



INSTITUT DE FRANCE  
Académie des sciences

# *Comptes Rendus*

---

# *Mathématique*


Sourav Das

**A complete monotonicity property of the multiple gamma function**

Volume 358, issue 8 (2020), p. 917-922.

<<https://doi.org/10.5802/crmath.115>>

© Académie des sciences, Paris and the authors, 2020.  
*Some rights reserved.*

 This article is licensed under the  
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.  
<http://creativecommons.org/licenses/by/4.0/>



*Les Comptes Rendus. Mathématique* sont membres du  
Centre Mersenne pour l'édition scientifique ouverte  
[www.centre-mersenne.org](http://www.centre-mersenne.org)



Number Theory / *Théorie des nombres*

# A complete monotonicity property of the multiple gamma function

Sourav Das<sup>Ⓢ</sup> <sup>a</sup>

<sup>a</sup> Department of Mathematics, National Institute of Technology Jamshedpur,  
Jharkhand-831014, India

E-mail: [souravdasmath@gmail.com](mailto:souravdasmath@gmail.com), [souravdas.math@nitjsr.ac.in](mailto:souravdas.math@nitjsr.ac.in)

**Abstract.** We consider the following functions

$$f_n(x) = 1 - \ln x + \frac{\ln G_n(x+1)}{x} \text{ and } g_n(x) = \frac{\sqrt[n]{G_n(x+1)}}{x}, \quad x \in (0, \infty), \quad n \in \mathbb{N},$$

where  $G_n(z) = (\Gamma_n(z))^{(-1)^{n-1}}$  and  $\Gamma_n$  is the multiple gamma function of order  $n$ . In this work, our aim is to establish that  $f_{2n}^{(2n)}(x)$  and  $(\ln g_{2n}(x))^{(2n)}$  are strictly completely monotonic on the positive half line for any positive integer  $n$ . In particular, we show that  $f_2(x)$  and  $g_2(x)$  are strictly completely monotonic and strictly logarithmically completely monotonic respectively on  $(0, 3]$ . As application, we obtain new bounds for the Barnes G-function.

**2020 Mathematics Subject Classification.** 33B15, 26D07.

*Manuscript received 2nd August 2020, revised and accepted 8th September 2020.*

## 1. Introduction

E. W. Barnes introduced multiple gamma functions  $\Gamma_n$  around 1899 in a series of papers [2, 3], which generalize the Euler's gamma function  $\Gamma$ .  $\Gamma_n$  appears in functional equations for the Selberg zeta functions associated to higher rank symmetric spaces.  $\Gamma_n$  are also useful to study the determinant of Laplacians on the  $n$ -dimensional unit sphere [5, 6]. For these reasons, the multiple gamma functions have recently attracted many researchers' concern.  $\Gamma_n$  plays vital role in mathematical physics, quantum physics, theoretical physics, number theory, approximation theory and in many branches of applied science and engineering. One of the most applicable research area of recent interest for the gamma functions is fractional calculus. It is important to note that  $\Gamma_n$  are of higher order transcendency (known as transcendentally transcendent function) and cannot be obtained by solving algebraic equations and algebraic differential equations. O. Hölder proved this result for the Euler's gamma function in 1887.  $\Gamma_n$  is characterized by the generalized Bohr–Møllerup theorem, which is given below.

**Theorem 1.** [13]  $\Gamma_n$  ( $n \in \mathbb{N}$ ) satisfies the following relations:

- (i)  $\Gamma_n(z) = \frac{\Gamma_{n+1}(z)}{\Gamma_{n+1}(z+1)}$  for  $z \in \mathbb{C}$ ,
- (ii)  $1/\Gamma_n(x)$  is  $\mathbb{C}^\infty$  on  $\mathbb{R}$ ,
- (iii)  $(-1)^n \frac{d^{n+1}}{dx^{n+1}} \log \Gamma_n(x) \geq 0$  for  $x > 0$ ,
- (iv)  $\Gamma_n(1) = 1$ ,
- (v)  $\Gamma_1(z) = \Gamma(z)$  and  $\Gamma_0(z) = 1/z$ .

