



INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus

Mathématique

Guangdong Jing and Penghui Wang

A note on “Problem of eigenvalues of stochastic Hamiltonian systems with boundary conditions”

Volume 359, issue 1 (2021), p. 99-104

Published online: 15 February 2021

<https://doi.org/10.5802/crmath.103>



This article is licensed under the
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.
<http://creativecommons.org/licenses/by/4.0/>



Les Comptes Rendus. Mathématique sont membres du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569



Stochastic differential equations / *Équations différentielles stochastiques*

A note on “Problem of eigenvalues of stochastic Hamiltonian systems with boundary conditions”

Guangdong Jing^a and Penghui Wang^{✉*, a}

^a School of Mathematics, Shandong University Jinan, Shandong 250100, The People's Republic of China

E-mails: 201511240@mail.sdu.edu.cn (G. Jing), phwang@sdu.edu.cn (P. Wang)

Abstract. The eigenvalue problem of stochastic Hamiltonian systems with boundary conditions was studied by Peng [4] in 2000. For the one-dimensional case, denoting by $\{\lambda_n\}_{n=1}^\infty$ all the eigenvalues of such an eigenvalue problem, Peng proved that $\lambda_n \rightarrow +\infty$ as $n \rightarrow \infty$. In this short note, we prove that the growth order of λ_n is the same as n^2 . Apart from the interest of this result in itself, the statistic periodicity of solutions of FBSEs can be estimated directly by corresponding coefficients and time duration.

2020 Mathematics Subject Classification. 34L15, 60H10.

Funding. Guangdong Jing is partially supported by NSFC (No.11871308). Penghui Wang is partially supported by NSFC (No.11471189, No.11871308).

Manuscript received 23rd October 2019, revised 17th April 2020 and 28th July 2020, accepted 28th July 2020.

1. Introduction and main results

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space, on which a standard one-dimensional Brownian motion $B = \{B_t\}_{t \geq 0}$ is defined, and $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration of B augmented by all the \mathbb{P} -null sets in \mathcal{F} . Let $T > 0$ be any fixed time horizon.

In [4], Peng considered the following eigenvalue problem of stochastic Hamiltonian system with boundary conditions:

$$\begin{cases} dx_t &= [H_{21}^\lambda x_t + H_{22}^\lambda y_t + H_{23}^\lambda z_t] dt + [H_{31}^\lambda x_t + H_{32}^\lambda y_t + H_{33}^\lambda z_t] dB_t, & t \in [0, T], \\ -dy_t &= [H_{11}^\lambda x_t + H_{12}^\lambda y_t + H_{13}^\lambda z_t] dt - z_t dB_t, & t \in [0, T], \\ x(0) &= 0, & y(T) = 0, \end{cases} \quad (1)$$

* Corresponding author.

where $H^\lambda = H - \lambda \tilde{H}$,

$$H = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} \tilde{H}_{11} & \tilde{H}_{12} & \tilde{H}_{13} \\ \tilde{H}_{21} & \tilde{H}_{22} & \tilde{H}_{23} \\ \tilde{H}_{31} & \tilde{H}_{32} & \tilde{H}_{33} \end{bmatrix},$$

$H_{ij}^\lambda = H_{ij} - \lambda \tilde{H}_{ij}$, $H_{ij} = H_{ij}^T$, $\tilde{H}_{ij} = \tilde{H}_{ij}^T$, $i, j = 1, 2, 3$, which are constant matrices.

Definition 1. A real number λ is called an eigenvalue of linear stochastic Hamiltonian system with boundary conditions (1) if there exists a nontrivial solution (x, y, z) of (1). This solution is called an eigenfunction corresponding to λ . All eigenfunctions associated with the eigenvalue λ constitute a linear subspace of $M^2(0, T; \mathbb{R}^n)$, called the eigenfunction subspace corresponding to λ .

The above eigenvalue problem is a stochastic analogue of classical eigenvalue problem of mechanic systems, and it is closely related to the existence of solutions to Forward–Backward Stochastic Differential Equations (FBSDEs in short). Please refer to [1–5] and references therein for the well-posedness of FBSDEs, among which the *monotonicity condition* is an important sufficient condition for the existence and uniqueness of solutions to FBSDEs. In particular, for linear FBSDE (1) with $\tilde{H} = 0$, the monotonicity condition has the following concrete form:

$$\begin{bmatrix} -H_{11} & -H_{12} & -H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} \leq -\alpha I_{3n}, \tag{2}$$

where $\alpha > 0$ is a constant.

