [3] et une supercongruence conjecturée par Z.-W. Sun [25].

1. Introduction

In the past few years, a lot of researchers worked on congruences for sums of binomial coefficients (see, for instance, [7,12–15,20,26,27]). In 2011, Sun and Tauraso [27] proved that, for any prime $p > 3$,

$$\sum_{k=0}^{p-1} \binom{2k}{k} \equiv_{p^2} \left(\frac{p}{3}\right), \tag{1.1}$$

where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Legendre symbol.

Pan and Sun [17] proved that, for any odd prime p,

$$\sum_{n=0}^{p-1} (3n+1)\binom{2n}{n} \equiv_p \left(\frac{p}{3}\right).$$

Then Apagodu [3] gave the following conjecture.

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Conjecture 1.1. *For any odd prime p, we have:*

$$\sum_{n=0}^{p-1}(5n+1)\binom{4n}{2n} \equiv_p -\left(\frac{p}{3}\right).$$

In this paper, we first prove the above conjecture and give another congruence.

Theorem 1.1. *Conjecture 1.1 is true. And we prove that, for each odd prime p,*

$$\sum_{n=0}^{p-1}(3n+1)\binom{4n}{2n} \equiv_p -\frac{1}{5}\left(\frac{p}{5}\right).$$

Recall that the Euler numbers and the Bernoulli numbers are given by

$$E_0 = 1, \text{ and } E_n = -\sum_{k=1}^{\lfloor n/2 \rfloor}\binom{n}{2k}E_{n-2k} \text{ (for } n \in \mathbb{Z}^+ = \{1, 2, \ldots\}),$$

$$B_0 = 1, \sum_{k=0}^{n-1}\binom{n}{k}B_k = 0 \quad (n = 2, 3, \ldots).$$

The well-known Catalan–Larcombe–French numbers $P_0, P_1, P_2, \ldots$ (cf. [8]) are given by

$$P_n = \sum_{k=0}^{n}\frac{\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2}{\binom{n}{k}},$$

which arose from the theory of elliptic integrals (see [11]). It is known that $(n+1)P_{n+1} = (24n(n+1)+8)P_n - 128n^2P_{n-1}$ for all $n \in \mathbb{Z}^+$. The sequence $(P_n)_{n\geq 0}$ is also related to the theory of modular forms. See D. Zagier [29].

Many researchers worked on the Catalan–Larcombe–French numbers, (see [9,8,13]). For instance, in 2017, the author proved that

$$\sum_{k=0}^{p-1}\frac{P_k}{8^k} \equiv 1 + 2(-1)^{(p-1)/2}p^2E_{p-3} \pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{P_k}{16^k} \equiv (-1)^{(p-1)/2} - p^2E_{p-3} \pmod{p^3}.$$

In [23], Sun proved that

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{(p-1)/2} + p^2E_{p-3} \pmod{p^3},$$

which plays an important role for proving the above two supercongruences involving P_n.

The famous Domb numbers are defined by

$$D(n) = \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k}.$$

This ubiquitous sequence (see $A002895$ of Sloane [19]) not only arises in the theory of third-order Apéry-like differential equations [2], odd moments of Bessel functions in quantum field theory [4], uniform random walks in the plane [5], new series for $1/\pi$ [6], interacting systems on crystal lattices [29] and the enumeration of abelian squares of length $2n$ over an alphabet with 4 letters [18], but if

$$F(z) = \frac{\eta^4(z)\eta^4(3z)}{\eta^2(2z)\eta^2(6z)} \text{ and } t(z) = \left(\frac{\eta(6z)\eta(2z)}{\eta(z)\eta(3z)}\right)^6,$$

then (see [6])

$$F(z)=\sum_{n=0}^{\infty}(-1)^n D(n)t^n(z).$$

There are also many researchers working on the Domb numbers (see, [16,21]). For example, the author and Wang [16] confirmed a conjecture of Sun [22]: for any prime $p>3$, we have:

$$D(p-1)\equiv 64^{p-1}-\frac{p^3}{6}B_{p-3} \pmod{p^4}.$$

Motivated by the above work, we will work on the sequence of numbers defined by

$$C_n=\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}^2,$$

which are the coefficients of the solutions to the Calabi–Yau equations. We know that the Calabi–Yau-type equation $\mathcal{D}y=0$, with

