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Monotonicity and sharp inequalities related to complete (p, q)-elliptic integrals of the first kind

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Dedicated to Professir Sen-Lin Xu retired from USTC in China on his 80th birthday anniversary

Abstract. With the aid of the monotone L'Hôpital rule, the authors verify monotonicity of some functions involving complete (p, q)-elliptic integrals of the first kind and the inverse of generalized hyperbolic tangent function, derive several sharp inequalities of complete (p, q)-elliptic integrals of the first kind, and generalize some known sharp approximation of complete elliptic integrals of the first kind.

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1. Motivation and main results

For $z \in \mathbb{C}$ and $n \in \{0\} \cup \mathbb{N}$, the rising factorial $(z)_n$ is defined [17] by

$$(z)_n = \prod_{\ell=0}^{n-1} (z+\ell) = \begin{cases} z(z+1)\cdots(z+n-1), & n \ge 1; \\ 1, & n = 0. \end{cases}$$

It can also be called the Pochhammer symbol or shifted factorial. The hypergeometric function $F(a,b;c;z)$ is defined [23, p. 108, (5.3)] by

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad |z| < 1 \tag{1}$$

for $a,b,c \in \mathbb{C}$ with $c \neq 0,-1,-2,\ldots$.

The complete elliptic integrals of the first and second kinds $\mathscr{K}(r)$ and $\mathscr{E}(r)$ can be expressed by

$$\mathscr{K}(r) = \int_0^{\pi/2} \frac{\mathrm{d}\phi}{\sqrt{1-r^2\sin^2\phi}} = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right)$$

and

$$\mathscr{E}(r) = \int_0^{\pi/2} \sqrt{1-r^2\sin^2\phi}\,\mathrm{d}\phi = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right).$$

See [14, Section 3.4] and [23, p. 128, Exercise 5.2].

Let $F_{p,q} : [0,1] \to \left[0, \frac{\pi_{p,q}}{2}\right]$ be defined [10,21] by

$$F_{p,q}(x) = \arcsin_{p,q}(x) = \int_0^x \left(1-t^q\right)^{-1/p} \mathrm{d}t, \quad x \in [0,1]$$

and let $\pi_{p,q} = 2\arcsin_{p,q}(1)$. Then

$$\pi_{p,q} = \frac{2}{q}\int_0^1 \frac{t^{1/p-1}}{(1-t)^{1/q}}\mathrm{d}t = \frac{2}{q} B\left(1-\frac{1}{p}, \frac{1}{q}\right) = \frac{2\pi}{q\sin\left(\frac{\pi}{p}\right)},$$

where

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t, \quad \Re(x), \Re(y) > 0$$

denotes the classical beta function. The inverse function $F_{p,q}^{-1} : \left[0, \frac{\pi_{p,q}}{2}\right] \to [0,1]$ is called generalized (p,q)-sine function, denoted by $\sin_{p,q}$. It is clear that $\sin_{2,2} = \sin$.

The complete (p,q)-elliptic integrals of the first and second kinds were defined in [12,22] by

$$\mathscr{K}_{p,q}(r) = \int_0^{\pi_{p,q}/2} \left(1-r^q\sin_{p,q}^q t\right)^{1/p-1}\mathrm{d}t \quad \text{and} \quad \mathscr{E}_{p,q}(r) = \int_0^{\pi_{p,q}/2} \left(1-r^q\sin_{p,q}^q t\right)^{1/p}\mathrm{d}t$$

for $p,q \in (1,\infty)$ and $r \in [0,1)$. It is obvious that $\mathscr{K}_{2,2}(r) = \mathscr{K}(r)$ and $\mathscr{E}_{2,2}(r) = \mathscr{E}(r)$ are classical complete elliptic integrals of the first and second kinds.

For $p,q \in (1,\infty)$ and $r \in [0,1)$, the complete (p,q)-elliptic integrals of the first and second kinds can be represented [11,12,22] in terms of the hypergeometric functions $F(a,b;c;z)$ by

$$\begin{cases} \mathscr{K}_{p,q}(r) = \dfrac{\pi_{p,q}}{2} F\left(1-\dfrac{1}{p}, \dfrac{1}{q}; 1-\dfrac{1}{p}+\dfrac{1}{q}; r^q\right); \\ \mathscr{K}_{p,q}(0) = \dfrac{\pi_{p,q}}{2}, \quad \mathscr{K}_{p,q}(1) = \infty \end{cases} \tag{2}$$

and

$$\begin{cases} \mathscr{E}_{p,q}(r) = \dfrac{\pi_{p,q}}{2} F\left(-\dfrac{1}{p}, \dfrac{1}{q}; 1-\dfrac{1}{p}+\dfrac{1}{q}; r^q\right); \\ \mathscr{E}_{p,q}(0) = \dfrac{\pi_{p,q}}{2}, \quad \mathscr{E}_{p,q}(1) = 1. \end{cases} \tag{3}$$