For the case

$$\tilde{H} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & H_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3}$$

by using the blow-up time (as usual, the *blow-up time* denotes the endpoints of the maximum existing interval of the solution to certain ODEs) of the solution for the related Riccati equation, Peng proved the following

Theorem 2. [4, Theorem 3.2] For (1) of one dimension with perturbation (3), assume that (2) is satisfied as well as $H_{23} = -H_{33}H_{13}$. Then there exist $\{\lambda_n\}_{n=1}^{+\infty}$, all the eigenvalues of the problem (1), such that $\lambda_n \rightarrow +\infty$ as $n \rightarrow +\infty$. Moreover, the eigenfunction space corresponding to each λ_n is of one dimension.

Such a theorem was generalized to the eigenvalue problem of stochastic Hamiltonian system driven Poisson process in [6].

The existence of eigenvalues is given in the above Theorem 2. Then it is natural and meaningful to ask that whether those eigenvalues have any relationship with the coefficients of systems and how they tend to infinity. Towards solving these problems we have the following

Theorem 3. Under the same assumptions in Theorem 2,

$$\lambda_n = O(n^2), \quad \text{as } n \rightarrow +\infty.$$

In detail,

$$\frac{\pi^2}{-2H_{11}H_{22}T^2} \leq \liminf_{n \rightarrow +\infty} \frac{\lambda_n}{n^2} \leq \limsup_{n \rightarrow +\infty} \frac{\lambda_n}{n^2} \leq \frac{4\pi^2}{-H_{11}H_{22}T^2}.$$

Remark 4. The result about order in Theorem 3 can be considered as an analogue of the well-known result in deterministic case.

In literature, the study of FBSDEs is mainly focused on the existence and uniqueness of solutions and hardly any on properties of solutions. From the point of view of eigenvalue problem of stochastic Hamiltonian system with boundary conditions, what is different is that some concrete characteristic such as statistic periodicity and stochastic oscillations of solutions of FBSDEs can be given in [4]. By the proof of [4, Theorem 3.2], the eigenfunction associated to the n^{th} eigenvalue has the statistic periods of n . Combining with Theorem 3, we have the following Corollary 5.

Corollary 5. *Let λ be an eigenvalue of the stochastic Hamiltonian system in Theorem 3, for sufficiently large n , if*

$$\lambda < \frac{n^2 \pi^2}{-2H_{11}H_{22}T^2}, \quad \left(\text{resp. } \lambda > \frac{4n^2 \pi^2}{-H_{11}H_{22}T^2} \right),$$

the statistic periods of the associate eigenfunctions (i.e., the solutions of FBSDEs) is less (resp. greater) than n .

The rest of the paper is organized as follows. In Section 2, we recall some preliminary results and give several lemmata. The proof of Theorem 3 is given in Section 3.

2. Preliminaries and several lemmata

For one-dimensional case with perturbation (3), the eigenvalue problem of stochastic Hamiltonian system with boundary conditions (1) is rewritten as

$$\begin{cases} dx_t &= [H_{21}x_t + (1-\lambda)H_{22}y_t + H_{23}z_t] dt + [H_{31}x_t + H_{32}y_t + H_{33}z_t] dB_t, \quad t \in [0, T], \\ -dy_t &= [H_{11}x_t + H_{12}y_t + H_{13}z_t] dt - z_t dB_t, \quad t \in [0, T], x(0) = 0, \quad y(T) = 0. \end{cases} \quad (4)$$