It is well-known that  $(\Gamma_n(z))^{-1}$  is an entire function with zeros at  $z = -k$ ,  $k \in \mathbb{N} \cup \{0\}$  with multiplicities given by

$$\binom{n+k-1}{n-1} \quad n \in \mathbb{N}, k \in \mathbb{N} \cup \{0\}. \tag{1}$$

Using (1), the following explicit form of  $\Gamma_n$ , in terms of Weierstrass canonical product can be obtained [7]:

$$\Gamma_n(1+z) = \exp [Q_n(z)] \prod_{k=1}^{\infty} \left( \left( 1 + \frac{z}{k} \right)^{-\binom{n+k-2}{n-1}} \exp \left[ \binom{n+k-2}{n-1} \left( \sum_{j=1}^n \frac{(-1)^{j-1} z^j}{j k^j} \right) \right] \right), \tag{2}$$

where  $Q_n(z)$  is a polynomial of degree  $n$  given by

$$\begin{aligned} Q_n(z) &:= (-1)^{n-1} \left[ -zA_n(1) + \sum_{k=1}^{n-1} \frac{p_k(z)}{k!} \left( f_{n-1}^{(k)}(1) \right) \right], \\ f_n(z) &:= -zA_n(1) + \sum_{k=1}^{n-1} \frac{p_k(z)}{k!} \left( f_{n-1}^{(k)}(1) \right) + A_n(z), \\ A_n(z) &:= \sum_{k=1}^{\infty} (-1)^{n-1} \binom{n+k-2}{n-1} \left[ -\log \left( 1 + \frac{z}{k} \right) + \sum_{j=1}^n \frac{(-1)^{j-1} z^j}{j k^j} \right] \end{aligned}$$

and

$$p_n(z) = \frac{1}{n+1} \sum_{k=1}^{n+1} \binom{n+1}{k} B_{n+1-k} z^k, \quad n \in \mathbb{N},$$

with  $B_k$  being the Bernoulli numbers.

By introducing a hierarchy of functions

$$G_n(z) = (\Gamma_n(z))^{(-1)^{n-1}}, \tag{3}$$

M. F. Vignéras [14] redefined multiple gamma functions which satisfy the conditions of generalized Bohr–Mollerup theorem [7, 12, 14].

**Theorem 2.** [7] For all  $n \in \mathbb{N}$ , there exists a unique meromorphic function  $G_n(z)$  satisfying each of the following properties:

- (i)  $G_n(z+1) = G_n(z)G_{n-1}(z)$  for  $z \in \mathbb{C}$ ;
- (ii)  $G_n(1) = 1$ ;
- (iii) For  $x \geq 0$ ,  $G_n(x+1)$  are infinitely differentiable and

$$\frac{d^{n+1}}{dx^{n+1}} \log G_n(x+1) \geq 0;$$

- (iv)  $G_0(z) = z$ .

The reciprocal of the double gamma function is the well-known Barnes G-function  $G(z) = G_2(z) = \frac{1}{\Gamma_2(z)}$ . Note that  $G_1(z) = \Gamma_1(z) = \Gamma(z)$ ,  $G_2(z) = 1/\Gamma_2(z) = G(z)$ ,  $G_3(z) = \Gamma_3(z)$  and so on. The multiple psi function  $\Psi_n$  can be defined as the logarithmic derivative of  $\Gamma_n$ . i.e.  $\Psi_n = \frac{\Gamma'_n}{\Gamma_n}$ . The  $m^{\text{th}}$  order derivative of  $\Psi_n$  (where,  $m, n \in \mathbb{N}$ ) is known as the poly multiple gamma function and denoted by  $\Psi_n^{(m)}$ . Similarly, one can define multiple psi function  $\psi_n = \frac{G'_n}{G_n}$  involving  $G_n$  and poly multiple gamma function  $\Phi_n^{(m)}$  (involving  $G_n$ ) as the  $m^{\text{th}}$  order derivative of  $\Phi_n$ , where,  $m, n \in \mathbb{N}$ . Using (3), it can be easily verify that  $\Phi_n$  and  $\Psi_n$  satisfy the following relations:

$$\Phi_n(z) = (-1)^{n-1} \Psi_n(z) \quad \text{and} \quad \Phi_n^{(m)}(z) = (-1)^{n-1} \Psi_n^{(m)}(z), \quad n, m \in \mathbb{N}. \tag{4}$$

For further information on multiple gamma functions we refer to [1–3, 7–10] and references cited therein.