In [3], the double inequality

$$\frac{\pi}{2}\left(\frac{\operatorname{arctanh} r}{r}\right)^{1/2} < \mathscr{K}(r) < \frac{\pi}{2}\frac{\operatorname{arctanh} r}{r} \tag{4}$$

was obtained, where $\operatorname{arctanh} r$ denotes the inverse of hyperbolic tangent function. In [20], the double inequality

$$\left(\alpha+\frac{\pi}{2r}\right)\operatorname{arctanh} r < \mathscr{K}(r) < \left(\beta+\frac{\pi}{2r}\right)\operatorname{arctanh} r \tag{5}$$

was proved to be valid if and only if $\alpha \le 1-\frac{\pi}{2}$ and $\beta \ge 0$. In [15] and [16, Section 9], among other things, the inequalities

$$\frac{\pi \arcsin r}{2r} < \mathscr{K}(r) < \frac{\pi}{4r}\ln\frac{1+r}{1-r} \tag{6}$$

and

$$\mathscr{E}(r) < \frac{16-4r^2-3r^4}{4\left(4+r^2\right)}\mathscr{K}(r) \tag{7}$$

were derived from the Čebyšev integral inequality [18]. In [1], the double inequality

$$\frac{\pi}{2}\left(\frac{\operatorname{arctanh} r}{r}\right)^{\alpha_1} < \mathscr{K}(r) < \frac{\pi}{2}\left(\frac{\operatorname{arctanh} r}{r}\right)^{\beta_1} \tag{8}$$

was sharpened by $\alpha_1 = \frac{3}{4}$ and $\beta_1 = 1$. In [7], among other things, it was obtained that

$$\frac{\pi}{2}-\frac{1}{2}\ln\frac{(1+r)^{r-1}}{(1-r)^{r+1}} < \mathscr{E}(r) < \frac{\pi-1}{2}+\frac{1-r^2}{4r}\ln\frac{1+r}{1-r}. \tag{9}$$

In [35], the double inequalities

$$\frac{\pi\sqrt{6+2\sqrt{1-r^2}-3r^2}}{4\sqrt{2}} \le \mathscr{E}(r) \le \frac{\pi\sqrt{10-2\sqrt{1-r^2}-5r^2}}{4\sqrt{2}} \tag{10}$$

and

$$\frac{\pi\sqrt{32-r^4-32r^2}}{8\sqrt{2}\sqrt[4]{\left(1-r^2\right)^3}} \le \mathscr{K}(r) \le \frac{\pi\sqrt{r^4-32r^2+32}}{8\sqrt{2}\sqrt[4]{\left(1-r^2\right)^3}} \tag{11}$$

were established. In [24], we discussed monotonicity and some inequalities related to complete elliptic integrals of the second kind $\mathscr{E}(r)$.

We observe that

(1) because $\operatorname{arctanh} r = \frac{1}{2}\ln\frac{1+r}{1-r}$, the upper bounds in (4) and (6) and the best possible bounds in (5) and (8) are the same one which cannot compare with the upper bound in (11) on $(0,1)$;
(2) the lower bound in (8) is clearly better than the corresponding one in (4), the lower bounds in (5) and (6) cannot compare with each other on $(0,1)$, the lower bounds in (5) and (8) cannot compare with each other on $(0,1)$, the lower bound in (8) is better than the corresponding one in (6), and the lower bound in (11) cannot compare with the corresponding ones in (4) to (8);
(3) the lower bound in (10) is better than the corresponding one in (9) and the upper bounds in (10) and (9) cannot compare with each other on $(0,1)$.

These observations can be verified by plotting via the Wolfram Mathematica 11.1.

In [36], it was obtained that, for $p \in (1,\infty)$,

$$\frac{\operatorname{arctanh}_p r}{r} < \mathscr{K}_p(r) < \frac{\pi_p}{2}\frac{\operatorname{arctanh}_p r}{r}, \tag{12}$$

where $\mathscr{K}_p(r) = \mathscr{K}_{p,p}(r)$ and

$$\operatorname{arctanh}_p r = rF\left(\frac{1}{p},1;1+\frac{1}{p};r^p\right). \tag{13}$$

In [25], we investigated monotonicity and some inequalities related to generalized Grötzsch ring function

$$\frac{\pi}{2\sin(\pi q)}\frac{\mathscr{K}_{1/q,1/q}\big((1-r^2)^q\big)}{\mathscr{K}_{1/q,1/q}\big(r^{2q}\big)}, \quad q\in\Big(0,\frac{1}{2}\Big].$$