As given in [4, Subsection 4.2], through Legendre transformation, the dual Hamiltonian \tilde{H} of the original Hamiltonian H corresponding to (4) is

$$\tilde{H} = \begin{bmatrix} H_{33}^{-1}H_{32}^2 - \rho H_{22} & H_{33}^{-1}H_{32}H_{31} - H_{21} & -H_{33}^{-1}H_{32} \\ H_{33}^{-1}H_{32}H_{31} - H_{21} & H_{33}^{-1}H_{31}^2 & -H_{33}^{-1}H_{31} \\ -H_{33}^{-1}H_{32} & -H_{33}^{-1}H_{31} & H_{33}^{-1} \end{bmatrix},$$

where $\rho = 1 - \lambda$, and the relation between solution (x, y, z) of original Hamiltonian system and solution $(\tilde{x}, \tilde{y}, \tilde{z})$ of dual Hamiltonian system is:

$$\begin{cases} x(t) &= \tilde{y}(t), \quad y(t) = \tilde{x}(t), \\ z(t) &= -H_{33}^{-1}H_{32}\tilde{x}(t) - H_{33}^{-1}H_{31}\tilde{y}(t) + H_{33}^{-1}\tilde{z}(t). \end{cases}$$

In Peng [4], the idea to study the eigenvalue problem of stochastic Hamiltonian system is to deal with the blow-up time of the following Riccati equations with terminal conditions [4, (6.2)]:

$$\begin{cases} \frac{dk}{dt} &= -(2H_{21} + H_{13}^2)k - H_{11} - (\rho H_{22} - H_{33}H_{13}^2)k^2, \quad t \leq T, \\ k(T) &= 0, \end{cases} \quad (5)$$

and dual Riccati equations with terminal conditions [4, (6.4)]:

$$\begin{cases} \frac{d\tilde{k}}{dt} &= (2H_{21} + H_{13}^2)\tilde{k} + H_{11}\tilde{k}^2 + (\rho H_{22} - H_{33}H_{13}^2), \quad t \leq T, \\ \tilde{k}(T) &= 0. \end{cases} \quad (6)$$

The following two lemmata 6 and 7 from [4] are needed.

Lemma 6 (See [4, Lemma 6.1]). *For the Riccati equation (5), the blow-up time t_ρ is continuous and strictly decreasing with respect to ρ when $\rho < \rho_0$, where $\rho_0 = H_{22}^{-1}H_{33}H_{13}^2$. Besides,*

$$\lim_{\rho \rightarrow -\infty} t_\rho = T, \quad \lim_{\rho \rightarrow \rho_0^-} t_\rho = -\infty. \quad (7)$$

Lemma 7 (See [4, Lemma 6.2]). For the dual Riccati equation (6), the blow-up time \tilde{t}_ρ is continuous and strictly decreasing with respect to ρ when $\rho < \rho_0$. Besides,

$$\lim_{\rho \rightarrow -\infty} \tilde{t}_\rho = T, \quad \lim_{\rho \rightarrow \rho_0} \tilde{t}_\rho = -\infty. \tag{8}$$

To simplify the notation, denote

$$\begin{cases} q(\rho) = -(\rho H_{22} - H_{33} H_{13}^2), & r = -H_{11}, \\ \tilde{r}(\rho) = (\rho H_{22} - H_{33} H_{13}^2), & \tilde{q} = H_{11}, \\ \tilde{p} = -p = 2H_{21} + H_{13}^2, \\ \rho_* = (4H_{11}H_{22})^{-1} (2H_{21} + H_{13}^2)^2. \end{cases} \tag{9}$$

By (2),

$$r = -H_{11} < 0, \quad H_{22} < 0, \quad H_{33} < 0.$$

Besides, by [4, Page 278], ρ_0 is the critical point, which implies that all the eigenvalues of problem (4) are located in $(-\infty, \rho_0)$. Moreover, by Theorem 2, there are at most finite eigenvalues $\{\lambda_i\}_{i=1}^m$ of problem (4), such that $1 - \lambda_i \in [\rho_0 + \rho_*, \rho_0)$. Since we merely try to depict the growth order of λ_n in this paper, it is reasonable for us to only check those $\rho \in (-\infty, \rho_0 + \rho_*)$. Actually, that $\rho < \rho_0 + \rho_*$ guarantees

$$1 - \frac{p^2}{4rq(\rho)} = 1 - \frac{\tilde{p}^2}{4\tilde{r}(\rho)\tilde{q}} > 0. \tag{10}$$

In what follows, for simplicity, we sometimes omit the ρ in $q(\rho)$ and $\tilde{r}(\rho)$.

