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A nonlinear Korn inequality on a surface with an explicit estimate of the constant

Une inégalité de Korn non linéaire sur une surface avec une majoration explicite de la constante

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Abstract. A nonlinear Korn inequality on a surface estimates a distance between a surface $\theta(\omega)$ and another surface $\phi(\omega)$ in terms of distances between their fundamental forms in the space $L^p(\omega)$, $1 < p < \infty$.

We establish a new inequality of this type. The novelty is that the immersion θ belongs to a specific set of mappings of class \mathcal{C}^1 from $\bar{\omega}$ into \mathbb{R}^3 with a unit vector field also of class \mathcal{C}^1 over $\bar{\omega}$.

Résumé. Une inégalité de Korn non linéaire sur une surface estime une distance entre une surface $\theta(\omega)$ et une autre surface $\phi(\omega)$ en fonction des distances entre leur formes fondamentales dans l'espace $L^p(\omega)$, $1 < p < \infty$.

Nous établissons une nouvelle inégalité de ce type. La nouveauté réside dans l'appartenance de l'immersion θ à un ensemble particulier d'applications de classe \mathcal{C}^1 de $\bar{\omega}$ dans \mathbb{R}^3 avec un champ de vecteurs normaux unitaires aussi de classe \mathcal{C}^1 dans $\bar{\omega}$.

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1. Notation and definitions

Vector and matrix fields are denoted by boldface letters.

Given any open set $\Omega \subset \mathbb{R}^n$, $n \geq 1$, any subset $V \subset Y$ of a finite-dimensional vector space Y , and any integer $\ell \geq 0$, the notation $\mathcal{C}^\ell(\Omega; V)$ designates the set of all fields $\mathbf{v} = (v_i) : \Omega \rightarrow Y$ such that $\mathbf{v}(x) \in V$ for all $x \in \Omega$ and $v_i \in \mathcal{C}^\ell(\Omega)$. Likewise, given any real number $p > 1$, the notation $L^p(\Omega; V)$, resp. $W^{\ell,p}(\Omega; V)$, designates the set of all fields $\mathbf{v} = (v_i) : \Omega \rightarrow Y$ such that $\mathbf{v}(x) \in V$ for almost all $x \in \Omega$ and $v_i \in L^p(\Omega)$, resp. $v_i \in W^{\ell,p}(\Omega)$.

The space of all real matrices with k rows and ℓ columns is denoted $\mathbb{M}^{k \times \ell}$. We also let

$$\mathbb{M}^k := \mathbb{M}^{k \times k}, \mathbb{S}^k := \{ \mathbf{A} \in \mathbb{M}^k; \mathbf{A} = \mathbf{A}^T \},$$

$$\mathbb{S}_>^k := \{ \mathbf{A} \in \mathbb{S}^k; \mathbf{A} \text{ is positive-definite} \}, \text{ and } \mathbb{O}_+^k := \{ \mathbf{A} \in \mathbb{M}^k; \mathbf{A}\mathbf{A}^T = \mathbf{I} \text{ and } \det \mathbf{A} = 1 \}.$$

A $k \times \ell$ matrix whose column vectors are the vectors $\mathbf{v}_1, \dots, \mathbf{v}_\ell \in \mathbb{R}^k$ is denoted $(\mathbf{v}_1 | \dots | \mathbf{v}_\ell)$. If $\mathbf{A} \in \mathbb{S}_>^k$, there exists a unique matrix $\mathbf{U} \in \mathbb{S}_>^k$ such that $\mathbf{U}^2 = \mathbf{A}$; this being the case, we let $\mathbf{A}^{1/2} := \mathbf{U}$.