Let $\gamma = 0.57721566\ldots$ stands for Euler–Mascheroni's constant, let $\psi(z) = [\ln\Gamma(z)]' = \frac{\Gamma'(z)}{\Gamma(z)}$ be the logarithmic derivative of the classical Euler gamma function which can be defined (see [13] and [23, Chapter 3]) by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t, \quad \Re(z)>0 \quad \text{or by} \quad \Gamma(z) = \lim_{n\to\infty}\frac{n!n^z}{\prod_{k=0}^n(z+k)}, \quad z\in\mathbb{C}\setminus\{0,-1,-2,\ldots\},$$

and let

$$R(x,y) = \psi(x)-\psi(y)-2\gamma, \quad x,y\in(0,\infty)$$

denote the Ramanujan constant function [6]. In [2, Theorem 2.2], the function $\frac{\mathscr{K}(r)}{\ln\left[c/\sqrt{1-r^2}\right]}$ is showed to be decreasing if and only if $1<c\le 4$ and to be increasing if and only if $c\ge e^2$.

In this paper, with the aid of the monotone L'Hôpital rule, we will use a new and concise method to generalize the above inequalities and monotonicity for functions involving $\mathscr{K}(r)$ and $\mathscr{E}(r)$ to those involving complete (p,q)-elliptic integrals $\mathscr{K}_{p,q}(r)$ and $\mathscr{E}_{p,q}(r)$, to reveal monotonicity of several functions involving $\mathscr{K}_{p,q}(r)$, $\mathscr{E}_{p,q}(r)$, and the inverse of generalized hyperbolic tangent function, and to improve inequalities (4), (5), (8), and (12).

Our main results can be stated as the following theorems.

Theorem 1. *For $r\in(0,1)$ and $p,q\in(1,\infty)$, let $F(r) = \frac{\mathscr{K}_{p,q}(r)}{\ln[c/(1-r^q)^{1/q}]}$. Then the function $F(r)$*

(1) *increases on $(0,1)$ if and only if $c\ge\exp\Big(\frac{q(p-1)+p}{q(p-1)}\Big)$;*
(2) *decreases on $(0,1)$ if and only if $1\le c\le\exp\Big(\frac{R(1-1/p,1/q)}{q}\Big)$;*

and, consequently, when $1\le c\le\exp\Big(\frac{R(1-1/p,1/q)}{q}\Big)$,

$$\ln\frac{c}{(1-r^q)^{1/q}} < \mathscr{K}_{p,q}(r) < \frac{\pi_{p,q}}{2\ln c}\ln\frac{c}{(1-r^q)^{1/q}}. \tag{14}$$

Theorem 2. *For $r\in(0,1)$ and $q\in(1,\infty)$,*

(1) *when $p\ge2$, the function $G(r) = \frac{r\mathscr{K}_{p,q}^2(r)}{\operatorname{arctanh}_q r}$ increases and maps $(0,1)$ onto $\Big(\frac{\pi_{p,q}^2}{4},\infty\Big)$. Consequently, for $r\in(0,1)$, $p\in[2,\infty)$, and $q\in(1,\infty)$, we have*

$$\frac{\pi_{p,q}}{2}\left(\frac{\operatorname{arctanh}_q r}{r}\right)^{1/2} < \mathscr{K}_{p,q}(r); \tag{15}$$

(2) *when $p>1$, the function $W(r) = \frac{r\mathscr{K}_{p,q}(r)}{\operatorname{arctanh}_q r}$ decreases and maps $(0,1)$ onto $\Big(1,\frac{\pi_{p,q}}{2}\Big)$. Consequently, for $r\in(0,1)$ and $p,q\in(1,\infty)$, we have*

$$\frac{\operatorname{arctanh}_q r}{r} < \mathscr{K}_{p,q}(r) < \frac{\pi_{p,q}}{2}\frac{\operatorname{arctanh}_q r}{r}. \tag{16}$$

Theorem 3. *For $r\in(0,1)$ and $p,q\in(1,\infty)$, the function $H(r) = \frac{\frac{\pi_{p,q}}{2}\operatorname{arctanh}_q r - r\mathscr{K}_{p,q}(r)}{r^q\operatorname{arctanh}_q r}$ increases and maps $(0,1)$ onto $\Big(\frac{\pi_{p,q}}{2}\frac{1}{pq(q+1)},\frac{\pi_{p,q}}{2}-1\Big)$. Consequently, for $r\in(0,1)$ and $p,q\in(1,\infty)$, we have*

$$\frac{\pi_{p,q}}{2}\frac{\operatorname{arctanh}_q r}{r}\big(1-\alpha_2 r^q\big) < \mathscr{K}_{p,q}(r) < \frac{\pi_{p,q}}{2}\frac{\operatorname{arctanh}_q r}{r}\big(1-\beta_2 r^q\big), \tag{17}$$

where $\alpha_2 = 1-\frac{2}{\pi_{p,q}}$ and $\beta_2 = \frac{1}{pq(q+1)}$ are the best possible constants in the sense that they can not be replaced by any bigger and smaller constants respectively.