$$\mathcal{D}=\theta^4-2^4z(2\theta+1)^2(2\theta^2+2\theta+1)+2^{10}z^2(\theta+1)^2(2\theta+1)(2\theta+3),$$

has the solution [1, Appendix A, case #3*]:

$$y_0=\sum_{n=0}^{\infty}z^n\cdot\binom{2n}{n}C_n.$$

Sun [25] proved the following congruence involving C_n

$$\sum_{n=0}^{p-1}\frac{n}{32^n}C_n\equiv 0 \pmod{p^3}.$$

He gave the following conjecture:

Conjecture 1.2. *Let p be an odd prime. Then*

$$\sum_{n=0}^{p-1}\frac{n}{32^n}C_n\equiv -2p^3E_{p-3} \pmod{p^4}.$$

In this paper, we confirm this conjecture.

Theorem 1.2. *Conjecture 1.2 is true.*

We end this introduction by giving the organization of this paper. We shall prove Theorem 1.1 in Section 2, and Section 3 is devoted to prove Theorem 1.2.

2. Proof of Theorem 1.1

Lemma 2.1. *Let p be an odd prime. Then*

$$\binom{2k}{k}\equiv_p\binom{(p-1)/2}{k}(-4)^k.$$

Proof. It is easy to see that

$$\begin{aligned}\binom{(p-1)/2}{k}(-4)^k&=\frac{\left(\frac{p-1}{2}\right)\left(\frac{p-1}{2}-1\right)\cdots\left(\frac{p-1}{2}-k+1\right)}{k!}(-4)^k\\&=\frac{(1-p)(3-p)\cdots(2k-1-p)}{k!}2^k\\&\equiv_p\frac{1\cdot 3\cdots(2k-1)}{k!}2^k=\binom{2k}{k}.\end{aligned}$$

Now we finish the proof of Lemma 2.1. □

We shall separate the left-hand side of Conjecture 1.1 into two parts, one is $\vartheta_1 = \sum_{n=0}^{p-1}\binom{4n}{2n}$, the other is $\vartheta_2 = \sum_{n=0}^{p-1} n\binom{4n}{2n}$. We only consider $p > 5$, the cases $p = 3$ and $p = 5$ can be checked directly. We calculate ϑ_1 first. It is easy to see the identity as follows:

$$\vartheta_1 = \frac{1}{2}\left(\sum_{k=0}^{2p-1}\binom{2k}{k} + \sum_{k=0}^{2p-1}(-1)^k\binom{2k}{k}\right).$$

By the Lucas congruence and (1.1), we have:

$$\begin{aligned}\sum_{k=0}^{2p-1}\binom{2k}{k} &= \sum_{k=0}^{p-1}\binom{2k}{k} + \sum_{k=p}^{2p-1}\binom{2k}{k} = \sum_{k=0}^{p-1}\binom{2k}{k} + \sum_{k=0}^{p-1}\binom{2p+2k}{p+k} \\ &\equiv_p \sum_{k=0}^{p-1}\binom{2k}{k} + \sum_{k=0}^{p-1}\binom{2}{1}\binom{2k}{k} = 3\sum_{k=0}^{p-1}\binom{2k}{k} \equiv_p 3\left(\frac{p}{3}\right),\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{2p-1}(-1)^k\binom{2k}{k} &= \sum_{k=0}^{p-1}(-1)^k\binom{2k}{k} + \sum_{k=p}^{2p-1}(-1)^k\binom{2k}{k} \\ &= \sum_{k=0}^{p-1}(-1)^k\binom{2k}{k} + \sum_{k=0}^{p-1}(-1)^{p+k}\binom{2p+2k}{p+k} \\ &\equiv_p \sum_{k=0}^{p-1}(-1)^k\binom{2k}{k} - \sum_{k=0}^{p-1}(-1)^k\binom{2}{1}\binom{2k}{k} \\ &= -\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}.\end{aligned}$$