The Euclidean norm in \mathbb{R}^3 is denoted $|\cdot|$. Spaces of matrices are equipped with the Frobenius norm, also denoted $|\cdot|$. The spaces $L^p(\Omega)$, $L^p(\Omega; \mathbb{R}^k)$, and $L^p(\Omega; \mathbb{M}^{k \times \ell})$, are respectively equipped with the norms denoted and defined by

$$\|u\|_{L^p(\Omega)} := \left(\int_{\Omega} |u(x)|^p dx \right)^{1/p}, \quad \|\mathbf{v}\|_{L^p(\Omega)} := \left(\int_{\Omega} |\mathbf{v}(x)|^p dx \right)^{1/p},$$

$$\text{and } \|\mathbf{A}\|_{L^p(\Omega)} := \left(\int_{\Omega} |\mathbf{A}(x)|^p dx \right)^{1/p}.$$

A *domain* Ω in \mathbb{R}^n , $n \geq 2$, is a connected and open subset of \mathbb{R}^n that is bounded and has a Lipschitz-continuous boundary, the set Ω being locally on the same side of its boundary (cf. Adams [1], Maz'ya [10], or Nečas [11]).

Given an open subset Ω of \mathbb{R}^n and any integer $\ell \geq 0$, the notation $\mathcal{C}^\ell(\overline{\Omega})$ designates the space of all functions $u \in \mathcal{C}^\ell(\Omega)$ such that u and all their partial derivatives up to order ℓ possess continuous extensions to the closure $\overline{\Omega}$ of Ω . If $\Omega \subset \mathbb{R}^n$ is a domain, then $\mathcal{C}^\ell(\overline{\Omega}) = \{f|_{\overline{\Omega}}; f \in \mathcal{C}^\ell(\mathbb{R}^n)\}$, where $f|_{\overline{\Omega}}$ denotes the restriction of the function f to the set $\overline{\Omega}$, thanks to Whitney's extension theorem: cf. Whitney [12]; see also Ciarlet & Mardare [5].

Given a connected open subset Ω of \mathbb{R}^n , the *geodesic distance* between two points $x, y \in \Omega$ is defined by

$$\text{dist}_\Omega(x, y) := \inf \left\{ \ell \in \mathbb{R}; \text{ there exists a path } c \in \mathcal{C}^1([0, \ell]; \mathbb{R}^n) \right.$$

$$\left. \text{such that } c(0) = x, c(\ell) = y, c(s) \in \Omega \text{ and } |c'(s)| = 1 \text{ for all } s \in [0, \ell] \right\}. \quad (1)$$

If $\Omega \subset \mathbb{R}^n$ is a domain, then there exists a constant $C_\Omega \geq 1$ such that

$$\text{dist}_\Omega(x, y) \leq C_\Omega |x - y| \text{ for all } x, y \in \overline{\Omega}; \quad (2)$$

see e.g. Anicic, Le Dret & Raoult [2, Proposition 5.1].

A generic point in an open subset ω of \mathbb{R}^2 is denoted $y = (y_1, y_2)$ and partial derivative operators with respect to y_1 and y_2 are denoted ∂_1 and ∂_2 .

2. Main result

An *immersion of class \mathcal{C}^1* from a two-dimensional domain $\omega \subset \mathbb{R}^2$ into the three-dimensional Euclidean space \mathbb{R}^3 is a mapping $\phi : \omega \rightarrow \mathbb{R}^3$ of class \mathcal{C}^1 such that the two vector fields $\partial_1 \phi : \omega \rightarrow \mathbb{R}^3$ and $\partial_2 \phi : \omega \rightarrow \mathbb{R}^3$ are linearly independent at each point of ω . This means that the image

of ω by $\boldsymbol{\phi}$ is a surface in \mathbb{R}^3 whose tangent plane at its point $\boldsymbol{\phi}(y)$, $y \in \omega$, is spanned by the two vectors $\partial_1\boldsymbol{\phi}(y)$ and $\partial_2\boldsymbol{\phi}(y)$. Consequently,

$$\mathbf{v}(\boldsymbol{\phi}) := \frac{\partial_1\boldsymbol{\phi} \wedge \partial_2\boldsymbol{\phi}}{|\partial_1\boldsymbol{\phi} \wedge \partial_2\boldsymbol{\phi}|} \tag{3}$$

is a continuous unit vector field from ω into \mathbb{R}^3 that is normal to the surface $\boldsymbol{\phi}(\omega)$.