2. Lemmas and their proofs

For proving our main results, we need the following known results and lemmas.

In [22], the following two derivatives were given:

$$\frac{\mathrm{d}\mathscr{K}_{p,q}(r)}{\mathrm{d}r}=\frac{\mathscr{E}_{p,q}(r)-\left(1-r^q\right)\mathscr{K}_{p,q}(r)}{r(1-r^q)} \tag{18}$$

and

$$\frac{\mathrm{d}\mathscr{E}_{p,q}(r)}{\mathrm{d}r}=-\frac{q}{p}\frac{\mathscr{K}_{p,q}(r)-\mathscr{E}_{p,q}(r)}{r}. \tag{19}$$

Lemma 4 (cf. [4]). *For $a,b\in\mathbb{R}$ with $a<b$, let f and g be continuous on $[a,b]$, differentiable on (a,b), and $g'\ne 0$ on (a,b). If the ratio $\frac{f'}{g'}$ is increasing on (a,b), then both of the functions $\frac{f(x)-f(a)}{g(x)-g(a)}$ and $\frac{f(x)-f(b)}{g(x)-g(b)}$ are increasing with respect to $x\in(a,b)$.*

Lemma 5 (cf. [1]). *Suppose that the power series*

$$R(x)=\sum_{n=0}^{\infty}r(n)x^n \quad \text{and} \quad S(x)=\sum_{n=0}^{\infty}s(n)x^n$$

converge for $|x|<1$. If $s(n)>0$ for $n\ge 0$ and the sequence $\frac{r(n)}{s(n)}$ increases with respect to $n\ge 0$, then the ratio $\frac{R(x)}{S(x)}$ increases with respect to $x\in(0,1)$.

Lemma 6 (cf. [11, Theorem 1]). *For $r\in(0,1)$ and $p,q\in(1,\infty)$, the following conclusions are valid:*

(1) *the function $\frac{\mathscr{E}_{p,q}(r)-\left(1-r^q\right)\mathscr{K}_{p,q}(r)}{r^q\mathscr{K}_{p,q}(r)}$ is decreasing and maps from $(0,1)$ onto $\left(1,\frac{(p-1)q}{pq+p-q}\right)$;*

(2) *the function $\frac{\mathscr{K}_{p,q}(r)-\mathscr{E}_{p,q}(r)}{r^q\mathscr{K}_{p,q}(r)}$ is increasing and maps $(0,1)$ onto $\left(\frac{p}{pq+p-q},1\right)$. Consequently,*

$$1-r^q<\frac{\mathscr{E}_{p,q}(r)}{\mathscr{K}_{p,q}(r)}<1-\frac{p}{(p-1)q+p}r^q. \tag{20}$$

Lemma 7. *For $r\in(0,1)$ and $p,q\in(1,\infty)$, the function $f(r)=\frac{1}{q}\ln\left(1-r^q\right)+\frac{r^q\mathscr{K}_{p,q}(r)}{\mathscr{E}_{p,q}(r)-(1-r^q)\mathscr{K}_{p,q}(r)}$ decreases and maps $(0,1)$ onto $\left(\frac{R(1-1/p,1/q)}{q},\frac{q(p-1)+p}{q(p-1)}\right)$. Consequently,*

$$\frac{R(1-1/p,1/q)}{q}<\frac{1}{q}\ln\left(1-r^q\right)+\frac{r^q\mathscr{K}_{p,q}(r)}{\mathscr{E}_{p,q}(r)-(1-r^q)\mathscr{K}_{p,q}(r)}<\frac{q(p-1)+p}{q(p-1)}. \tag{21}$$