Note that $\binom{2k}{k} \equiv 0 \pmod{p}$ for each k such that $p/2 < k < p$. So, by Lemma 2.1, we have:

$$\sum_{k=0}^{2p-1}(-1)^k\binom{2k}{k} \equiv_p -\sum_{k=0}^{\frac{p-1}{2}}\binom{\frac{p-1}{2}}{k}4^k = -5^{\frac{p-1}{2}} \equiv_p -\left(\frac{5}{p}\right) = -\left(\frac{p}{5}\right).$$

Hence,

$$\vartheta_1 \equiv_p \frac{1}{2}\left(3\left(\frac{p}{3}\right) - \left(\frac{p}{5}\right)\right). \tag{2.1}$$

Now we turn to calculate ϑ_2; like the identity of ϑ_1, we have the following identity:

$$\vartheta_2 = \frac{1}{4}\left(\sum_{k=0}^{2p-1}k\binom{2k}{k} + \sum_{k=0}^{2p-1}(-1)^k k\binom{2k}{k}\right).$$

It is easy to see that

$$\begin{aligned}\sum_{k=0}^{2p-1}k\binom{2k}{k} &= \sum_{k=0}^{p-1}k\binom{2k}{k} + \sum_{k=p}^{2p-1}k\binom{2k}{k} \\ &= \sum_{k=0}^{p-1}k\binom{2k}{k} + \sum_{k=0}^{p-1}(p+k)\binom{2p+2k}{p+k}\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{2p-1}(-1)^k k\binom{2k}{k} &= \sum_{k=0}^{p-1}(-1)^k k\binom{2k}{k} + \sum_{k=p}^{2p-1}(-1)^k k\binom{2k}{k} \\ &= \sum_{k=0}^{p-1}(-1)^k k\binom{2k}{k} + \sum_{k=0}^{p-1}(-1)^{p+k}(p+k)\binom{2p+2k}{p+k}.\end{aligned}$$

Then, by Lucas congruence, we have:

$$\begin{aligned}\sum_{k=0}^{2p-1} k\binom{2k}{k} &\equiv_p \sum_{k=0}^{p-1} k\binom{2k}{k} + \sum_{k=0}^{p-1} k\binom{2}{1}\binom{2k}{k} = 3\sum_{k=0}^{p-1} k\binom{2k}{k} \\ &\equiv_p 3\sum_{k=0}^{(p-1)/2} k\binom{(p-1)/2}{k}(-4)^k \\ &\equiv_p 6\sum_{k=0}^{(p-3)/2} \binom{(p-3)/2}{k}(-4)^k \equiv_p -2\left(\frac{-3}{p}\right) = -2\left(\frac{p}{3}\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{2p-1} (-1)^k k\binom{2k}{k} &\equiv_p \sum_{k=0}^{p-1} (-1)^k k\binom{2k}{k} - \sum_{k=0}^{p-1} (-1)^k k\binom{2}{1}\binom{2k}{k} \\ &= -\sum_{k=0}^{p-1} (-1)^k k\binom{2k}{k} \equiv_p -\sum_{k=0}^{(p-1)/2} k\binom{(p-1)/2}{k}4^k \\ &\equiv_p 2\cdot 5^{(p-3)/2} \equiv_p \frac{2}{5}\left(\frac{p}{5}\right).\end{aligned}$$

Therefore,

$$\vartheta_2 \equiv_p \frac{1}{4}\left(-2\left(\frac{p}{3}\right) + \frac{2}{5}\left(\frac{p}{5}\right)\right) = -\frac{1}{2}\left(\frac{p}{3}\right) + \frac{1}{10}\left(\frac{p}{5}\right). \tag{2.2}$$

Combining (2.1) and (2.2), we immediately obtain that

$$\sum_{n=0}^{p-1}(5n+1)\binom{4n}{2n} = \vartheta_1 + 5\vartheta_2 \equiv_p -\left(\frac{p}{3}\right)$$

and

$$\sum_{n=0}^{p-1}(3n+1)\binom{4n}{2n} \equiv_p -\frac{1}{5}\left(\frac{p}{5}\right).$$