Given an immersion $\boldsymbol{\phi} : \omega \rightarrow \mathbb{R}^3$ of class \mathcal{C}^1 , we let

$$\nabla\boldsymbol{\phi} := (\partial_1\boldsymbol{\phi} \mid \partial_2\boldsymbol{\phi}) \text{ and } \mathbf{A}(\boldsymbol{\phi}) := \nabla\boldsymbol{\phi}^T \nabla\boldsymbol{\phi}. \tag{4}$$

Note that $\nabla\boldsymbol{\phi}$ is field of 3×2 matrices whose column vectors are the partial derivatives of $\boldsymbol{\phi}$ and that $\mathbf{A}(\boldsymbol{\phi})$ is a field of 2×2 positive-definite symmetric matrices whose components are the covariant components of the *first fundamental form* associated with the immersion $\boldsymbol{\phi}$.

Given an immersion $\boldsymbol{\phi} : \omega \rightarrow \mathbb{R}^3$ of class \mathcal{C}^1 such that the unit vector field $\mathbf{v}(\boldsymbol{\phi}) : \omega \rightarrow \mathbb{R}^3$ is also of class \mathcal{C}^1 , we let

$$\nabla\mathbf{v}(\boldsymbol{\phi}) = (\partial_1\mathbf{v}(\boldsymbol{\phi}) \mid \partial_2\mathbf{v}(\boldsymbol{\phi})) \text{ and } \mathbf{B}(\boldsymbol{\phi}) := \nabla\boldsymbol{\phi}^T \nabla\mathbf{v}(\boldsymbol{\phi}). \tag{5}$$

Note that $\nabla\mathbf{v}(\boldsymbol{\phi})$ is field of 3×2 matrices whose column vectors are the partial derivatives of the vector field $\mathbf{v}(\boldsymbol{\phi})$ and that $\mathbf{B}(\boldsymbol{\phi})$ is a field of 2×2 symmetric matrices whose components are the covariant components of the *second fundamental form* associated with the immersion $\boldsymbol{\phi}$.

The above definitions and notations apply as well to immersions $\boldsymbol{\phi} : \omega \rightarrow \mathbb{R}^3$ and their associated unit vector fields $\mathbf{v}(\boldsymbol{\phi}) : \omega \rightarrow \mathbb{R}^3$ that are both of class \mathcal{C}^1 up to the boundary of ω , or of class $W^{1,p}$ in ω , $1 < p < \infty$. This being the case, we let

$$\mathcal{C}_+^1(\bar{\omega}; \mathbb{R}^3) := \{\boldsymbol{\phi} \in \mathcal{C}^1(\bar{\omega}; \mathbb{R}^3); |\partial_1\boldsymbol{\phi} \wedge \partial_2\boldsymbol{\phi}| > 0 \text{ in } \bar{\omega}, \mathbf{v}(\boldsymbol{\phi}) \in \mathcal{C}^1(\bar{\omega}; \mathbb{R}^3)\} \tag{6}$$

and

$$W_+^{1,p}(\omega; \mathbb{R}^3) := \{\boldsymbol{\phi} \in W^{1,p}(\omega; \mathbb{R}^3); |\partial_1\boldsymbol{\phi} \wedge \partial_2\boldsymbol{\phi}| > 0 \text{ a.e. in } \omega, \mathbf{v}(\boldsymbol{\phi}) \in W^{1,p}(\omega; \mathbb{R}^3)\}. \tag{7}$$

The objective of this Note is to indicate how to establish a *nonlinear Korn inequality with an explicit estimate of the constant* that appears in it for mappings $\boldsymbol{\phi} \in W_+^{1,p}(\omega; \mathbb{R}^3)$ and $\boldsymbol{\theta} \in \mathcal{C}_+^1(\bar{\omega}; \mathbb{R}^3)$; see Theorem 1 below. We will show in particular that the estimate for the constant depends on $\boldsymbol{\theta}$ only via two scalar parameters, denoted ρ and δ in what follows, which are related to the assumption that $\boldsymbol{\theta}$ is an immersion such that $\boldsymbol{\theta}$ and $\mathbf{v}(\boldsymbol{\theta})$ are continuously differentiable vector fields over $\bar{\omega}$.