Proof. By virtue of (18) and (19), differentiating $f(r)$ gives

$$\begin{aligned}
f'(r)&=-\frac{r^q}{r(1-r^q)}+\frac{r^q}{r(1-r^q)}\frac{\begin{pmatrix} q\left(1-r^q\right)\mathscr{K}_{p,q}\left[\mathscr{E}_{p,q}(r)-\left(1-r^q\right)\mathscr{K}_{p,q}(r)\right]\\ +\left[\mathscr{E}_{p,q}(r)-\left(1-r^q\right)\mathscr{K}_{p,q}(r)\right]^2\\ -(1-q/p)\left(1-r^q\right)\mathscr{K}_{p,q}\left[\mathscr{K}_{p,q}(r)-\mathscr{E}_{p,q}(r)\right]\\ -(q-1)r^q\left(1-r^q\right)\mathscr{K}_{p,q}^2(r)\end{pmatrix}}{\left[\mathscr{E}_{p,q}(r)-(1-r^q)\mathscr{K}_{p,q}(r)\right]^2}\\
&=\frac{r^q\mathscr{K}_{p,q}(r)}{r\left[\mathscr{E}_{p,q}(r)-(1-r^q)\mathscr{K}_{p,q}(r)\right]^2}\begin{pmatrix} q\left[\mathscr{E}_{p,q}(r)-\left(1-r^q\right)\mathscr{K}_{p,q}(r)\right]\\ -(1-q/p)\left[\mathscr{K}_{p,q}(r)-\mathscr{E}_{p,q}(r)\right]\\ -(q-1)r^q\mathscr{K}_{p,q}(r)\end{pmatrix}\\
&=\frac{r^q\mathscr{K}_{p,q}(r)}{r\left[\mathscr{E}_{p,q}(r)-(1-r^q)\mathscr{K}_{p,q}(r)\right]^2}\left(r^q\mathscr{K}_{p,q}-\frac{q(p-1)+p}{p}\left[\mathscr{K}_{p,q}(r)-\mathscr{E}_{p,q}(r)\right]\right)\\
&=\frac{\left(r^q\mathscr{K}_{p,q}(r)\right)^2}{r\left[\mathscr{E}_{p,q}(r)-(1-r^q)\mathscr{K}_{p,q}(r)\right]^2}\left[1-\frac{q(p-1)+p}{p}\frac{\mathscr{K}_{p,q}-\mathscr{E}_{p,q}}{r^q\mathscr{K}_{p,q}(r)}\right].
\end{aligned}$$

From the second item of Lemma 6, it follows that $f'(r) < 0$ which means that $f(r)$ decreases.

By the first item in Lemma 6 and [11, Theorem 3], we acquire the limits $f(0^+) = \frac{q(p-1)+p}{q(p-1)}$ and

$$\begin{aligned}
f(1^-) &= \lim_{r\to1^-}\left[\frac{\ln(1-r^q)}{q} + \frac{r^q\mathscr{K}_{p,q}(r)}{\mathscr{E}_{p,q}(r)-(1-r^q)\mathscr{K}_{p,q}(r)}\right]\\
&= \lim_{r\to1^-}\frac{r^q}{\mathscr{E}_{p,q}(r)-(1-r^q)\mathscr{K}_{p,q}(r)}\left[\mathscr{K}_{p,q}+\frac{\ln(1-r^q)}{q}\right]\\
&\quad+\lim_{r\to1^-}\left[1-\frac{r^q}{\mathscr{E}_{p,q}(r)-(1-r^q)\mathscr{K}_{p,q}(r)}\right]\frac{\ln(1-r^q)}{q}\\
&= \frac{R(1-1/p,1/q)}{q}+\lim_{r\to1^-}\left[\mathscr{E}_{p,q}(r)-(1-r^q)\mathscr{K}_{p,q}-r^q\right]\frac{\ln(1-r^q)}{q}\\
&= \frac{R(1-1/p,1/q)}{q}+\lim_{r\to1^-}\frac{\mathscr{E}_{p,q}(r)-r^q}{(1-r^q)^{1/q}}(1-r^q)^{1/q}\frac{\ln(1-r^q)}{q}\\
&= \frac{R(1-1/p,1/q)}{q}.
\end{aligned}$$

The double inequality (21) follows from monotonicity of $f(r)$. The proof of Lemma 7 is complete. □

Lemma 8. *For $r\in(0,1)$ and $q\in(1,\infty)$, the function $\Phi(r)=\frac{(1-r^q)\operatorname{arctanh}_q r}{r}$ decreases and maps $(0,1)$ onto $(0,1)$.*

Proof. In [36], it was obtained that $\left(\operatorname{arctanh}_q r\right)' = \frac{1}{1-r^q}$. Employing this result and the formula (13) yields

$$r\Phi'(r)=\frac{-q\left(r^{q-1}\operatorname{arctanh}_q r+1\right)r-\left(1-r^q\right)\operatorname{arctanh}_q r}{r}=1-\left[(q-1)r^q+1\right]\frac{\operatorname{arctanh}_q r}{r}<0.$$

Therefore, the function $\Phi(r)$ decreases.