So the proof of Theorem 1.1 is complete. □

3. Proof of Theorem 1.2

Lemma 3.1. ([23, Lemma 2.1]) *Let p be an odd prime. Then, for any $k = 1, \ldots, p-1$, we have:*

$$k\binom{2k}{k}\binom{2(p-k)}{p-k} \equiv (-1)^{\lfloor 2k/p\rfloor -1}2p \pmod{p^2}.$$

Lemma 3.2. ([24, Lemma 3.1]) *For any $n = 0, 1, 2, \ldots$, we have:*

$$\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}^2 = \sum_{k=0}^{n}\binom{2k}{k}^3\binom{k}{n-k}(-16)^{n-k}.$$

Lemma 3.3. (Sun [23, (1.4)]) *For any prime $p > 3$, we have:*

$$\sum_{k=1}^{(p-1)/2}\frac{4^k}{k^2\binom{2k}{k}} \equiv (-1)^{(p-1)/2}4E_{p-3} \pmod{p}.$$

Proof of Theorem 1.2. The $p=3$ case is easy to check. So we just need to prove that for $p>3$. First by Lemma 3.2, we have:

$$\sum_{n=0}^{p-1}\frac{n}{32^n}C_n=\sum_{n=0}^{p-1}\frac{n}{32^n}\sum_{k=0}^{n}\binom{2k}{k}^3\binom{k}{n-k}(-16)^{n-k}$$
$$=\sum_{k=0}^{p-1}\binom{2k}{k}^3\sum_{n=k}^{p-1}\frac{n}{32^n}\binom{k}{n-k}(-16)^{n-k}$$
$$=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{32^k}\sum_{n=0}^{p-1-k}(n+k)\binom{k}{n}\left(-\frac{1}{2}\right)^n.$$

Then we divide the sum into two parts for $p-1-k\geq k$ and $p-1-k<k$. Set

$$\theta_1=\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{32^k}\sum_{n=0}^{p-1-k}(n+k)\binom{k}{n}\left(-\frac{1}{2}\right)^n$$

and

$$\theta_2=\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}^3}{32^k}\sum_{n=0}^{p-1-k}(n+k)\binom{k}{n}\left(-\frac{1}{2}\right)^n.$$

Thus,

$$\sum_{n=0}^{p-1}\frac{n}{32^n}C_n=\theta_1+\theta_2. \tag{3.1}$$

Now we calculate θ_1. Recall that we have $p-1-k\geq k$; thus,

$$\sum_{n=0}^{p-1-k}(n+k)\binom{k}{n}\left(-\frac{1}{2}\right)^n=\sum_{n=0}^{k}(n+k)\binom{k}{n}\left(-\frac{1}{2}\right)^n$$
$$=\frac{k}{2^k}+k\sum_{n=1}^{k}\binom{k-1}{n-1}\left(-\frac{1}{2}\right)^n$$
$$=\frac{k}{2^k}-\frac{k}{2^k}=0.$$

Hence

$$\theta_1=0. \tag{3.2}$$

Then we turn to compute θ_2; now $p-1-k<k$; by Lemma 3.1, we have:

$$\theta_2=\sum_{k=1}^{(p-1)/2}\frac{\binom{2p-2k}{p-k}^3}{32^{p-k}}\sum_{n=0}^{k-1}(n+p-k)\binom{p-k}{n}\left(-\frac{1}{2}\right)^n$$
$$\equiv-\frac{p^3}{4}\sum_{k=1}^{(p-1)/2}\frac{32^k}{k^3\binom{2k}{k}^3}\sum_{n=0}^{k-1}(n-k)\binom{-k}{n}\left(-\frac{1}{2}\right)^n \pmod{p^4}.$$

Note that $\binom{-k}{n}=(-1)^n\binom{n+k-1}{n}$; we have:

$$\sum_{n=0}^{k-1}(n-k)\binom{-k}{n}\left(-\frac{1}{2}\right)^n$$
$$=-k\sum_{n=0}^{k-1}\binom{-k}{n}\left(-\frac{1}{2}\right)^n-k\sum_{n=1}^{k-1}\binom{-k-1}{n-1}\left(-\frac{1}{2}\right)^n$$
$$=-k\sum_{n=0}^{k-1}\binom{n+k-1}{n}\left(\frac{1}{2}\right)^n+\frac{k}{2}\sum_{n=0}^{k-2}\binom{n+k}{n}\frac{1}{2^n}.$$