Note that the nonlinear Korn inequality of Theorem 1 constitutes an improvement, when $n = 3$, over two previous results by the authors about hypersurfaces in \mathbb{R}^n , $n \geq 3$: see [7, Theorem 3.1 and Lemma 3.2], or [6, Lemma 2].

The definition of the constant C_ω in the next statement is justified by relations (1)-(2) in Section 1.

Theorem 1. *Given any domain $\omega \subset \mathbb{R}^2$ and any real numbers $p > 1$, $1 \geq \rho > 0$ and $\delta > 0$, there exists a constant $C = C(\omega, p, \rho, \delta)$ such that*

$$\begin{aligned} & \inf_{\mathbf{R} \in \mathbb{O}_+^3} \left(\|\mathbf{v}(\boldsymbol{\phi}) - \mathbf{R}\mathbf{v}(\boldsymbol{\theta})\|_{L^p(\omega)} + \|\nabla\boldsymbol{\phi} - \mathbf{R}\nabla\boldsymbol{\theta}\|_{\mathbb{L}^p(\omega)} + \|\nabla\mathbf{v}(\boldsymbol{\phi}) - \mathbf{R}\nabla\mathbf{v}(\boldsymbol{\theta})\|_{\mathbb{L}^p(\omega)} \right) \\ & \leq C \left\| \inf_{\mathbf{R} \in \mathbb{O}_+^3} (|\mathbf{v}(\boldsymbol{\phi}) - \mathbf{R}\mathbf{v}(\boldsymbol{\theta})| + |\nabla\boldsymbol{\phi} - \mathbf{R}\nabla\boldsymbol{\theta}| + |\nabla\mathbf{v}(\boldsymbol{\phi}) - \mathbf{R}\nabla\mathbf{v}(\boldsymbol{\theta})|) \right\|_{L^p(\omega)} \\ & \leq C\sqrt{3} \left(\| \mathbf{A}(\boldsymbol{\phi})^{1/2} - \mathbf{A}(\boldsymbol{\theta})^{1/2} \|_{\mathbb{L}^p(\omega)} + \| \mathbf{A}(\boldsymbol{\phi})^{-1/2} \mathbf{B}(\boldsymbol{\phi}) - \mathbf{A}(\boldsymbol{\theta})^{-1/2} \mathbf{B}(\boldsymbol{\theta}) \|_{\mathbb{L}^p(\omega)} \right) \end{aligned}$$

for all mappings $\phi \in W_+^{1,p}(\omega; \mathbb{R}^3)$ and $\theta \in \mathcal{C}_{\rho,\delta,\mu}^1(\bar{\omega}; \mathbb{R}^3)$, where

$$\mu > 0 \text{ is any real number such that } \mu \leq \frac{\rho^{11}}{468(1 + C_\omega)},$$

$$C_\omega \geq 1 \text{ is any constant such that } \text{dist}_\omega(y, \tilde{y}) \leq C_\omega |y - \tilde{y}| \text{ for all } y, \tilde{y} \in \omega,$$

and

$$\begin{aligned} \mathcal{C}_{\rho,\delta,\mu}^1(\bar{\omega}; \mathbb{R}^3) := & \left\{ \theta \in \mathcal{C}_+^1(\bar{\omega}; \mathbb{R}^3); \inf_{y \in \bar{\omega}} |\partial_1 \theta(y) \wedge \partial_2 \theta(y)| \geq \rho, \sup_{y \in \bar{\omega}} |\nabla \theta(y)| \leq \frac{1}{\rho}, \sup_{y \in \bar{\omega}} |\nabla \mathbf{v}(\theta)(y)| \leq \frac{1}{\rho}, \right. \\ & \left. \sup_{\substack{y, \tilde{y} \in \bar{\omega}, \\ |y - \tilde{y}| \leq \delta}} |\nabla \theta(y) - \nabla \theta(\tilde{y})| \leq \mu, \sup_{\substack{y, \tilde{y} \in \bar{\omega}, \\ |y - \tilde{y}| \leq \delta}} |\nabla \mathbf{v}(\theta)(y) - \nabla \mathbf{v}(\theta)(\tilde{y})| \leq \mu \right\}. \end{aligned}$$

The proof of the Theorem 1 is sketched in Section 3 below; the details are given in [9].