If a positive function  $g(x)$  has derivatives of all orders on an interval  $I$  and satisfy the following nonnegativity condition

$$(-1)^k g^{(k)}(x) \geq 0 \quad \text{for all } x \in I \quad \text{and} \quad k \geq 0, \tag{5}$$

then the function  $g(x)$  is called completely monotonic on  $I$ . If the inequality (5) is strict for any  $x \in I$ , then  $g(x)$  is known as strictly completely monotonic on  $I$ . A function  $g(x)$  is called logarithmically completely monotonic if

$$(-1)^k (\ln g(x))^{(k)} \geq 0 \quad \text{for all } x \in I \quad \text{and} \quad k \geq 1. \tag{6}$$

If the inequality (6) is strict, then  $g(x)$  is called strictly logarithmically completely monotonic on  $I$ .

In [11], Qi and Chen proved that the function

$$f(x) = 1 - \ln x + \frac{\ln \Gamma(x+1)}{x}$$

is strictly completely monotonic on  $(0, \infty)$ . In addition, they [11] established that the function  $g(x) = \sqrt[x]{\Gamma(x+1)}/x$  is strictly logarithmically completely monotonic on the positive half line. They [11] also proved the following result.

**Theorem 3.** [11] *A (strictly) logarithmically completely monotonic function is also (strictly) completely monotonic.*

The above results motivate us to discuss the complete monotonicity properties of the following functions:

$$f_n(x) = 1 - \ln x + \frac{\ln G_n(x+1)}{x} \quad \text{and} \quad g_n(x) = \frac{\sqrt[x]{G_n(x+1)}}{x}, \quad x \in (0, \infty), \quad n \in \mathbb{N}.$$

## 2. Main Results

Let  $m \geq n$  be any natural number. Then taking logarithm on both sides of (2) and differentiating  $m + 1$  times, we have (see also [8])

$$\Psi_n^{(m)}(x) = (-1)^{m-1} \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} \frac{m!}{(x+k)^{m+1}}, \quad x > 0, \tag{7}$$

where  $(\alpha)_n$  is the Pochhammer symbol defined as  $(\alpha)_0 = 1$ ,  $(\alpha)_n = \prod_{k=1}^n (\alpha + k - 1)$ ,  $n \in \mathbb{N}$ . Combining (4) and (7), we obtain

$$\Phi_n^{(m)}(x) = (-1)^{m+n} \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} \frac{m!}{(x+k)^{m+1}}, \quad m \geq n \geq 1, \quad x > 0. \tag{8}$$

This leads to the following result.

**Theorem 4.** Let  $n$  and  $p$  be natural numbers. Then  $\psi_n^{(n)}(x)$  and  $(-1)^p \psi_n^{(n+p)}(x)$  are positive for any positive real number  $x$ .

Now, we proceed to prove the main result.

**Theorem 5.** Let  $f_n(x)$  and  $g_n(x)$  be defined as

$$f_n(x) = 1 - \ln x + \frac{1}{x} \ln G_n(x + 1), \tag{9}$$

$$g_n(x) = \frac{\sqrt[n]{G_n(x + 1)}}{x}, \quad x \in (0, \infty), \quad n \in \mathbb{N}. \tag{10}$$

Then  $f_{2n}^{(2n)}(x)$  and  $(\ln g_{2n})^{(2n)}(x)$  are strictly completely monotonic on  $(0, \infty)$ .