By taking $m=n$ and $x=1/2$ in, e.g., [10, (1.1)], we obtain that

$$\sum_{n=0}^{k}\binom{n+k}{n}\frac{1}{2^n}=2^k.$$

So,

$$\begin{aligned}&\sum_{n=0}^{k-1}(n-k)\binom{-k}{n}\left(-\frac{1}{2}\right)^n\\&=-k2^{k-1}+\frac{k}{2}\left(\sum_{n=0}^{k}\binom{n+k}{n}\frac{1}{2^n}-\binom{2k-1}{k-1}\frac{1}{2^{k-1}}-\binom{2k}{k}\frac{1}{2^k}\right)\\&=-k2^{k-1}+\frac{k}{2}\left(2^k-\binom{2k}{k}\frac{2}{2^k}\right)=-\frac{k}{2^k}\binom{2k}{k}.\end{aligned}$$

Hence,

$$\theta_2\equiv\frac{p^3}{4}\sum_{k=1}^{(p-1)/2}\frac{32^k}{k^3\binom{2k}{k}^3}\frac{k}{2^k}\binom{2k}{k}=\frac{p^3}{4}\sum_{k=1}^{(p-1)/2}\frac{16^k}{k^2\binom{2k}{k}^2}\pmod{p^4}.$$

Set $n=(p-1)/2$; then we have:

$$\sum_{k=1}^{(p-1)/2}\frac{16^k}{k^2\binom{2k}{k}^2}\equiv\sum_{k=1}^{n}\frac{1}{k^2\binom{n}{k}^2}=\frac{1}{n^2}\sum_{k=0}^{n-1}\frac{1}{\binom{n-1}{k}^2}\equiv 4\sum_{k=0}^{n-1}\frac{1}{\binom{n-1}{k}^2}\pmod{p}.$$

We have the following identity in [28],

$$\sum_{k=0}^{n-1}\frac{1}{\binom{n-1}{k}^2}=\frac{2n^2}{n+1}\sum_{k=1}^{n}\frac{1}{k\binom{2n+1-k}{n-k}}.$$

So,

$$\begin{aligned}\sum_{k=1}^{(p-1)/2}\frac{16^k}{k^2\binom{2k}{k}^2}&\equiv 4\sum_{k=1}^{n}\frac{1}{k\binom{-k}{n-k}}=4\sum_{k=1}^{n}\frac{(-1)^{n-k}}{k\binom{n-1}{k-1}}=4n\sum_{k=1}^{n}\frac{(-1)^{n-k}}{k^2\binom{n}{k}}\\&\equiv -2(-1)^n\sum_{k=1}^{n}\frac{4^k}{k^2\binom{2k}{k}}\pmod{p}.\end{aligned}$$