The restriction in Theorem 1 that θ belongs to the subset $\mathcal{C}_{\rho,\delta,\mu}^1(\bar{\omega}; \mathbb{R}^3)$ of the set $\mathcal{C}_+^1(\bar{\omega}; \mathbb{R}^3)$, rather than to the set $\mathcal{C}_+^1(\bar{\omega}; \mathbb{R}^3)$ itself, is essential (i.e., not merely an artefact of the proof). However, this inconvenient is alleviated by the fact that, as $\rho \rightarrow 0^+$ and $\delta \rightarrow 0^+$, the subset $\mathcal{C}_{\rho,\delta,\mu}^1(\bar{\omega}; \mathbb{R}^3)$ becomes as large in $\mathcal{C}_+^1(\bar{\omega}; \mathbb{R}^3)$ as one wants. More specifically, for each $\mu > 0$,

$$\mathcal{C}_+^1(\bar{\omega}; \mathbb{R}^3) = \lim_{\rho \rightarrow 0^+} \left(\lim_{\delta \rightarrow 0^+} \mathcal{C}_{\rho,\delta,\mu}^1(\bar{\omega}; \mathbb{R}^3) \right),$$

where the limits above are defined as the union of an increasing sequence of sets.

3. Sketch of the proof of Theorem 1

The proof is broken for clarity into six steps, numbered (i) to (vi).

Proof. As in the statement of the Theorem 1, let there be given a domain $\omega \subset \mathbb{R}^2$, a constant C_ω such that

$$\text{dist}_\omega(y, \tilde{y}) \leq C_\omega |y - \tilde{y}| \text{ for all } y, \tilde{y} \in \omega,$$

four real numbers $p > 1, 1 \geq \rho > 0, \delta > 0, \mu > 0$, and two mappings

$$\theta \in \mathcal{C}_{\rho,\delta,\mu}^1(\bar{\omega}; \mathbb{R}^3) \text{ and } \phi \in W_+^{1,p}(\omega; \mathbb{R}^3).$$

Then let $\lambda := \rho/3, \eta := 13\mu\rho^{-8}/3, \varepsilon := \rho^6/3$, let $\Omega^\varepsilon := \omega \times (-\varepsilon, \varepsilon)$, and let $\Theta : \Omega^\varepsilon \rightarrow \mathbb{R}^3$ and $\Phi : \Omega^\varepsilon \rightarrow \mathbb{R}^3$ be the mappings defined by

$$\Theta(x) := \theta(y) + x_3 \mathbf{v}(\theta)(y) \text{ for all } x = (y, x_3) \in \Omega^\varepsilon$$

and

$$\Phi(x) := \phi(y) + x_3 \mathbf{v}(\phi)(y) \text{ for almost all } x = (y, x_3) \in \Omega^\varepsilon.$$

Note that the above definition of the constants λ, η and ε is justified by the estimates established in Step (iv) below.

Step (i). There exists a constant $C_1(p, \rho) > 0$ such that

$$\begin{aligned} & \inf_{\mathbf{R} \in \mathbb{O}_+^3} \|\nabla \Phi - \mathbf{R} \nabla \Theta\|_{\mathbb{L}^p(\Omega^\varepsilon)} \\ & \geq C_1(p, \rho) \inf_{\mathbf{R} \in \mathbb{O}_+^3} \left(\|\mathbf{v}(\phi) - \mathbf{R} \mathbf{v}(\theta)\|_{\mathbb{L}^p(\omega)} + \|\nabla \phi - \mathbf{R} \nabla \theta\|_{\mathbb{L}^p(\omega)} + \|\nabla \mathbf{v}(\phi) - \mathbf{R} \nabla \mathbf{v}(\theta)\|_{\mathbb{L}^p(\omega)} \right). \end{aligned}$$

The proof of this inequality relies on Clarkson’s inequalities in the space $L^p(\Omega^\varepsilon)$ (see, e.g., Adams [1, Theorem 2.28]) and follows an argument previously used in Ciarlet, Malin & Mardare [4, Proof of Theorem 4.2].