The following two Lemmata 8 and 9 depict the solutions of (5) and (6), which are essential in proving Theorem 3.

Lemma 8. The blow-up time t_ρ of solution k of (5) satisfies

$$\sqrt{rq(\rho) - \frac{p^2}{4}} (T - t_\rho) + \arctan \frac{-p}{\sqrt{4rq(\rho) - p^2}} = \frac{\pi}{2}. \tag{11}$$

Proof. Under (10), $pk + r + qk^2 < 0$ and $1 - \frac{p^2}{4rq} > 0$. Then

$$\frac{dk}{\frac{p}{r}k + \frac{q}{r}k^2 + 1} = \frac{dk}{\left(\sqrt{\frac{q}{r}}k - \frac{p}{2\sqrt{rq}}\right)^2 + \left(1 - \frac{p^2}{4rq}\right)} = rdt.$$

Combined with terminal condition $k(T) = 0$,

$$k = -\frac{\sqrt{4rq - p^2}}{2q} \tan \left[\sqrt{rq - \frac{p^2}{4}} (T - t) + \arctan \frac{-p}{\sqrt{4rq - p^2}} \right] - \frac{p}{2q}.$$

Therefore, for any fixed ρ ($\rho < \rho_0 + \rho_*$), the blow-up time t_ρ of k satisfies

$$\sqrt{rq - \frac{p^2}{4}} (T - t_\rho) + \arctan \frac{-p}{\sqrt{4rq - p^2}} = \frac{\pi}{2}.$$

□

Lemma 9. The blow-up time \tilde{t}_ρ of solution \tilde{k} of (6) satisfies

$$\sqrt{\tilde{q}\tilde{r}(\rho) - \frac{\tilde{p}^2}{4}} (\tilde{t}_\rho - T) + \arctan \frac{\tilde{p}}{\sqrt{4\tilde{q}\tilde{r}(\rho) - \tilde{p}^2}} = -\frac{\pi}{2}. \tag{12}$$

Proof. Under (10), $\tilde{r} + \tilde{p}\tilde{k} + \tilde{q}\tilde{k}^2 > 0$ and $1 - \frac{\tilde{p}^2}{4\tilde{r}\tilde{q}} > 0$. Then

$$\frac{d\tilde{k}}{1 + \frac{\tilde{p}}{\tilde{r}}\tilde{k} + \frac{\tilde{q}}{\tilde{r}}\tilde{k}^2} = \frac{d\tilde{k}}{\left(\sqrt{\frac{\tilde{q}}{\tilde{r}}}\tilde{k} + \frac{\tilde{p}}{2\sqrt{\tilde{q}\tilde{r}}}\right)^2 + \left(1 - \frac{\tilde{p}^2}{4\tilde{q}\tilde{r}}\right)} = \tilde{r}dt.$$

Combined with terminal condition $\tilde{k}(T) = 0$,

$$\tilde{k} = \frac{\sqrt{4\tilde{q}\tilde{r} - \tilde{p}^2}}{2\tilde{q}} \tan \left[\frac{\sqrt{4\tilde{q}\tilde{r} - \tilde{p}^2}}{2}(t - T) + \arctan \frac{\tilde{p}}{\sqrt{4\tilde{q}\tilde{r} - \tilde{p}^2}} \right] - \frac{\tilde{p}}{2\tilde{q}}.$$

For any fixed ρ ($\rho < \rho_0 + \rho_*$), the blow-up time \tilde{t}_ρ of solution \tilde{k} satisfies

$$\sqrt{\tilde{q}\tilde{r} - \frac{\tilde{p}^2}{4}}(\tilde{t}_\rho - T) + \arctan \frac{\tilde{p}}{\sqrt{4\tilde{q}\tilde{r} - \tilde{p}^2}} = -\frac{\pi}{2}.$$

□

3. Proof of Theorem 3

In this section, we prove Theorem 3.