Therefore, with the help of Lemma 3.3, we finally obtain

$$\sum_{k=1}^{(p-1)/2}\frac{16^k}{k^2\binom{2k}{k}^2}\equiv -8E_{p-3}\pmod{p}$$

and hence

$$\theta_2\equiv -2p^3E_{p-3}\pmod{p^4}.$$

This, with (3.1) and (3.2), yields that

$$\sum_{n=0}^{p-1}\frac{n}{32^n}C_n\equiv -2p^3E_{p-3}\pmod{p^4},$$

as desired. □

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References

[1] G. Almkvist, C. van Enckevort, D. van Straten, W. Zudilin, Tables of Calabi–Yau equations, preprint, arXiv:math/0507430, 2010.
[2] G. Almkvist, D. van Straten, W. Zudilin, Generalizations of Clausen's formula and algebraic transformations of Calabi–Yau differential equations, Proc. Edinb. Math. Soc. 54 (2) (2011) 273–295.
[3] M. Apagodu, Elementary proof of congruences involving sum of binomial coefficients, Int. J. Number Theory 14 (2018) 1547–1557.
[4] D. Bailey, J. Borwein, D. Broadhurst, M. Glasser, Elliptic integral evaluations of Bessel moments and applications, J. Phys. A 41 (2008) 205203, 46 p.
[5] J. Borwein, D. Nuyens, A. Straub, J. Wan, Some arithmetic properties of short random walk integrals, Ramanujan J. 26 (1) (2011) 109–132.
[6] H. Chan, S. Chan, Z. Liu, Domb's numbers and Ramanujan-Sato type series for $1/\pi$, Adv. Math. 186 (2) (2004) 396–410.
[7] V.J.W. Guo, Proof of a supercongruence conjectured by Z.-H. Sun, Integral Transforms Spec. Funct. 25 (2014) 1009–1015.
[8] F. Jarvis, H.A. Verrill, Supercongruences for the Catalan–Larcombe–French numbers, Ramanujan J. 22 (2010) 171–186.
[9] X.-J. Ji, Z.-H. Sun, Congruences for Catalan–Larcombe–French numbers, Publ. Math. (Debr.) 90 (2017) 387–406.
[10] T.H. Koornwinder, M.J. Schlosser, On an identity by Chaundy and Bullard. I, Indag. Math. 19 (2) (2008) 239–261.
[11] P. Larcombe, D. French, On the 'other' Catalan numbers: a historical formulation re-examined, Congr. Numer. 143 (2000) 33–64.
[12] J.-C. Liu, On two conjectural supercongruences of Apagodu and Zeilberger, J. Differ. Equ. Appl. 22 (2016) 1791–1799.
[13] G.-S. Mao, Proof of two conjectural supercongruences involving Catalan–Larcombe–French numbers, J. Number Theory 179 (2017) 88–96.
[14] G.-S. Mao, H. Pan, Supercongruences on some binomial sums involving Lucas sequences, J. Math. Anal. Appl. 448 (2017) 1061–1078.
[15] G.-S. Mao, Z.-W. Sun, Two congruences involving harmonic numbers with applications, Int. J. Number Theory 12 (2016) 527–539.
[16] G.-S. Mao, J. Wang, On some congruences involving Domb numbers and harmonic numbers, Int. J. Number Theory 15 (10) (2019) 2179–2200, https://doi.org/10.1142/S1793042119501203.
[17] H. Pan, Z.-W. Sun, A combinatorial identity with applications to Catalan numbers, Discrete Math. 306 (2006) 1921–1940.
[18] L. Richmond, J. Shallit, Counting Abelian squares, Electron. J. Comb. 16 (1) (2009) 72, 9 p.
[19] N. Sloane, The on-line encyclopedia of integer sequences, available at http://oeis.org.
[20] Z.-H. Sun, Generalized Legendre polynomials and related supercongruences, J. Number Theory 143 (2014) 293–319.
[21] Z.-H. Sun, Congruences for Domb and Almkvist–Zudilin numbers, Integral Transforms Spec. Funct. 26 (8) (2015) 642–659.
[22] Z.-H. Sun, Congruences for Apéry-like numbers, arXiv:1803.10051v5, 2018.
[23] Z.W. Sun, Super congruences and Euler numbers, Sci. China Math. 54 (12) (2011) 2509–2535.
[24] Z.W. Sun, On sums involving products of three binomial coefficients, Acta Arith. 156 (2012) 123–141.
[25] Z.W. Sun, Some new series for $1/\pi$ and related congruences, J. Nanjing Univ. Math. Biq. 3 (2014) 150–164.
[26] Z.-W. Sun, R. Tauraso, New congruences for central binomial coefficients, Adv. Appl. Math. 45 (2010) 125–148.
[27] Z.-W. Sun, R. Tauraso, On some new congruences for binomial coefficients, Int. J. Number Theory 7 (2011) 645–662.
[28] B. Sury, T.-M. Wang, F.-Z. Zhao, Identities involving reciprocals of binomial coefficients, J. Integer Seq. 7 (2004), Article 04.2.8.
[29] D. Zagier, Integral solutions of Apery-like recurrence equations, in: Groups and Symmetries: From Neolithic Scots to John McKay, in: CRM Proc. Lecture Notes, vol. 47, Amer. Math. Soc., Providence, RI, USA, 2009, pp. 349–366.