Step (ii). There exists a constant $C_2(p, \rho) > 0$ such that

$$\begin{aligned} & \left\| \inf_{\mathbf{R} \in \mathbb{O}_+^3} |\nabla \Phi - \mathbf{R} \nabla \Theta| \right\|_{L^p(\Omega^\varepsilon)} \\ & \leq C_2(p, \rho) \left\| \inf_{\mathbf{R} \in \mathbb{O}_+^3} \left(|\mathbf{v}(\phi) - \mathbf{R} \mathbf{v}(\theta)| + |\nabla \phi - \mathbf{R} \nabla \theta| + |\nabla \mathbf{v}(\phi) - \mathbf{R} \nabla \mathbf{v}(\theta)| \right) \right\|_{L^p(\omega)}. \end{aligned}$$

The proof of this inequality uses either Jensen’s inequality if $p > 2$, or the inequality $(a + b + c)^{p/2} \leq a^{p/2} + b^{p/2} + c^{p/2}$ if $p \leq 2$ for some appropriate nonnegative real numbers a, b and c , followed by an appropriate application of Fubini’s theorem.

Step (iii). The following assertions hold:

$$\mathbf{A}(\theta) \in \mathcal{C}^0(\bar{\omega}; \mathbb{S}_>^2), \quad \mathbf{A}(\theta)^{-1} \in \mathcal{C}^0(\bar{\omega}; \mathbb{S}_>^2), \quad \mathbf{B}(\theta) \in \mathcal{C}^0(\bar{\omega}; \mathbb{S}^2), \quad \Theta \in \mathcal{C}^1(\bar{\Omega}^\varepsilon; \mathbb{R}^3),$$

and

$$\mathbf{A}(\phi)^{1/2} \in L^p(\omega; \mathbb{S}_>^2), \quad \mathbf{A}(\phi)^{-1/2} \mathbf{B}(\phi) \in L^p(\omega; \mathbb{M}^2), \quad \Phi \in W^{1,p}(\Omega^\varepsilon; \mathbb{R}^3).$$

These assertions are straightforward generalisations of similar ones established in Ciarlet, Gratie & Mardare [3] for $p = 2$, and for this reason their proof is omitted.

Step (iv). The mapping Θ satisfies the following properties:

$$\det \nabla \Theta(x) \geq \lambda \quad \text{and} \quad |\nabla \Theta(x)| \leq \frac{1}{\lambda} \quad \text{for all } x \in \bar{\Omega}^\varepsilon,$$

and

$$|\nabla \Theta(x) - \nabla \Theta(\tilde{x})| \leq \eta \quad \text{for all } x, \tilde{x} \in \bar{\Omega}^\varepsilon \text{ such that } |x - \tilde{x}| \leq \delta.$$

Using the estimates in terms of ρ of the partial derivatives of θ and $\mathbf{v}(\theta)$ appearing in the definition of the set $\mathcal{C}_{\rho, \delta, \mu}^1(\bar{\omega}; \mathbb{R}^3)$ (see the statement of Theorem 1), we first deduce from Weingarten’s equations that

$$|\nabla \Theta(x)| \leq \frac{7}{3\rho} \quad \text{and} \quad \det \nabla \Theta(x) \geq \frac{11\rho}{18} \quad \text{for all } x = (y, x_3) \in \bar{\Omega}^\varepsilon,$$

then we deduce from the definition of the vector field $\mathbf{v}(\theta)$ in terms of the partial derivatives of θ (see relation (3)) that

$$|\mathbf{v}(y) - \mathbf{v}(\tilde{y})| \leq \frac{3}{\rho^8} |\nabla \theta(y) - \nabla \theta(\tilde{y})| \quad \text{for all } y, \tilde{y} \in \bar{\omega}.$$