It is straightforward to derive $\Phi(0^+)=1$ and $\Phi(1^-)=0$. The proof of Lemma 8 is complete. □

Lemma 9. *For $r\in(0,1)$, $q\in(1,\infty)$, and $p\in[2,\infty)$, the function $\phi(r)=2\mathscr{E}_{p,q}(r)-\left(1-r^q\right)\mathscr{K}_{p,q}(r)$ increases and maps $(0,1)$ onto $\left(\frac{\pi_{p,q}}{2},2\right)$. Consequently,*

$$\frac{\pi_{p,q}}{4}<\mathscr{E}_{p,q}(r)-\frac{1-r^q}{2}\mathscr{K}_{p,q}(r)<1.$$

Proof. Utilizing the derivative formulas (18) and (19) and differentiating give

$$\phi'(r)=\left(1-\frac{2q}{p}\right)\frac{\mathscr{K}_{p,q}(r)-\mathscr{E}_{p,q}(r)}{r}+(q-1)\frac{r^q\mathscr{K}_{p,q}(r)}{r}=\frac{r^q\mathscr{K}_{p,q}(r)}{r}\varphi(r),$$

where

$$\varphi(r)=\left(1-\frac{2q}{p}\right)\frac{\mathscr{K}_{p,q}(r)-\mathscr{E}_{p,q}(r)}{r^q\mathscr{K}_{p,q}(r)}+q-1.$$

When $p\ge 2q$, from the second item of Lemma 6, it follows readily that $\varphi(r)>0$.

When $2\le p<2q$, by the second item of Lemma 6, it follows that

$$\varphi(r)>0\Longleftrightarrow\inf_{0<r<1}\left[\left(1-\frac{2q}{p}\right)\frac{\mathscr{K}_{p,q}(r)-\mathscr{E}_{p,q}(r)}{r^q\mathscr{K}_{p,q}(r)}+q-1\right]>0\Longleftrightarrow q\left(1-\frac{2}{p}\right)\ge 0.$$

Accordingly, when $p\ge2$ and $q>1$, the function $\phi(r)$ is increasing.

By virtue of (2) and (3), it is easy to deduce the limits $\phi(0^+)=\frac{\pi_{p,q}}{2}$ and $\phi(1^-)=2$. The proof of Lemma 9 is complete. □

3. Proofs of main results

Now we start out to prove our main results.

Proof of Theorem 1. By (18), direct differentiating $F(r)$ gives

$$\begin{aligned}\left[\ln\frac{c}{(1-r^q)^{1/q}}\right]^2 F'(r) &= \frac{\mathscr{E}_{p,q}(r)-\left(1-r^q\right)\mathscr{K}_{p,q}(r)}{r(1-r^q)}\ln\frac{c}{(1-r^q)^{1/q}}-\frac{r^q\mathscr{K}_{p,q}(r)}{r(1-r^q)}\\ &= \frac{\mathscr{E}_{p,q}(r)-\left(1-r^q\right)\mathscr{K}_{p,q}(r)}{r(1-r^q)}\left[\ln c-\left(\frac{1}{q}\ln\left(1-r^q\right)+\frac{r^q\mathscr{K}_{p,q}(r)}{\mathscr{E}_{p,q}(r)-(1-r^q)\mathscr{K}_{p,q}(r)}\right)\right]\end{aligned}$$

By Lemma 7, we have

$$\begin{aligned}F'(r)<0 &\Longleftrightarrow \inf_{0<r<1}\left[\ln c-\left(\frac{\ln\left(1-r^q\right)}{q}+\frac{r^q\mathscr{K}_{p,q}(r)}{\mathscr{E}_{p,q}(r)-(1-r^q)\mathscr{K}_{p,q}(r)}\right)\right]<0\\ &\Longleftrightarrow c\le \inf_{0<r<1}\exp\left(\frac{\ln\left(1-r^q\right)}{q}+\frac{r^q\mathscr{K}_{p,q}(r)}{\mathscr{E}_{p,q}(r)-(1-r^q)\mathscr{K}_{p,q}(r)}\right)\Longleftrightarrow c\le\exp\left(\frac{R(1-1/p,1/q)}{q}\right)\end{aligned}$$

and

$$\begin{aligned}F'(r)>0 &\Longleftrightarrow \sup_{0<r<1}\left[\ln c-\left(\frac{\ln\left(1-r^q\right)}{q}+\frac{r^q\mathscr{K}_{p,q}(r)}{\mathscr{E}_{p,q}(r)-(1-r^q)\mathscr{K}_{p,q}(r)}\right)\right]>0\\ &\Longleftrightarrow c\ge \sup_{0<r<1}\exp\left(\frac{\ln\left(1-r^q\right)}{q}+\frac{r^q\mathscr{K}_{p,q}(r)}{\mathscr{E}_{p,q}(r)-(1-r^q)\mathscr{K}_{p,q}(r)}\right)\Longleftrightarrow c\ge\exp\left(\frac{q(p-1)+p}{q(p-1)}\right)\end{aligned}$$

The double inequality (14) follows from monotonicity of $F(r)$. The proof of Theorem 1 is complete. □