**Proof.** Let  $n$  and  $p$  be any positive integer. Then with the help of Leibnitz's rule,

$$(u(x)v(x))^{(n)} = \sum_{k=0}^n \binom{n}{k} u^{(k)}(x)v^{(n-k)}(x),$$

we obtain

$$\begin{aligned} f_n^{(n+p)}(x) &= \sum_{k=0}^{n+p} \binom{n+p}{k} \left(\frac{1}{x}\right)^{(n+p-k)} (\ln G_n(x + 1))^{(k)} - \frac{(-1)^{n+p-1}(n+p-1)!}{x^{n+p}} \\ &= \left(\frac{1}{x}\right)^{(n+p)} \ln G_n(x + 1) + \sum_{k=1}^{n+p} \binom{n+p}{k} \left(\frac{1}{x}\right)^{(n+p-k)} (\Phi_n(x + 1))^{(k-1)} \\ &\quad + \frac{(-1)^n(n+p)!}{(n+p)x^{n+p}} \\ &= \frac{(-1)^{n+p}(n+p)!}{x^{n+p+1}} \ln G_n(x + 1) + \sum_{k=1}^{n+p} \frac{(n+p)!}{k!} \frac{(-1)^{n+p-k}}{x^{n+p-k+1}} (\Phi_n(x + 1))^{(k-1)} \\ &\quad + \frac{(-1)^{n+p}(n+p)!}{(n+p)x^{n+p}} \\ &\triangleq (-1)^{n+p} \frac{(n+p)!}{x^{n+p+1}} g_{n,p}(x), \end{aligned} \tag{11}$$

where

$$g'_{n,p}(x) = \frac{(-1)^{n+p}}{(n+p)!} x^{n+p} \Phi_n^{(n+p)}(x + 1) + \frac{1}{n+p}. \tag{12}$$

Using Theorem 4, we have  $g'_{n,p}(x) > 0$ , which implies that  $g_{2n,p}(x)$  is strictly increasing on  $(0, \infty)$ . Hence,  $g_{2n,p}(x) > g_{2n,p}(0) = 0$ . Consequently,  $(-1)^p f_{2n}^{(2n+p)}(x) > 0$  for  $p \geq 0$ . This proves that  $f_{2n}^{(2n)}$  is strictly completely monotonic.

To prove the remaining part of the Theorem 5, let us consider the following function:

$$h_n(x) = \sqrt[n]{G_n(x + 1)}/x.$$

Then

$$\ln h_n(x) = f_n(x) - 1 \implies (\ln h_{2n}(x))^{(2n)} = f_{2n}^{(2n)}(x),$$

which proves the Theorem 5. □

**Remark 6.** We note the following points.

- (i) Using the series expansion [7] of  $\ln G(x + 1)$ , one can verify that  $f_2(x) > 0$  for any  $x \in (0, \infty)$  and

$$f'_2(x) = \begin{cases} < 0, & \text{if } x \in (0, 3] \\ > 0, & \text{if } x \in [3.1, \infty). \end{cases} \tag{13}$$

Figure 1 verifies our claim. Therefore,  $f_2(x)$  is strictly completely monotonic on  $(0, 3]$  and consequently using Theorem 3, we obtain that  $g_2(x)$  is strictly logarithmically completely monotonic on  $(0, 3]$ .

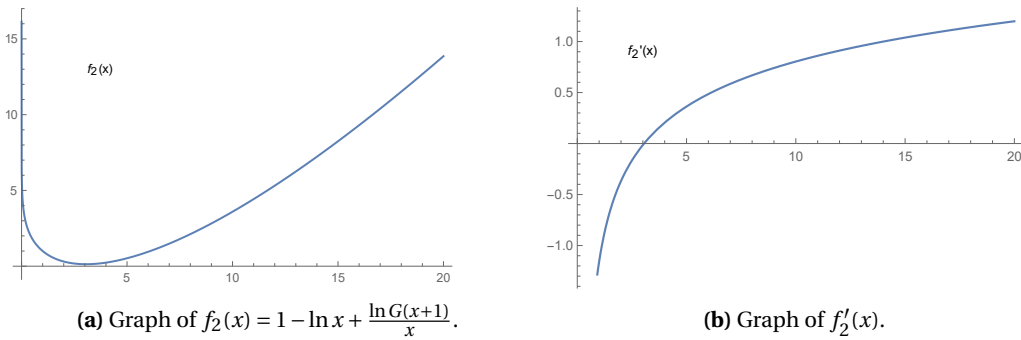
(ii) From Theorem 5, we have  $f_2^{(2)}(x) > 0$ . Again, we have  $f_2(x) > 0$  for all  $x > 0$ . Hence for any positive real  $x$ , the Barnes G-function satisfies the following inequality:

$$\left(\frac{x}{e}\right)^x < G(x+1) < \exp\left[\frac{x}{2}(2\Phi_2(x+1) - 1) - \frac{x^2}{2}\Phi_2^{(1)}(x+1)\right]. \tag{14}$$