Combined with the definition of the mapping Θ in terms of θ and the definition of the parameter ε in terms of ρ , the last inequality implies that, for each $x = (y, x_3) \in \bar{\Omega}^\varepsilon$ and each $\tilde{x} = (\tilde{y}, \tilde{x}_3) \in \bar{\Omega}^\varepsilon$,

$$\begin{aligned} |\nabla \Theta(x) - \nabla \Theta(\tilde{x})| & \leq |\nabla \theta(y) - \nabla \theta(\tilde{y})| + \varepsilon |\nabla \mathbf{v}(y) - \nabla \mathbf{v}(\tilde{y})| + |\mathbf{v}(y) - \mathbf{v}(\tilde{y})| \\ & \leq \left(1 + \frac{3}{\rho^8} \right) |\nabla \theta(y) - \nabla \theta(\tilde{y})| + \frac{\rho^6}{3} |\nabla \mathbf{v}(y) - \nabla \mathbf{v}(\tilde{y})|. \end{aligned}$$

Assume next that x and \tilde{x} satisfy $|x - \tilde{x}| \leq \delta$, so that, in particular, $|y - \tilde{y}| \leq \delta$. Then we infer from the definition of the space $\mathcal{C}_{\rho, \delta, \mu}^1(\bar{\omega}; \mathbb{R}^3)$ and from the previous estimate that

$$|\nabla \Theta(x) - \nabla \Theta(\tilde{x})| \leq \left(1 + \frac{3}{\rho^8} + \frac{\rho^6}{3} \right) \mu \leq \frac{13\mu}{3\rho^8}.$$

Step (v). Assume that the given constant $\mu > 0$ satisfies $\mu \leq \frac{\rho^{11}}{468(1+C_\omega)}$. Then there exists a constant $C_3(\omega, p, \rho, \delta)$ depending only on ω, p, ρ, δ such that

$$\inf_{\mathbf{R} \in \mathbb{O}_+^3} \|\nabla \Phi - \mathbf{R} \nabla \Theta\|_{L^p(\Omega^\varepsilon)} \leq C_3(\omega, p, \rho, \delta) \left\| \inf_{\mathbf{R} \in \mathbb{O}_+^3} |\nabla \Phi - \mathbf{R} \nabla \Theta| \right\|_{L^p(\Omega^\varepsilon)}. \tag{8}$$

First, the inequalities established in (iv) imply that the mapping Θ belong to the set:

$$\mathcal{C}_{\lambda, \delta, \eta}^1(\overline{\Omega^\varepsilon}; \mathbb{R}^3) := \left\{ \Theta \in \mathcal{C}^1(\overline{\Omega^\varepsilon}; \mathbb{R}^3); \inf_{x \in \overline{\Omega^\varepsilon}} \det \nabla \Theta(x) \geq \lambda, \sup_{x \in \overline{\Omega^\varepsilon}} |\nabla \Theta(x)| \leq \frac{1}{\lambda}, \sup_{\substack{x, \tilde{x} \in \overline{\Omega^\varepsilon} \\ |x - \tilde{x}| \leq \delta}} |\nabla \Theta(x) - \nabla \Theta(\tilde{x})| \leq \eta \right\}.$$

Secondly, by (iii),

$$\Phi \in W^{1,p}(\Omega^\varepsilon; \mathbb{R}^3).$$

Thirdly, the definition of the set Ω^ε in terms of ω , the definition of the geodesic distance in Ω^ε (see relation (1)), and the definition of the constant C_ω (see the statement of Theorem 1), together show that, for each $x = (y, x_3) \in \overline{\Omega^\varepsilon}$ and each $\tilde{x} = (\tilde{y}, \tilde{x}_3) \in \overline{\Omega^\varepsilon}$,

$$\text{dist}_{\Omega^\varepsilon}(x, \tilde{x}) \leq \text{dist}_\omega(y, \tilde{y}) + |x_3 - \tilde{x}_3| \leq C_\omega |y - \tilde{y}| + |x_3 - \tilde{x}_3| \leq (1 + C_\omega) |x - \tilde{x}|.$$