Proof of Theorem 2. Let $g_1(r)=r\mathscr{K}^2_{p,q}(r)$ and $g_2(r)=\operatorname{arctanh}_q r$. Then $G(r)=\frac{g_1(r)}{g_2(r)}$ and $g_1(0)=g_2(0)=0$. Making use of [36] and (18), we have

$$\begin{aligned}\frac{g_1'(r)}{g_2'(r)} &= \left(1-r^q\right)\mathscr{K}_{p,q}(r)\left[\mathscr{K}_{p,q}(r)+2\frac{\mathscr{E}_{p,q}(r)-\left(1-r^q\right)\mathscr{K}_{p,q}(r)}{(1-r^q)}\right]\\ &= \mathscr{K}_{p,q}(r)\left[2\mathscr{E}_{p,q}(r)-\left(1-r^q\right)\mathscr{K}_{p,q}(r)\right].\end{aligned}$$

From Lemma 9, it follows that the function $G(r)$ is increasing on $(0,1)$. By the L'Hôpital rule, it follows that $G(0^+)=\frac{\pi^2_{p,q}}{4}$ and $G(1^-)=\infty$.

It is obvious that inequality (15) follows from monotonicity of $G(r)$.

Let $g_3(r)=r\mathscr{K}_{p,q}(r)$ and $g_4(r)=\operatorname{arctanh}_q r$. Then $W(r)=\frac{g_3(r)}{g_4(r)}$ and $g_3(0)=g_4(0)=0$. Since

$$\frac{g_3'(r)}{g_4'(r)}=\frac{\mathscr{K}_{p,q}(r)+\left[\mathscr{E}_{p,q}(r)-\left(1-r^q\right)\mathscr{K}_{p,q}(r)\right]/\left(1-r^q\right)}{1/(1-r^q)}=\mathscr{E}_{p,q}(r),$$

by Lemma 4, the function $W(r)$ is decreasing on $(0,1)$. By Lemma 4, those formulas in (3), and the L'Hôpital rule, we can obtain readily that $W(0^+)=\frac{\pi_{p,q}}{2}$ and $W(1^-)=1$.

The double inequality (16) follows from monotonicity of $W(r)$ directly. □

Proof of Theorem 3. Let the functions $h_1(r)=\frac{\pi_{p,q}}{2}\operatorname{arctanh}_q r-r\mathscr{K}_{p,q}(r)$ and $h_2(r)=r^q\operatorname{arctanh}_q r$. Then $H(r)=\frac{h_1(r)}{h_2(r)}$ and $h_1(0)=h_2(0)=0$. Using (18) and differentiating yield

$$\frac{h_1'(r)}{h_2'(r)}=\frac{\begin{pmatrix}\dfrac{\pi_{p,q}}{2}-\left(1-r^q\right)\mathscr{K}_{p,q}(r)\\ -\left[\mathscr{E}_{p,q}(r)-\left(1-r^q\right)\mathscr{K}_{p,q}(r)\right]\end{pmatrix}}{r^q}\frac{1}{1+q\frac{(1-r^q)\operatorname{arctanh}_q r}{r}}=\frac{h_3(r)}{h_4(r)},$$

where

$$h_3(r)=\frac{\frac{\pi_{p,q}}{2}-\left(1-r^q\right)\mathcal{K}_{p,q}(r)-\left[\mathcal{E}_{p,q}(r)-\left(1-r^q\right)\mathcal{K}_{p,q}(r)\right]}{r^q}$$

and $h_4(r)=1+q\frac{(1-r^q)\operatorname{arctanh}_q r}{r}$. By Lemma 8, the function $h_4(r)$ is decreasing on $(0,1)$.

Let $a=1-\frac{1}{p}$, $b=1-\frac{1}{p}+\frac{1}{q}$, $a_n=\frac{(a)_n\left(\frac{1}{q}\right)_n}{(b)_n n!}$, and $b_n=\frac{a_n}{n+b}$. By virtue of (1) to (3), we have

$$\begin{aligned}\mathcal{E}_{p,q}(r)-\left(1-r^q\right)\mathcal{K}_{p,q}(r)&=\frac{\pi_{p,q}}{2}\left[\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{p}\right)_n\left(\frac{1}{q}\right)_n}{(b)_n n!}r^{qn}-\sum_{n=0}^{\infty}\frac{(a)_n\left(\frac{1}{q}\right)_n}{(b)_n n!}r^{qn}+\sum_{n=0}^{\infty}\frac{(a)_n\left(\frac{1}{q}\right)_n}{(b)_n n!}r^{q(n+1)}\right]\\&=\frac{\pi_{p,q}}{2}\sum_{n=1}^{\infty}\left[\frac{(a)_{n-1}\left(\frac{1}{q}\right)_{n-1}}{(b)_{n-1}(n-1)!}-\frac{(a)_{n-1}\left(\frac{1}{q}\right)_n}{(b)_n n!}\right]r^{qn}\\&=\frac{a\pi_{p,q}}{2}\sum_{n=1}^{\infty}\frac{n(a)_{n-1}\left(\frac{1}{q}\right)_{n-1}}{(b)_n n!}r^{qn}=\frac{a\pi_{p,q}}{2}\sum_{n=0}^{\infty}b_n r^{q(n+1)}\end{aligned}$$