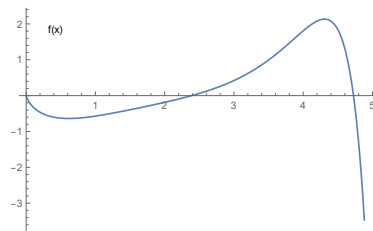
(iii) In [4], Batir established various bounds for  $G(x+1)$ . One of such bounds is given below:

$$(\Gamma(x))^{x/2} x^{-x} (2\pi)^{x/2} e^{\left(-\frac{x}{2} - \frac{x^2}{2}\right)} < G(x+1) < \left(\frac{\Gamma(x)}{\Gamma(x/2)}\right)^x (8\pi)^{x/2} e^{\left(-\frac{x}{2} - \frac{x^2}{2}\right)}. \tag{15}$$

Comparing the lower bounds in (14) and (15) using Mathematica software, we can verify that lower bound in (14) is sharper than the lower bound in (15) in the interval  $[2.4, 4.7]$ . Figure 2 verifies our claim.



**Figure 1.** Graph of  $f_2(x)$  and  $f_2'(x)$ .



**Figure 2.** Graph of  $f(x) = \left(\frac{x}{e}\right)^x - (\Gamma(x))^{x/2} x^{-x} (2\pi)^{x/2} e^{\left(-\frac{x}{2} - \frac{x^2}{2}\right)}$ .

*Acknowledgement*

The author wishes to thank the reviewers for suggestions that helped to improve the paper.

**References**

- [1] V. S. Adamchik, "The multiple gamma function and its application to computation of series", *Ramanujan J.* **9** (2005), no. 3, p. 271-288.
- [2] E. W. Barnes, "The theory of the G-function", *Quart. J.* **31** (1900), p. 264-314.
- [3] ———, "On the theory of the multiple Gamma function", *Trans. Camb. Philos. Soc.* **19** (1904), p. 374-439.
- [4] N. Batir, "Inequalities for the double gamma function", *J. Math. Anal. Appl.* **351** (2009), no. 1, p. 182-185.
- [5] J. Choi, "Determinant of Laplacian on  $S^3$ ", *Math. Japon.* **40** (1994), no. 1, p. 155-166.

- [6] ———, “Determinants of the Laplacians on the  $n$ -dimensional unit sphere  $\mathbf{S}^n$ ”, *Adv. Differ. Equ.* **2013** (2013), article ID 236 (12 pages).
- [7] ———, “Multiple gamma functions and their applications”, in *Analytic number theory, approximation theory, and special functions*, Springer, 2014, p. 93-129.
- [8] S. Das, “Inequalities involving the multiple psi function”, *C. R. Math. Acad. Sci. Paris* **356** (2018), no. 3, p. 288-292.
- [9] S. Das, H. L. Pedersen, A. Swaminathan, “Pick functions related to the triple Gamma function”, *J. Math. Anal. Appl.* **455** (2017), no. 2, p. 1124-1138.
- [10] S. Das, A. Swaminathan, “Bounds for triple gamma functions and their ratios”, *J. Inequal. Appl.* **2016** (2016), article ID 210 (11 pages).
- [11] F. Qi, C.-P. Chen, “A complete monotonicity property of the gamma function”, *J. Math. Anal. Appl.* **296** (2004), no. 2, p. 603-607.
- [12] K. Ueno, M. Nishizawa, “The multiple gamma function and its  $q$ -analogue”, in *Quantum groups and quantum spaces (Warsaw, 1995)*, Banach Center Publications, Institute of Mathematics of the Polish Academy of Sciences, 1995, p. 429-441.
- [13] I. Vardi, “Determinants of Laplacians and multiple gamma functions”, *SIAM J. Math. Anal.* **19** (1988), no. 2, p. 493-507.
- [14] M.-F. Vignéras, “L’équation fonctionnelle de la fonction zêta de Selberg du groupe modulaire  $\mathrm{PSL}(2, \mathbf{Z})$ ”, in *Journées Arithmétiques de Luminy (Colloq. Internat. CNRS, Centre Univ. Luminy, Luminy, 1978)*, Astérisque, vol. 61, Société Mathématique de France, 1979, p. 235-249.