Fourthly, the assumption on μ made in (v) implies that

$$\eta \leq \frac{\lambda^3}{4(1 + C_\omega)}.$$

The four observations above together imply that the assumptions of [8, Theorem 1 (a)] are satisfied by the domain Ω^ε and by the mappings Θ and Φ from Ω^ε into \mathbb{R}^3 . Thus inequality (8) holds as a consequence of this theorem.

Step (vi). Combining the three inequalities established in steps (i), (ii) and (v) above yields the inequality

$$\begin{aligned} \inf_{\mathbf{R} \in \mathbb{O}_+^3} \left(\|\mathbf{v}(\phi) - \mathbf{R} \mathbf{v}(\theta)\|_{L^p(\omega)} + \|\nabla \phi - \mathbf{R} \nabla \theta\|_{L^p(\omega)} + \|\nabla \mathbf{v}(\phi) - \mathbf{R} \nabla \mathbf{v}(\theta)\|_{L^p(\omega)} \right) \\ \leq C \left\| \inf_{\mathbf{R} \in \mathbb{O}_+^3} (|\nabla \phi - \mathbf{R} \nabla \theta| + |\mathbf{v}(\phi) - \mathbf{R} \mathbf{v}(\theta)| + |\nabla \mathbf{v}(\phi) - \mathbf{R} \nabla \mathbf{v}(\theta)|) \right\|_{L^p(\omega)}, \end{aligned}$$

where $C := (C_1(p, \rho))^{-1} C_2(p, \rho) C_3(\omega, p, \rho, \delta)$. This establishes the first inequality of Theorem 1.

Using the polar decomposition of the 3×3 matrix fields

$$(\nabla \theta | \mathbf{v}(\theta)) \text{ and } (\nabla \phi | \mathbf{v}(\phi))$$

and a method similar to one used in the proof of [7, Theorem 3.1], we next show that the following inequality holds almost everywhere in ω :

$$\begin{aligned} \inf_{\mathbf{R} \in \mathbb{O}_+^3} \left(|\mathbf{v}(\phi) - \mathbf{R} \mathbf{v}(\theta)|^2 + |\nabla \phi - \mathbf{R} \nabla \theta|^2 + |\nabla \mathbf{v}(\phi) - \mathbf{R} \nabla \mathbf{v}(\theta)|^2 \right) \\ \leq |\mathbf{A}(\phi)^{1/2} - \mathbf{A}(\theta)^{1/2}|^2 + |\mathbf{A}(\phi)^{-1/2} \mathbf{B}(\phi) - \mathbf{A}(\theta)^{1/2} \mathbf{B}(\theta)|^2. \end{aligned}$$

Consequently,

$$\left\| \inf_{\mathbf{R} \in \mathbb{O}_+^3} (|\mathbf{v}(\boldsymbol{\phi}) - \mathbf{R}\mathbf{v}(\boldsymbol{\theta})| + |\nabla\boldsymbol{\phi} - \mathbf{R}\nabla\boldsymbol{\theta}| + |\nabla\mathbf{v}(\boldsymbol{\phi}) - \mathbf{R}\nabla\mathbf{v}(\boldsymbol{\theta})|) \right\|_{L^p(\omega)} \leq \sqrt{3} \left(\|\mathbf{A}(\boldsymbol{\phi})^{1/2} - \mathbf{A}(\boldsymbol{\theta})^{1/2}\|_{\mathbb{L}^p(\omega)} + \|\mathbf{A}(\boldsymbol{\phi})^{-1/2}\mathbf{B}(\boldsymbol{\phi}) - \mathbf{A}(\boldsymbol{\theta})^{-1/2}\mathbf{B}(\boldsymbol{\theta})\|_{\mathbb{L}^p(\omega)} \right).$$

This establishes the second inequality of Theorem 1. \square

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