and

$$\begin{aligned}\frac{\pi_{p,q}}{2}-\left(1-r^q\right)\mathcal{K}_{p,q}(r)&=\frac{\pi_{p,q}}{2}-\left(1-r^q\right)\frac{\pi_{p,q}}{2}\sum_{n=0}^{\infty}\frac{(a)_n\left(\frac{1}{q}\right)_n}{(b)_n n!}r^{qn}\\&=\frac{\pi_{p,q}}{2}\left[1-\sum_{n=0}^{\infty}\frac{(a)_n\left(\frac{1}{q}\right)_n}{(b)_n n!}r^{qn}+\sum_{n=0}^{\infty}\frac{(a)_n\left(\frac{1}{q}\right)_n}{(b)_n n!}r^{q(n+1)}\right]\\&=\frac{\pi_{p,q}}{2}\left[-\sum_{n=1}^{\infty}\frac{(a)_n\left(\frac{1}{q}\right)_n}{(b)_n n!}r^{qn}+\sum_{n=0}^{\infty}b_n r^{q(n+1)}\right]\\&=\frac{\pi_{p,q}}{2}\left[-\sum_{n=0}^{\infty}\frac{(a)_{n+1}\left(\frac{1}{q}\right)_{n+1}}{(b)_{n+1}(n+1)!}r^{q(n+1)}+\sum_{n=0}^{\infty}b_n r^{q(n+1)}\right]\\&=\frac{\pi_{p,q}}{2}\sum_{n=0}^{\infty}c_n r^{q(n+1)},\end{aligned}$$

where $c_n=\left[1-\frac{(n+a)\left(n+\frac{1}{q}\right)}{(n+b)(n+1)}\right]b_n$. Furthermore, by (1), we have

$$h_3(r)=\frac{\pi_{p,q}}{2}\sum_{n=0}^{\infty}(b_n+c_n)r^{qn}=\frac{\pi_{p,q}}{2}\sum_{n=0}^{\infty}\frac{1}{p}\left(n+\frac{1}{q}\right)r^{qn}>0.$$

By Lemma 7, the function $h_3(r)$ is increasing on $(0,1)$. By Lemmas 4 to 6, the function $H(r)$ is increasing.

By the L'Hôpital rule, it follows that

$$H(0^+)=\lim_{r\to0^+}H(r)=\lim_{r\to0^+}\frac{h_1'(r)}{h_2'(r)}=\lim_{r\to0^+}\frac{h_3(r)}{h_4(r)}=\frac{\pi_{p,q}}{2}\frac{b_0+c_0}{1+q}=\frac{\pi_{p,q}}{2pq(q+1)}$$

and

$$H(1^-)=\lim_{r\to1^-}H(r)=\lim_{r\to0^+}\frac{h_1'(r)}{h_2'(r)}=\lim_{r\to0^+}\frac{h_3(r)}{h_4(r)}=\frac{\pi_{p,q}}{2}-1.$$

Thus, the double inequality (17) holds. The proof of Theorem 3 is complete. □

4. Remarks

Remark 10. When $p=q=2$, Theorem 1 reduces to [2, Theorem 2.2 (5)].

Remark 11. When $p=q=2$, inequalities (15) and (16) in Theorem 2 become (4). When $p=q$, inequality (16) becomes (12).

Remark 12. When $p = q$, the double inequality (17) improves the double inequality (12).

If setting $p = q = 2$ in the double inequality (17) in Theorem 3, then

$$\frac{\pi}{2}\frac{\operatorname{arctanh} r}{r}\left[1-\left(1-\frac{2}{\pi}\right)r^2\right] < \mathscr{K}(r) < \frac{\pi}{2}\frac{\operatorname{arctanh} r}{r}\left(1-\frac{1}{12}r^2\right)$$

for $r \in (0,1)$. This double inequality improves the double inequalities (4) and (5).

Remark 13. When $p = q = 2$, the inequality (20) becomes

$$1-r^2 < \frac{\mathscr{E}(r)}{\mathscr{K}(r)} < 1-\frac{r^2}{2}$$

whose upper bound is worse than (7). This means that the inequality (7) is much better.

Remark 14. Interested readers who are curious about this paper not only just want to know the main research content of this paper, but also want to know the research background and research progress in this field. Enriching references as many as possible is very important for these readers. So we list several recently published papers [5,8,9,19,26–34], which are closely related to the topic of this paper, to the list of references of this paper.

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