Proof of Theorem 3. By [4, Section 6, Proof of Theorem 3.2.], the n^{th} eigenvalue λ_n of (4) is uniquely determined by

$$t_{\rho_n}^{2n-1} = 0, \quad \lambda_n = 1 - \rho_n, \quad \forall n \in \mathbb{N}_+,$$

where

$$\begin{aligned} t_{\rho_n}^{2j-1} &= T - j(T - t_{\rho_n}) - (j-1)(T - \tilde{t}_{\rho_n}), & j &= 1, 2, \dots, n, \\ t_{\rho_n}^{2j-2} &= T - (j-1)(T - t_{\rho_n}) - (j-1)(T - \tilde{t}_{\rho_n}), & t_{\rho_1}^1 &= t_{\rho_1}. \end{aligned}$$

By Lemma 8 and Lemma 9, we obtain

$$\sqrt{rq - \frac{p^2}{4}}(T - t_\rho) + \arctan \frac{-p}{\sqrt{4rq - p^2}} = \frac{\pi}{2}$$

and

$$\sqrt{\tilde{q}\tilde{r} - \frac{\tilde{p}^2}{4}}(\tilde{t}_\rho - T) + \arctan \frac{\tilde{p}}{\sqrt{4\tilde{q}\tilde{r} - \tilde{p}^2}} = -\frac{\pi}{2}.$$

Then

$$\lim_{n \rightarrow +\infty} \frac{T - t_{\rho_n}}{T - \tilde{t}_{\rho_n}} = 1$$

and

$$\sqrt{rq - \frac{p^2}{4}} = \frac{\frac{\pi}{2} + \arctan \frac{p}{\sqrt{4rq - p^2}}}{T - t_\rho} = \frac{\frac{\pi}{2} - \arctan \frac{p}{\sqrt{4rq - p^2}}}{T - \tilde{t}_\rho}. \tag{13}$$

Besides, following the method in [4, Subsection 6.2],

$$T - n(T - t_{\rho_n}) - (n-1)(T - \tilde{t}_{\rho_n}) = 0. \tag{14}$$

Then

$$\max\{T - t_{\rho_n}, T - \tilde{t}_{\rho_n}\} \geq \frac{T}{2n-1}$$

and for sufficiently large n ,

$$\max\{T - t_{\rho_n}, T - \tilde{t}_{\rho_n}\} \leq \frac{T}{2n-2}.$$

Then by (13), for sufficiently large n ,

$$\frac{(n-1)\pi}{T} \leq \sqrt{rq(\rho_n) - \frac{p^2}{4}} \leq \frac{(2n-1)\pi}{T}.$$

Moreover, by (9),

$$rq(\rho_n) - \frac{p^2}{4} = (-H_{11}H_{22})\lambda_n + H_{11}H_{22} - H_{11}H_{33}H_{13}^2 - \frac{(2H_{21} + H_{13}^2)^2}{4}.$$

Then

$$\frac{\pi^2}{-2H_{11}H_{22}T^2} \leq \liminf_{n \rightarrow +\infty} \frac{\lambda_n}{n^2} \leq \limsup_{n \rightarrow +\infty} \frac{\lambda_n}{n^2} \leq \frac{4\pi^2}{-H_{11}H_{22}T^2}.$$

□

References

- [1] Y. Hu, S. Peng, "Solution of forward-backward stochastic differential equations", *Probab. Theory Relat. Fields* **103** (1995), no. 2, p. 273-283.
- [2] J. Ma, P. Protter, J. Yong, "Solving forward-backward stochastic differential equations explicitly — a four step scheme", *Probab. Theory Relat. Fields* **98** (1994), no. 3, p. 339-359.
- [3] J. Ma, Z. Wu, D. Zhang, J. Zhang, "On well-posedness of forward-backward SDEs — a unified approach", *Ann. Appl. Probab.* **25** (2015), no. 4, p. 2168-2214.
- [4] S. Peng, "Problem of eigenvalues of stochastic Hamiltonian systems with boundary conditions", *Stoch. Proc. Appl.* **88** (2000), no. 2, p. 259-290.
- [5] S. Peng, Z. Wu, "Fully coupled forward-backward stochastic differential equations and applications to optimal control", *SIAM J. Control Optimization* **37** (1999), no. 3, p. 825-843.
- [6] H. Wang, Z. Wu, "Eigenvalues of stochastic Hamiltonian systems driven by Poisson process with boundary conditions", *Bound. Value Probl.* **2017** (2017), article no. 164.